ISLAMIC AZAD UNIVERSITY
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GENERAL DUAL FUZZY LINEAR SYSTEMS

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**General dual fuzzy linear systems**" by **Maryam Mosleh** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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To My dears:

My Husband "Mahmood"

My Mother and My Father

Contents

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Abstract

One of the major applications by using fuzzy number arithmetics is treating fuzzy linear systems. In this thesis, some new approaches for solving fuzzy linear systems, dual fuzzy linear systems and system of fuzzy polynomial equations, are presented. In this work, a numerical method for finding minimal solution of a $m \times n$, $Ax+f = Bx+c$ general duality fuzzy linear system based on pseudo inverse calculation is given when the matrix of coefficients is row full rank or column full rank, and A, B are real $m \times n$ matrices, the unknown vector x is a vector consisting of n fuzzy numbers and the constants f and c are vectors consisting of m fuzzy numbers. Initially we wrote system of fuzzy polynomial equations as parametric form and then solve it by Fixed point method and for finding the real root (if exist) of fuzzy polynomial, we applied the Adomian decomposition method.

Originality

The following chapters are proposed in this work:

Chapter 3: We define general dual fuzzy linear systems and propose a numerical method for finding minimal solution of these systems.

Chapter 4: we define system of fuzzy polynomial equations and apply a numerical method for finding fuzzy solution of these systems by using parametric form of fuzzy numbers.

Articles

1. Minimal solution of general dual fuzzy linear systems, *Journal of Chaos Solitons & Fractals,* 37 (2008) 1113-1124 (ISI).

2. Iterative method for fuzzy equations, *Journal of Soft Comut* 12 (2008) 935-939 (ISI).

3. General dual fuzzy linear systems, *Journal of Contemporary Mathematical Sciences*, 3 (2008) 1385-1394 (AMS).

Chapter 1

Fuzzy Mathematics

1.1 Introduction

Fuzziness is not a priori an obvious concept and demands some explanations. "Fuzziness" is what Black calls "vagueness" when he distinguishes it from "generality" and from "ambiguity". Generalizing refers to the application of a symbol to a multiplicity of objects in the field of reference, ambiguity to the association of a finite number of alternative meanings having the same phonetic form. But, the fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol applies.

More specifically, let X be a field of reference, also called a universe of discourse or universe for short, covering a definite range of objects. Consider a subset $\tilde{A}$ where transition between membership and nonmembership is gradual rather than abrupt. This "fuzzy subset" obviously has no well-defined boundaries. Fuzzy classes of objects are often encountered in real life. For instance, $\tilde{A}$ may be the set of tall men in a community X. Usually, there are members of X who are definitely tall, others who are definitely not tall, but there exist also borderline cases. Traditionally, the grade of membership 1 is assigned to the objects that completely belong to $\tilde{A}$-here the men who are definitely tall, conversely the objects that do not belong to $\tilde{A}$ at all are assigned a membership value 0. Quite naturally, the grades of membership of the borderline cases lie between 0 and 1. The more an element or object x belongs to $\tilde{A}$, the closer to 1 is its grade of membership $\mu_{\tilde{A}}(x)$. The use of a numerical scale such as the interval [0, 1] allows a convenient representation of the gradation in membership. Precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or a group. Moreover, they are context-dependent. The grades of membership reflect an "ordering" of the objects in the universe, induced by the predicate associated with $\tilde{A}$; this "ordering", when it exists, is more important than the membership values themselves. The membership assessment of objects can sometimes be made easier by the use of a similarity measure with respect to an ideal element. Note that a membership value $\mu_{\tilde{A}}(x)$ can be interpreted as the degree of compatibility of the predicate associated with $\tilde{A}$ and the object x. For concepts such as "tallness", related to a physical measurement scale, the assignment of membership values will often be less controversial than for more complex and subjective concepts such as "beauty".

The above approach, developed by Zadeh (1964), provides a tool for modeling human-centered systems. As a matter of fact, fuzziness seems to pervade most human perception and thinking processes. Parikh (1977) has pointed out that no nontrivial first-order-logic-like observational predicate (i.e., one pertaining to perception) can be defined on an observationally connected space; the only possible observational predicates on such a space are not classical predicates but "vague" ones. Moreover, according to Zadeh (1973), one of the most important facets of human thinking is the ability to summarize information "into labels of fuzzy sets which bear an approximate relation to the primary data". Linguistic descriptions, which are usually summary descriptions of complex situations, are fuzzy in essence.

It must be noticed that fuzziness differs from imprecision. In tolerance analysis imprecision refers to lack of knowledge about the value of a parameter and is thus expressed as a crisp tolerance interval. This interval is the set of possible values of the parameters. Fuzziness occurs when the interval has no sharp boundaries, i.e., is a fuzzy set $\tilde{A}$. Then, $\mu_{\tilde{A}}(x)$ is interpreted as the degree of possibility (Zadeh, 1978) that x is the value of the parameter fuzzily restricted by $\tilde{A}$.

The word fuzziness has also been used by Sugeno (1977) in a radically different context. Consider an arbitrary object x of the universe X; to each nonfuzzy subset A of X is assigned a value $g_x(A) \in [0,1]$ expressing the "grade of fuzziness" of the statement "x belongs to A". In fact this grade of fuzziness must be understood as a grade of certainty: according to the mathematical definition of g, $g_x(A)$ can be interpreted as the probability, the degree of subjective belief, the possibility that x belongs to A.

Generally, g is assumed increasing in the sense of set inclusion, but not necessarily additive as in the probabilistic case. The situation modeled by Sugeno is more a matter of guessing whether $x \in A$ rather than a problem of vagueness in the sense of Zadeh. The existence of two different points of view on "fuzziness" has been pointed out by MacVicar-Whelan (1977) and Skala. The monotonicity assumption for g seems to be more consistent with human guessing than does the additivity assumption. For instance, seeing a piece of Indian pottery in a shop, we may try to guess whether it is genuine or counterfeit; obviously, genuineness is a fuzzy concept. Hence x is the Indian pottery; A is the crisp set of genuine Indian artifacts; and $g_x(A)$ expresses, for instance, a subjective belief that the pottery is indeed genuine. The situation is slightly more complicated when we try to guess whether the pottery is old: actually, the set $\tilde{A}$ of old Indian pottery is fuzzy because "old" is a vague predicate.

1.2 Fuzzy Sets

Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X is often viewed as a characteristic function, μ_A from X to $\{0, 1\}$ such that

$$\mu_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in X - A. \end{cases}$$

If the valuation set, i.e., 0 and 1, is allowed to be the real interval $[0, 1]$, A is called a fuzzy set (Zadeh, 1965), and $\mu_A(x)$ is the grade of membership of x in A. Sometimes we denote $\mu_A(x)$ by $A(x)$. The closer the value of $\mu_A(x)$ is to 1, the more

x belongs to A. Clearly, A is a subset of X that has no sharp boundary. Hence A is completely characterized by the set of pairs

$$A = \{(x, \mu_A(x)) | x \in X\}.$$

A more convenient notation was proposed by Zadeh [33]. When X is a finite set $\{x_1, ..., x_n\}$ a fuzzy set on X is expressed as

$$A = \frac{\mu_A(x_1)}{x_1} + \cdots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}.$$

When X is not finite, we write

$$A = \int_x \mu_A(x).$$

Definition 1.2.1. (Equality of fuzzy sets) Two fuzzy sets A and B are said to be equal (denoted $A = B$) iff

$$\forall\ x \in X,\ \ \mu_A(x) = \mu_B(x).$$

Definition 1.2.2. (Support) The support of a fuzzy set A is the ordinary subset of X:

$$supp\ A = \{x \in X,\ \ \mu_A(x) > 0\}.$$

Definition 1.2.3. (Core) The core of a fuzzy set A is the set of all points with the membership degree one in A:

$$coreA = \{x \in X |\ \mu_A(x) = 1\}.$$

Definition 1.2.4. (Height of a fuzzy set) The height of A is $hgt(A) = sup_{x \in X}\ \mu_A(x)$, i.e., the least upper bound of $\mu_A(x)$.

Definition 1.2.5. (Normal fuzzy set) A is said to be normal if and only if $\exists x \in X,\ \mu_A(x) = 1$; this definition implies $hgt(A) = 1$.

Definition 1.2.6. (Empty fuzzy set) The empty set ϕ is defined as $\forall\ x \in X,\ \mu_\phi(x) = 0$; of course, $\forall\ X,\ \mu_X(x) = 1$.

Definition 1.2.7. ($h - Cuts$) When we want to exhibit an element $x \in X$ that typically belongs to a fuzzy set A, we may demand its membership value to be greater than some threshold $h \in [0, 1]$. The ordinary set of such elements is the $h - cut$ of A,

$$[A]_h = \begin{cases} \{\ x \in X \mid \mu_A(x) \geq h\ \} & if \quad h > 0 \\ cl(suppA) & if \quad h = 0 \end{cases}$$

where $cl(suppA)$ denotes the closure of the support of A. One also defines the strong h-cut as $[A]_{\overline{h}} = \{\ x \in X,\ \mu_A(x) > h\ \}$.

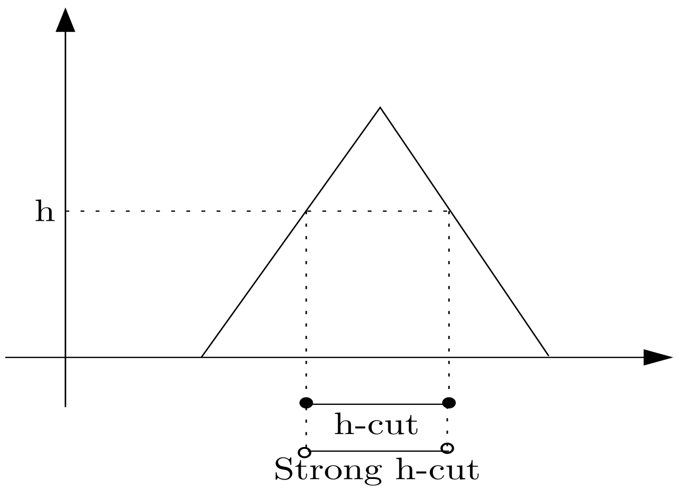

Figure 1.1: $h - Cut\ \ and\ \ Strong\ h - Cut.$

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its $h - cuts$ according to the formula [16]

$$\mu_A(x) = \sup_{h \in]0,1]} \min(h, \mu_{A_h}(x)).$$

It is easily checked that the following properties hold:

$$[A \cup B]_h = [A]_h \cup [B]_h, \qquad [A \cap B]_h = [A]_h \cap [B]_h.$$

Definition 1.2.8. (Convexity) A fuzzy set A of X is called convex if $[A]_h$ is a convex subset of X for each $h \in [0, 1]$. We now state a useful theorem that provides us with an alternative formulation of convexity of fuzzy sets. For the sake of simplicity, we restrict the theorem to fuzzy sets on $\mathbb{R}$, which are of primary interest in this text.

Theorem 1.2.1. *A fuzzy set A on $\mathbb{R}$ is convex if and only if*

$$A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{A(x_1), A(x_2)\} \tag{1.1}$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0, 1]$, where min *denotes the minimum operator.*

Proof. Assume that A is convex and let $h = A(x_1) \leq A(x_2)$. Hence $x_1, x_2 \in [A]_h$ and, moreover, $\lambda x_1 + (1 - \lambda)x_2 \in [A]_h$ for any $\lambda \in [0, 1]$ by the convexity of A. Consequently,

$$A(\lambda x_1 + (1 - \lambda)x_2) \geq h = A(x_1) = \min\{A(x_1), A(x_2)\}.$$

Conversely, assume that A satisfies (1.1). We need to prove that for any $h \in [0, 1]$, $[A]_h$ is convex. Now for any $x_1, x_2 \in [A]_h$(*i.e.*, $A(x_1) \geq h$, $A(x_2) \geq h$), and for any

$\lambda \in [0,1]$, by (1.1)

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\} \geq \min(h,h) = h,$$

i.e., $\lambda x_1 + (1-\lambda)x_2 \in [A]_h$. Therefore, $[A]_h$ is convex for any $h \in [0,1]$. Hence, A is convex. $\square$

Among the various types of fuzzy sets, of special significance are fuzzy sets that are defined on the set $\mathbb{R}$ of real numbers. Membership functions of these sets, which have the form

$$A : \mathbb{R} \to [0,1],$$

clearly have a quantitative meaning and may, under certain conditions, be viewed as fuzzy number that can be defined as follows.

Definition 1.2.9. (Fuzzy number) A fuzzy number is a map $A : \mathbb{R} \to I = [0,1]$ which satisfies:

(i) A is upper semi-continuous, i.e.,

$$\{x | A(x) < t\} \;\; is \;\; open \;\; for \; all \; t \in \mathbb{R}.$$

(ii) $A(x) = 0$ outside some interval $[c,d] \subset \mathbb{R}$,

(iii) There exist real numbers a, b such that $c \leq a \leq b \leq d$ where

1. $A(x)$ is monotonic increasing on $[c,a]$,
2. $A(x)$ is monotonic decreasing on $[b,d]$,
3. $A(x) = 1, \; a \leq x \leq b$.

In this work, we show the set of fuzzy numbers by E.

Definition 1.2.10. (Quasi fuzzy number) A quasi fuzzy number A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions

$$\lim_{t\to+\infty} A(t) = 0, \qquad \lim_{t\to-\infty} A(t) = 0.$$

Remark 1.2.1. Let A be a fuzzy number. Then $[A]_h$ is a closed convex (compact) subset of $\mathbb{R}$ for all $h \in [0,1]$.

Definition 1.2.11. (Triangular fuzzy number) A fuzzy number A is called triangular fuzzy number with center a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{\alpha} & if \quad a-\alpha \leq t \leq a \\ 1 - \frac{(t-a)}{\beta} & if \quad a \leq t \leq a+\beta \\ 0 & \quad otherwise \end{cases}$$

and we use for it the notation $A = (a, \alpha, \beta)$. It can easily be verified that

$$[A]_h = [a - (1-h)\alpha, a + (1-h)\beta], \ \forall h \in [0,1].$$

The support of A is $(a-\alpha, a+\beta)$. If $\alpha = \beta$ A is called symmetrical triangular fuzzy number. Let A be a symmetrical triangular fuzzy number, then we use for it the notation $A = (a, \alpha)$.

Figure 1.2 represents the triangular fuzzy number $A = (6, 1, 2)$.

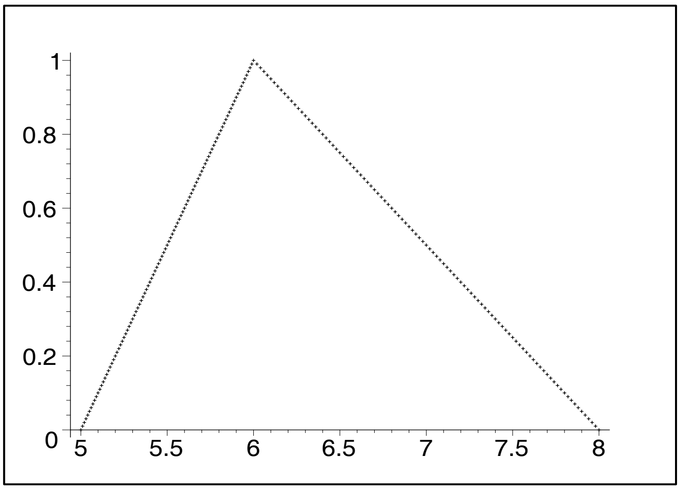

Figure 1.2: A triangular fuzzy number.

Definition 1.2.12. (Trapezoidal fuzzy number) A fuzzy number A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{\alpha} & if \quad a - \alpha \leq t \leq a, \\ 1 & if \quad a \leq t \leq b, \\ 1 - \frac{(t-b)}{\beta} & if \quad b \leq t \leq b + \beta, \\ 0 & otherwise, \end{cases}$$

and we use the notation $A = (a, b, \alpha, \beta)$. It can easily be shown that

$$[A]_h = [a - (1-h)\alpha, b + (1-h)\beta], \forall h \in [0, 1].$$

The support of A is $(a - \alpha, b + \beta)$.

Figure 1.3 represent the trapezoidal fuzzy number $A = (3, 5, 1, 2)$.

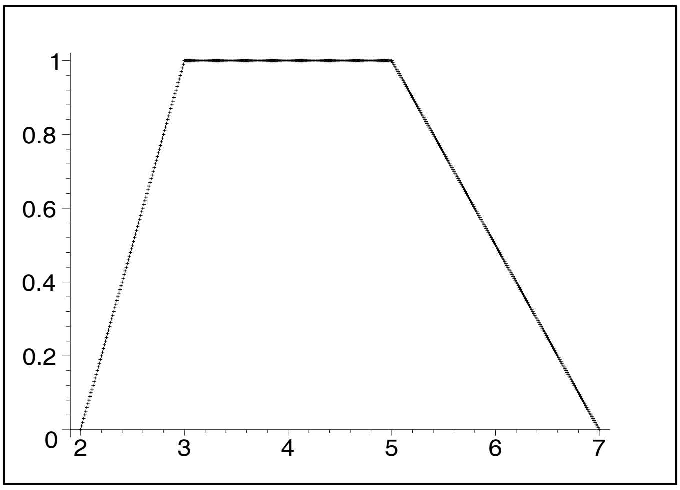

Figure 1.3: A trapezoidal fuzzy number.

Definition 1.2.13. (Parametric form) We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements [17]:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A crisp number λ is simply represented by $\underline{u}(r) = \overline{u}(r) = \lambda, 0 \leq r \leq 1$. [33] By appropriate definitions the fuzzy number space $\{\underline{u}(r), \overline{u}(r)\}$ becomes a convex cone, E.

Lemma 1.2.2. *Let v and w be fuzzy numbers and s be a real number. Then for*

$0 \le r \le 1$,

$$u = v \ \textit{if and only if} \ \underline{u}(r) = \underline{v}(r) \ \textit{and} \ \overline{u}(r) = \overline{v}(r),$$

$$v + w = (\underline{v}(r) + \underline{w}(r), \overline{v}(r) + \overline{w}(r)),$$

$$v - w = (\underline{v}(r) - \overline{w}(r), \overline{v}(r) - \underline{w}(r)),$$

$$v.w = (\min\{\underline{v}(r).\underline{w}(r), \underline{v}(r).\overline{w}(r), \overline{v}(r).\underline{w}(r), \overline{v}(r).\overline{w}(r)\},$$

$$\max\{\underline{v}(r).\underline{w}(r), \underline{v}(r).\overline{w}(r), \overline{v}(r).\underline{w}(r), \overline{v}(r).\overline{w}(r)\}),$$

$$sv = s(\underline{v}(r), \overline{v}(r)) = \begin{cases} (s\underline{v}(r), s\overline{v}(r)) \quad if \quad s \ge 0, \\ (s\overline{v}(r), s\underline{v}(r)) \quad if \quad s < 0. \end{cases}$$

See[31].

Definition 1.2.14. Any fuzzy number $A \in E$ can be described as

$$A(t) = \begin{cases} L(\frac{a-t}{\alpha}) & if \quad t \in [a-\alpha, a] \\ 1 & if \quad t \in [a, b] \\ R(\frac{t-b}{\beta}) & if \quad t \in [b, b+\beta] \\ 0 & otherwise \end{cases}$$

where $[a, b]$ is the core of A,

$$L : [0,1] \to [0,1], \quad R : [0,1] \to [0,1]$$

are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $R(1) = L(1) = 0$. We call the fuzzy interval of LR-type and refer to it by

$$A = (a, b, \alpha, \beta)_{LR}.$$

The support of A is $(a-\alpha, b+\beta)$.

Definition 1.2.15. (Fuzzy point) Let A be a fuzzy number. If supp(A)= $\{x_0\}$ then A is called a fuzzy point and we use the notation $A=\overline{x_0}$.

Figure 1.4 represents the fuzzy point $A=\overline{5}$.

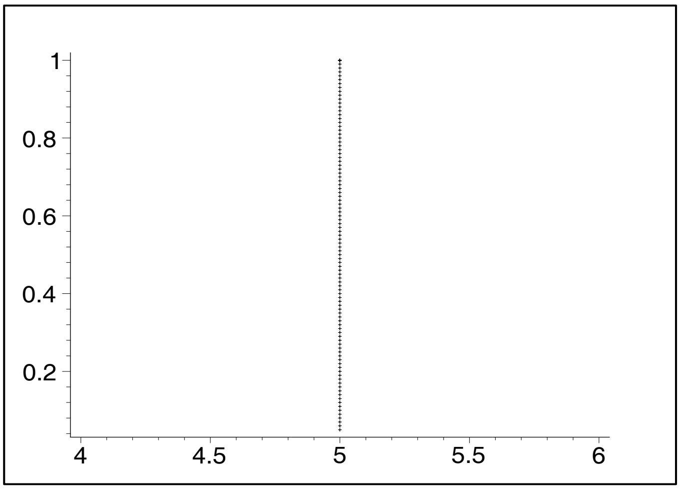

Figure 1.4: Fuzzy point $A=\overline{5}$.

Definition 1.2.16. The space E^n is all of fuzzy subsets U of $\mathbb{R}^n$ which satisfy the following conditions:

1. U is normal,
2. U is fuzzy convex,
3. U is upper semi-continuous,
4. $[U]_0$ is a bounded subset of $\mathbb{R}^n$,

when $n=1$, elements of E^1 are Fuzzy numbers.

1.3 The extension principle

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to *add*, *subtract*, *multiply* and *divide* with fuzzy quantities. The process of doing these operations is called *fuzzy arithmetic*.

We shall first introduce an important concept from fuzzy set theory called the *extension principle*, then use it to provide for these arithmetical operations on fuzzy numbers.

In general, the extension principle plays a fundamental role in enabling us to extend any point operations to involving fuzzy sets. In the sequel, we define this principle.

Definition 1.3.1. [20](extension principle) Assume X and Y are crisp sets and let f be a mapping from X to Y,

$$f : X \to Y,$$

such that for each $x \in X$, $f(x) = y \in Y$. Assume A is a fuzzy subset of X, using extension principle, we can define $f(A)$ as a fuzzy subset of Y such that

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Definition 1.3.2. (*sup-min extension n-place functions*) Let X_1, X_2,...,X_n and Y be a family of sets. Assume f is a mapping from the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$

into Y. Let A_1, A_2,...,A_n be fuzzy subsets of X_1, X_2,...,X_n, respectively, then we use the extension principle for the evaluation of $f(A_1, A_2, ..., A_n)$. $f(A_1, A_2, ..., A_n)$ is a fuzzy set such that

$$f(A_1, A_2, ..., A_n)(y) = \begin{cases} \sup\{\min\{A_1(x_1), A_2(x_2), \dots, A_n(x_n)\} \mid x \in f^{-1}(y)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad f^{-1}(y) \neq \emptyset, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

where $x = (x_1, x_2, ..., x_n)$.

Example 1.3.1. *Let $f : X \times X \to X$ be defined as*

$$f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2, \;\; \lambda_1, \;\; \lambda_2 \in \mathbb{R}.$$

Suppose A_1 and A_2 are fuzzy subsets of X. Then using the extension principle we get

$$f(A_1, A_2)(y) = \sup_{\lambda_1 x_1 + \lambda_2 x_2 = y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = \lambda_1 A_1 + \lambda_2 A_2$.

Definition 1.3.3. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be a function from $E^1(X)$ to $E^1(Y)$. Then f is called a fuzzy function (or mapping) and we use the notation

$$f : E(X) \to E(Y).$$

Let $A = (a_1, a_2, \alpha_1, \alpha_2)_{LR}$ and $B = (b_1, b_2, \beta_1, \beta_2)_{LR}$ be fuzzy intervals of $LR{-}type$. Using the (sup-min) extension principle, we can verify the following rules for addition and subtraction of fuzzy numbers of $LR - type$:

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_{LR}$$

$$A - B = (a_1 - b_1, a_2 - b_2, \alpha_1 + \beta_2, \alpha_2 + \beta_1)_{LR}$$

furthermore, if $\lambda \in \mathbb{R}$ is a real number then λA can be represented as

$$\lambda A = \begin{cases} (\lambda a_1, \lambda a_2, \lambda \alpha_1, \lambda \alpha_2)_{LR} & \text{if } \lambda \geq 0, \\ (\lambda a_2, \lambda a_1, \mid \lambda \mid \alpha_2, \mid \lambda \mid \alpha_1)_{LR} & \text{if } \lambda < 0. \end{cases}$$

In particular, if $A = (a_1, a_2, \alpha_1, \alpha_2)$ and $B = (b_1, b_2, \beta_1, \beta_2)$ are fuzzy numbers of trapezoidal form, then

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$A - B = (a_1 - b_1, a_2 - b_2, \alpha_1 + \beta_2, \alpha_2 + \beta_1).$$

If $A = (a, \alpha_1, \alpha_2)$ and $B = (b, \beta_1, \beta_2)$ are fuzzy numbers of triangular form, then

$$A + B = (a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$A - B = (a - b, \alpha_1 + \beta_2, \alpha_2 + \beta_1),$$

and if $A = (a, \alpha)$ and $B = (b, \beta)$ are fuzzy numbers of symmetrical triangular form, then

$$A + B = (a + b, \alpha + \beta)$$

$$A - B = (a - b, \alpha + \beta),$$

$$\lambda A = (\lambda a, \mid \lambda \mid \alpha).$$

The above results can be generalized to linear combination of fuzzy numbers.

Let A and B be fuzzy numbers with $[A]_h = [a_1(h), a_2(h)]$ and $[B]_h = [b_1(h), b_2(h)]$, $0 \leq h \leq 1$. Then it can easily be shown that

$$[A + B]_h = [a_1(h) + b_1(h), a_2(h) + b_2(h)],$$

$$[-A]_h = [-a_2(h), -a_1(h)],$$

$$[A - B]_h = [a_1(h) - b_2(h), a_2(h) - b_1(h)],$$

$$[\lambda A]_h = [\lambda a_1(h), \lambda a_2(h)] \quad if \quad \lambda \geq 0,$$

$$[\lambda A]_h = [\lambda a_2(h), \lambda a_1(h)] \quad if \quad \lambda < 0,$$

for all $h \in [0, 1]$, i.e. any h-level set of the extended sum of two fuzzy numbers is equal to the sum of their h-level sets. The following two theorems show that this property is valid for any continuous function.

Theorem 1.3.2. *[30] Let $f : X \to X$ be a continuous function and let A be a fuzzy number. Then,*

$$[f(A)]_h = f([A]_h),$$

where $f(A)$ is defined by the extension principle and

$$f([A]_h) = \{f(x) \mid x \in [A]_h\}.$$

If $[A]_h = [a_1(h), a_2(h)]$ and f is continuous and monotone increasing then from the above theorem we get

$$[f(A)]_h = f([A]_h) = f([a_1(h), a_2(h)]) = [f(a_1(h)), f(a_2(h))].$$

Theorem 1.3.3. *[30] Let $f : X \times X \to X$ be a continuous function and let A and B be fuzzy numbers. Then*

$$[f(A, B)]_h = f([A]_h, [B]_h)$$

where,

$$f([A]_h, [B]_h) = \{f(x_1, x_2) \mid x_1 \in [A]_h, x_2 \in [B]_h\}.$$

Let $f(x, y) = xy$ and let $[A]_h = [a_1(h), a_2(h)]$, $[B]_h = [b_1(h), b_2(h)]$ be the $h-$level sets of two fuzzy numbers A and B. Applying above theorem we get

$$[f(A, B)]_h = f([A]_h, [B]_h) = [A]_h[B]_h$$

The equation

$$[AB]_h = [A]_h[B]_h = [a_1(h)b_1(h), a_2(h)b_2(h)]$$

holds if and only if A and B are both nonnegative, i.e. $A(x) = B(x) = 0$ for $x \leq 0$. If B is nonnegative then we have

$$[A]_h[B]_h = [\min\{a_1(h)b_1(h), a_1(h)b_2(h)\}, \max\{a_2(h)b_1(h), a_2(h)b_2(h)\}].$$

In general case, we obtain a very complicated expression for the α-level sets of the product AB

$$[A]_h[B]_h = [\min\{a_1(h)b_1(h), a_1(h)b_2(h), a_2(h)b_1(h), a_2(h)b_2(h)\},$$

$$\max\{a_1(h)b_1(h), a_1(h)b_2(h), a_2(h)b_1(h), a_2(h)b_2(h)\}].$$

1.4 Hausdorff distance for fuzzy numbers

Let A and B be fuzzy numbers with $[A]_h = [a_1(h), a_2(h)]$ and $[B]_h = [b_1(h), b_2(h)]$.

Definition 1.4.1. (Hausdorff distance) The Hausdorff distance between two (nonempty) sets $X,\ Y \subseteq \mathbb{R}$ is given as

$$d_H(X,Y) = \max\{\beta(X,Y), \beta(Y,X)\},$$

where $\beta(X,Y) = \sup_{x\in X} \rho(x,Y)$ and $\rho(x,Y) = \inf_{y\in Y} \mid x-y \mid$. The generalization

$$d_H(A,B) \ \ = \ \ \sup_{h\in(0,1]} d_H([A]_h,[B]_h) \ \ \forall\ A,B \in E^1,$$

defines a distance measure [21]. It is clear that

$$d_H(A,B) = \sup_{h\in[0,1]} \max\{\mid a_1(h) - b_1(h) \mid, \mid a_2(h) - b_2(h) \mid\},$$

i.e. $d_H(A,B)$ is the maximal distance between h-level sets of A and B [29].

1.5 Operation on fuzzy numbers

Some previous works related to operations on fuzzy numbers are those of Jain [22], Nahmias [28], Mizumoto and Tanaka [27],[26] Baas and Kwakernaak [13].

1.5.1 Addition and Multiplication

Addition: Addition is an increasing operation. Hence, the extended addition $(\oplus)$ of fuzzy numbers gives a fuzzy number. Note that $-(M \oplus N) = (-M) \oplus (-N)$. $(\oplus)$ is commutative and associative but has no group structure. The identity of $(\oplus)$ is the nonfuzzy number 0. But M has no symmetrical element in the sense of a group structure. In particular, $M \oplus (-M) \neq 0, \quad \forall M \in E - \mathbb{R}$.

Multiplication: Multiplication is an increasing operation on $\mathbb{R}^+$ and a decreasing operation on $\mathbb{R}^-$. Hence, the product of fuzzy numbers ($\odot$) that are all either positive or negative gives a positive fuzzy number. Note that $-(M) \odot N = -(M \odot N)$, so that the factors can have different signs. ($\odot$) is commutative and associative. The set of positive fuzzy numbers is not a group for ($\odot$): although $\forall M, \quad M \odot 1 = M$, the product $M \odot M^{-1} \neq 1$ as soon as M is not a real number. M has no inverse in the sense of group structure.

1.5.2 Subtraction

Subtraction is neither increasing nor decreasing. However, it is easy to check that $M \ominus N = M \oplus (-N), \quad \forall (M, N) \in E^2$ so that $M \ominus N$ is a fuzzy number whenever M and N are.

1.5.3 Division

Division is neither increasing nor decreasing. But, since $M \oslash N = M \odot (N^{-1}), \forall (M, N) \in E^2$, $M \oslash N$ is a fuzzy number when M and N are positive or negative fuzzy numbers. The division of ordinary fuzzy numbers can be performed similar to multiplication, by decomposition.

Definition 1.5.1. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be a function from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$. Then f is said to be a fuzzy function (or mapping) and we use the notation

$$f : \mathcal{F}(X) \to \mathcal{F}(Y).$$

It should be noted, however, that a fuzzy function is not necessarily defined by Zadeh's extension principle. It can be any function which maps a fuzzy set $A \in \mathcal{F}(X)$ into a fuzzy set $B := f(A) \in \mathcal{F}(Y)$.

Definition 1.5.2. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets . A fuzzy mapping $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ is said to be monotonic increasing if from $A, A' \in \mathcal{F}(X)$ and $A \subset A'$ it follow that $f(A) \subset f(A')$.

Theorem 1.5.1. *Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets. Then every fuzzy mapping $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined by the extension principle is monotonic increasing.*

Proof. Let $A, A' \in \mathcal{F}(X)$ such that $A \subset A'$. Then using the definition of sup-min extension principle we get

$$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \leq \sup_{x \in f^{-1}(y)} A'(x) = f(A')(y)$$

for all $y \in Y$. $\square$

Example 1.5.2. *Let $f(x, y) = xy$ and let $[A]_h = [a_{1h}, a_{2h}]$ and $[B]_h = [b_{1h}, b_{2h}]$ be two fuzzy numbers. Applying theorem 1.5.1 we get*

$$[AB]_h = [A]_h[B]_h = [a_{1h}b_{1h}, a_{2h}b_{2h}],$$

hold if and only if A and B are both nonnegative, i.e. $A(x) = B(x) = 0$ for $x \leq 0$.

In general form we obtain a very complicated expression for the α level sets of the product AB

$$[AB]_h = [m_h, M_h],$$

where

$$m_h = \min\{a_{1h}b_{1h}, a_{1h}b_{2h}, a_{2h}b_{1h}, a_{2h}b_{2h}\},$$

$$M_h = \max\{a_{1h}b_{1h}, a_{1h}b_{2h}, a_{2h}b_{1h}, a_{2h}b_{2h}\},$$

for $h \in I = [0, 1]$.

Chapter 2

Fuzzy linear systems

2.1 Introduction

The main advantage of fuzzy models is their ability to describe expert knowledge in a descriptive, human like way, in the form of simple rules using linguistic variable. The theory of fuzzy sets allows the existence of uncertainty to vagueness (or fuzziness) rather than due to randomness. When using fuzzy sets, accuracy is traded for complexity-fuzzy logic models do not need an accurate definition or many systems (in term of the parameters).

Simultaneous linear equations play a major role in representing various systems in natural science, engineering, and social domain. Since in many applications at least some of the system's parameters and measurements are represented by exert experience in terms of fuzzy rather than crisp numbers, it is immensely important to develop mathematical models and numerical procedures that would appropriately deal with those general fuzzy terms.

2.2 Fuzzy linear system

Definition 2.2.1. [17] The $n \times n$ linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2, \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n, \end{cases} \tag{2.1}$$

where the coefficients matrix $A = (a_{ij}), 1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and $y_i \in E^1, 1 \leq i \leq n$ is called a fuzzy system of linear equations (FSLE).

Using the extension principle, the addition and the scalar multiplication of fuzzy numbers are defined by

$$(u + v)(x) = \sup_{x=s+t} min\{u(s), v(t)\},$$

$$(ku)(x) = u(x/k); \; k \neq 0,$$

for $u, v \in E^1, \; k \in \mathbb{R}$. Equivalently, for arbitrary $u = (\underline{u}, \overline{u}), \; v = (\underline{v}, \overline{v})$ and $k \in \mathbb{R}$, $0 \leq \alpha \leq 1$, we may define the addition and the scalar multiplication as

$$(\underline{u + v})(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha),$$

$$(\overline{u + v})(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha),$$

$$(\underline{ku})(\alpha) = k\underline{u}(\alpha), \quad (\overline{ku})(\alpha) = k\overline{u}(\alpha), \quad k \geq 0,$$

$$(\underline{ku})(\alpha) = k\overline{u}(\alpha), \quad (\overline{ku})(\alpha) = k\underline{u}(\alpha), \quad k \leq 0.$$

Definition 2.2.2. A fuzzy number vector $(x_1, x_2, \cdots, x_n)^T$ given by $x_i = (\underline{x_i}(\alpha), \overline{x_i}(\alpha)), 1 \leq i \leq n, 0 \leq \alpha \leq 1$, is called a solution of (2.1) if

$$min\{\sum_{j=1}^{n} a_{ij}u_j | u_j \in [x_j]_\alpha\} = \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y_i}, \quad i = 1, \dots, n, \tag{2.2}$$

$$max\{\sum_{j=1}^{n} a_{ij}u_j | u_j \in [x_j]_\alpha\} = \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y_i}, \quad i = 1, \dots, n, \tag{2.3}$$

where $[x_j]_\alpha$ is the α-level set of x_j.

For a particular i, if $a_{ij} > 0, \quad 1 \leq j \leq n$, we simply get

$$\sum_{j=1}^{n} a_{ij}\underline{x}_j = \underline{y}_j, \qquad \sum_{j=1}^{n} a_{ij}\overline{x}_j = \overline{y}_j. \tag{2.4}$$

In general, however, an arbitrary equation for either $\underline{y}_i$ or $\overline{y}_i$ may include a linear combination of $\underline{x}_j$ and $\overline{x}_j$'s. Consequently, in order to solve the system given by (2.1) one must solve a $2n \times 2n$ crisp linear system where the right-hand side column is the function vector $(\underline{y}_1, ..., \underline{y}_n, \overline{y}_1, ..., \overline{y}_n)^T$ [18].

Let us now rearrange the linear systems of equation (2.2) and (2.3) so that the unknowns are $\underline{x}_i, \overline{x}_i, \ 1 \leq i \leq n$, and the right-hand side column is

$$Y = (\underline{y}_1, \cdots, \underline{y}_n, -\overline{y}_1, \cdots, -\overline{y}_n)^T.$$

We get the $2n \times 2n$ linear system

$$\begin{cases} s_{1,1}\underline{x}_1 + s_{1,2}\underline{x}_2 + \cdots + s_{1,n}\underline{x}_n + s_{1,n+1}(-\overline{x}_1) + \cdots + s_{1,2n}(-\overline{x}_n) = \underline{y}_1, \\ \vdots \\ s_{n,1}\underline{x}_1 + s_{n,2}\underline{x}_2 + \cdots + s_{n,n}\underline{x}_n + s_{n,n+1}(-\overline{x}_1) + \cdots + s_{n,2n}(-\overline{x}_n) = \underline{y}_n, \\ s_{n+1,1}\underline{x}_1 + \cdots + s_{n+1,n}\underline{x}_n + s_{n+1,n+1}(-\overline{x}_1) + \cdots + s_{n+1,2n}(-\overline{x}_n) = -\overline{y}_1, \\ \vdots \\ s_{2n,1}\underline{x}_1 + s_{2n,2}\underline{x}_2 + \cdots + s_{2n,n}\underline{x}_n + s_{2n,n+1}(-\overline{x}_1) + \cdots + s_{2n,2n}(-\overline{x}_n) = -\overline{y}_n, \end{cases} \tag{2.5}$$

where s_{ij} are determined as follows:

$$\begin{aligned} a_{ij} \geq 0 &\implies s_{ij} = s_{i+n,j+n} = a_{ij}, \\ a_{ij} \leq 0 &\implies s_{i+n,j} = s_{i,j+n} = -a_{ij}, \end{aligned}$$

and any s_{ij} which is not determined is zero. Using matrix notation we get

$$SX = Y \tag{2.6}$$

where $S = (s_{ij}), \quad 1 \leq i, j \leq 2n$ and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \qquad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}. \tag{2.7}$$

Example 2.2.1. *Consider the 2×2 fuzzy linear system*

$$\begin{cases} x_1 & - & x_2 & = & y_1, \\ x_1 & + & 2x_2 & = & y_2. \end{cases}$$

The corresponding 4×4 system is

$$\begin{array}{cccccccccc} \underline{x}_1 & & & & & + & (-\overline{x}_2) & = & \underline{y}_1, \\ \underline{x}_1 & + & 2\underline{x}_2 & & & & & = & \underline{y}_2, \\ & & \underline{x}_2 & + & (-\overline{x}_1) & & & = & -\overline{y}_1, \\ & & & & (-\overline{x}_1) & + & 2(-\overline{x}_2) & = & -\overline{y}_2, \end{array}$$

i.e.

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

The structure of S implies that $S_{ij} \geq 0, \ \ 1 \leq i, j \leq n$ and that

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}$$

where S_1 contains the positive entries of A below the main diagonal and S_2 contains the absolute value of the entries of A over the main diagonal and $A = S_1 - S_2$.

The linear system (2.6) is now a $2n \times 2n$ crisp linear system and can be uniquely solved for X, if and only if the matrix S is nonsingular. We therefore must answer two following questions:

1. Is S nonsingular?

2. Do the components of the $2n$-dimensional solution vector X represent an n-dimensional solution fuzzy vector to the fuzzy system given by (2.1)?

If S is nonsingular, the answer to the second question is positive if and only if $(\underline{x}_i, \overline{x}_i)$ is a fuzzy number for all i.

The next example reveals the notable fact that S may be singular even if the original matrix A is not.

Example 2.2.2. *The matrix A of the linear fuzzy system*

$$\begin{cases} x_1 & - & x_2 & = & y_1 \\ x_1 & + & x_2 & = & y_2 \end{cases}$$

is nonsingular, while

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

is singular. In other words, a fuzzy linear system represented by a nonsingular matrix A may have no solution or an infinite number of solutions.

The next result eliminates the possibility of a unique fuzzy solution, whenever the crisp system is not uniquely solved, i.e. whenever A is singular.

Theorem 2.2.1 The matrix S is nonsingular if and only if the matrices $A = S_1 - S_2$ and $S_1 + S_2$ are both nonsingular.

Proof. By subtracting the $(n+i)$ th row of S from its i th row for $1 \leq i \leq n$, we obtain

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} \longrightarrow \begin{bmatrix} S_1 - S_2 & S_2 - S_1 \\ S_2 & S_1 \end{bmatrix} = \acute{S}_1.$$

Next, we add the j th column of S to its $(n+j)$ th column for $1 \leq j \leq n$ to obtain

$$\acute{S}_1 = \begin{bmatrix} S_1 - S_2 & S_2 - S_1 \\ S_2 & S_1 \end{bmatrix} \longrightarrow \begin{bmatrix} S_1 - S_2 & 0 \\ S_2 & S_1 + S_2 \end{bmatrix} = \acute{S}_2.$$

Clearly, $|S| = |\acute{S}_1| = |\acute{S}_2| = |S_1 + S_2||S_1 - S_2| = |S_1 - S_2||A|$. Therefore $|S| \neq 0$ if and only if $|A| \neq 0$ and $|S_1 - S_2| \neq 0$, which concludes the proof. $\square$

Corollary 2.2.1. If a crisp linear system does not have a unique solution, the associated fuzzy linear system does not have one either.

In order to solve the linear fuzzy system (2.6), we must now calculate S^{-1} (whenever it exists). The next result is taken from the theory of block matrices and provides the structure of S^{-1}.

Theorem 2.2.2 If S^{-1} exists it must have the same structure as S, i.e.

$$S^{-1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}.$$

Proof. Let S_{ij} denote the entry of S in the i th row and the j th column. If t_{ij} denotes the entry of S^{-1} at the same location then

$$t_{ij} = \frac{(-1)^{i+j}|S_{ji}|}{|S|}, 1 \leq i, j \leq n,$$

where S_{ji} is the matrix obtained by removing the j th row and the i th column of S. Consider now for example the entries $t_{i,n+j}$ and $t_{n+i,j}$ of S^{-1} for some $1 \leq i, j \leq n$. The associated matrices are $S_{n+j,i}$ and $S_{j,n+i}$, respectively. It can be easily shown that $S_{n+j,i}$ can be obtain from $S_{j,n+i}$ by interchanging rows and columns an even number of times. Therefore

$$t_{i,n+j} = (-1)^{i+n+j}\frac{|S_{n+j,i}|}{|S|} = (-1)^{i+n+j}\frac{|S_{j,n+i}|}{|S|} = t_{n+i,j}.$$

Similarly, $t_{ij} = t_{n+i,n+j}$ for arbitrary i and j and thus S^{-1} must have the same structure like S. $\Box$

In order to calculate E and D we write

$$SS^{-1} = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} D & E \\ E & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and get

$$S_1D + S_2E = I, \qquad S_2D + S_1E = 0$$

By adding and then subtracting the corresponding sides of $S_1D + S_2E = I$, $S_2D + S_1E = 0$ we obtain

$$D + E = (S_1 + S_2)^{-1}, \qquad D - E = (S_1 - S_2)^{-1},$$

and consequently

$$D = \frac{1}{2}[(S_1 + S_2)^{-1} + (S_1 - S_2)^{-1}],$$
$$E = \frac{1}{2}[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}].$$

Assuming that S (i.e. $S_1 + S_2$ and $S_1 - S_2$) is nonsingular we obtain

$$X = S^{-1}Y.$$

The solution vector is thus unique but may still not be an appropriate fuzzy vector. The following result provides necessary and sufficient conditions for the unique solution vector to be a fuzzy vector, given arbitrary input fuzzy vector Y.

Theorem 2.2.3 if S^{-1} is nonnegative and Y is an arbitrary fuzzy vector then the unique solution X of $X = S^{-1}Y$ is a fuzzy vector.

Proof. Let $S^{-1} = (t_{ij}) > 0, \quad 1 \leq i, j \leq 2n$ then

$$\underline{x}_i = \sum_{j=1}^{n} t_{ij}\underline{y}_i - \sum_{j=1}^{n} t_{i,n+j}\overline{y}_j, \qquad 1 \leq i \leq n, \tag{2.8}$$

$$-\overline{x}_i = \sum_{j=1}^{n} t_{n+i,j}\underline{y}_j - \sum_{j=1}^{n} t_{n+i,n+j}\overline{y}_j, \qquad 1 \leq i \leq n. \tag{2.9}$$

Due to the particular structure of S^{-1} we can replace (2.9) by

$$\overline{x}_i = -\sum_{j=1}^{n} t_{i,n+j}\underline{y}_j + \sum_{j=1}^{n} t_{i,j}\overline{y}_j, \quad 1 \leq i \leq n, \tag{2.10}$$

and by subtracting (2.8) from (2.10) we get

$$\overline{x}_i - \underline{x}_i = \sum_{j=1}^{n} t_{ij}(\overline{y}_j - \underline{y}_j) + \sum_{j=1}^{n} t_{i,n+j}(\overline{y}_j - \underline{y}_j), \quad 1 \leq i \leq n.$$

Thus, if Y is arbitrary input vector which represents a fuzzy vector, i.e. $\overline{y}_i - \underline{y}_i \geq 0, \ \ 1 \leq i \leq n$. $\square$

We now restrict the discussion to the triangular fuzzy numbers , i.e. $\underline{y}_i(\alpha), \overline{y}_i(\alpha)$ and consequently $\underline{x}_i(\alpha), \overline{x}_i(\alpha)$ are all linear functions of α. Having calculated X which solves $SX = Y$ we now define the fuzzy solution to the original system given by (2.1).

Definition 2.2.3. [17] Let $X = \{(\underline{x}_i(\alpha), \overline{x}_i(\alpha)) \mid \ 1 \leq i \leq n\}$ denote the unique solution of $SX = Y$. The fuzzy number vector $U = \{(\underline{u}_i(\alpha), \overline{u}_i(\alpha)), \ \ 1 \leq i \leq n\}$ defined by

$$\underline{u}_i(\alpha) = min\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), \underline{x}_i(1)\},$$

$$\overline{u}_i(\alpha) = max\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), \underline{x}_i(1)\}.$$

is called the fuzzy solution of $SX = Y$.

The use of $\underline{x}_i(1)$ is to eliminate the possibility of fuzzy numbers whose associated triangular possess an angle grater than $90°$. If $(\underline{x}_i(\alpha), \overline{x}_i(\alpha)), \ \ 1 \leq i \leq n$, are all fuzzy numbers then $\underline{u}_i(\alpha) = \underline{x}_i(\alpha), \ \overline{u}_i(\alpha) = \overline{x}_i(\alpha), 1 \leq i \leq n$ and U is called strong fuzzy solution. Otherwise, U is weak fuzzy solution.

Example 2.2.3 Consider the 2×2 fuzzy linear system

$$\begin{cases} x_1 - x_2 = (\alpha, 2-\alpha), \\ x_1 + 3x_2 = (4+\alpha, 7-2\alpha). \end{cases}$$

The extended 4×4 matrix is

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

and the solution is

$$X = \begin{bmatrix} \underline{x}_1(\alpha) \\ \underline{x}_2(\alpha) \\ -\overline{x}_1(\alpha) \\ -\overline{x}_2(\alpha) \end{bmatrix} = S^{-1}Y = \begin{bmatrix} 1.125 & -0.125 & 0.375 & -0.375 \\ -0.375 & 0.375 & -1.125 & 0.125 \\ 0.375 & -0.375 & 1.125 & -0.125 \\ -0.125 & 0.125 & -0.375 & 0.375 \end{bmatrix} \begin{bmatrix} \alpha \\ 4+\alpha \\ \alpha-2 \\ 2\alpha-7 \end{bmatrix}$$

i.e.

$$\underline{x}_1(\alpha) = 1.375 + 0.625\alpha, \qquad \overline{x}_1(\alpha) = 2.875 - 0.875\alpha,$$

$$\underline{x}_2(\alpha) = 0.875 + 0.125\alpha, \qquad \overline{x}_2(\alpha) = 1.375 - 0.375\alpha.$$

Here $\underline{x}_1 \leq \overline{x}_1$, $\underline{x}_2 \leq \overline{x}_2$; $\underline{x}_1$, $\underline{x}_2$ are monotonic decreasing functions. Therefore the fuzzy solution is $x_1 = (\underline{x}_1, \overline{x}_1)$, $x_2 = (\underline{x}_2, \overline{x}_2)$ and it is strong fuzzy solution.

Example 2.2.4 Consider the 3×3 fuzzy linear system

$$\begin{cases} x_1 + x_2 - x_3 = (\alpha, 2-\alpha), \\ x_1 - 2x_2 + x_3 = (2+\alpha, 3), \\ 2x_1 + x_2 + 3x_3 = (-2, -1-\alpha). \end{cases}$$

with

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{bmatrix}.$$

and

$$Y = (\alpha, 2 + \alpha, -2, \alpha - 2, -3, 1 + \alpha)^T.$$

The solution vector of $SX = Y$ is

$$X = \begin{bmatrix} -2.31 + 3.62\alpha \\ -0.62 - 0.77\alpha \\ 1.08 - 2.15\alpha \\ -4.69 + 3.38\alpha \\ 1.62 - 0.23\alpha \\ 2.92 - 1.85\alpha \end{bmatrix}.$$

i.e.

$$x_1 = (-2.31 + 3.62\alpha, 4.69 - 3.38\alpha),$$

$$x_2 = (-0.62 - 0.77\alpha, -1.62 + 0.23\alpha),$$

$$x_3 = (1.08 - 2.15\alpha, -2.92 + 1.85\alpha).$$

The fact that x_2 and x_3 are not fuzzy numbers is obvious. The fuzzy solution in this case is a weak solution given by

$$u_1 = (-2.31 + 3.62\alpha, 4.69 - 3.38\alpha),$$

$$u_2 = (-1.62 + 0.23\alpha, -0.62 - 0.77\alpha),$$

$$u_3 = (-2.92 + 1.85\alpha, 1.08 - 2.15\alpha).$$

2.3 A dual fuzzy linear system

Usually, there is no inverse element for an arbitrary fuzzy number $u \in E^1$, i.e. there exists no element $v \in E^1$ such that

$$u + v = 0.$$

Actually, for all fuzzy numbers $u \in E^1$ we have

$$u + (-u) \neq 0.$$

Therefore, the fuzzy linear system of equations

$$AX = BX + Y,$$

can not be equivalently replaced by the fuzzy linear system

$$(A - B)X = Y,$$

which had been investigated [18]. In the sequel, we will called the fuzzy linear system

$$AX = BX + Y,$$

where $A = (a_{ij})$, $B = (b_{ij})$, $1 \leq i, j \leq n$ are crisp coefficient matrices and Y a fuzzy vector, a dual fuzzy linear system.

Theorem 2.3.1. *Let $A = (a_{ij})$, $B = (b_{ij})$, $1 \leq i, j \leq n$, be nonnegative matrices. The dual linear system has a unique fuzzy solution if and only if the inverse of $A - B$ exists and has only nonnegative entries.*

Proof. We know the dual fuzzy linear system

$$\sum_{i=1}^{n} a_{ji} x_i = \sum_{i=1}^{n} b_{ji} x_i + y_j$$

is equivalent to (since $a_{ji} \geq 0$ and $b_{ji} \geq 0$ for all i, j)

$$\sum_{i=1}^{n} a_{ji} \underline{x}_i = \sum_{i=1}^{n} b_{ji} \underline{x}_i + \underline{y}_j,$$

$$\sum_{i=1}^{n} a_{ji} \overline{x}_i = \sum_{i=1}^{n} b_{ji} \overline{x}_i + \overline{y}_j.$$

It follows, that

$$\sum_{i=1}^{n} (a_{ji} - b_{ji}) \underline{x}_i = \underline{y}_j,$$

$$\sum_{i=1}^{n} (a_{ji} - b_{ji}) \overline{x}_i = \overline{y}_j.$$

If $(A - B)^{-1}$ exists, the equations have unique solutions $(\underline{x}_i)_1^n$ and $(\overline{x}_i)_1^n$; and clearly if $(A - B)^{-1} \geq 0$ for all i, j $(\underline{x}_i, \overline{x}_i)$ is a fuzzy number. $\square$

The following theorem guarantees the existence of a fuzzy solution for a general case. Consider dual fuzzy linear system, and transform its $n \times n$ coefficient A and B into $2n \times 2n$ matrices. Define matrices $S = (s_{ij}), T = (t_{i,j})$; $1 \leq i, j \leq n$ by

$$a_{ij} \geq 0 \Longrightarrow s_{ij} = a_{ij}, \qquad s_{i+n,j+n} = a_{ij},$$

$$a_{ij} < 0 \Longrightarrow s_{i,j+n} = -a_{ij}, \qquad s_{i+n,j} = -a_{ij},$$

$$b_{ij} \geq 0 \Longrightarrow t_{ij} = b_{ij}, \qquad t_{i+n,j+n} = b_{ij},$$

$$b_{ij} < 0 \Longrightarrow t_{i,j+n} = -b_{ij}, \qquad t_{i+n,j} = -b_{ij},$$

with all the remaining s_{ij}, t_{ij} taken to be zero.

Theorem 2.3.2. *The dual fuzzy linear system has a unique fuzzy solution if and only if the inverse matrix $S - T$ exists and is nonnegative.*

Proof. Using the form of Eq.(2.5), we obtain that the system $AX = BX + Y$ is equivalent to the system

$$SX = TX + Y$$

where X, Y are given by Eq.(2.7). Consequently,

$$(S - T)X = Y$$

and a solution exists if and only if $S - T$ is nonsingular. If in addition $(S - T)^{-1} \geq 0$, then by virtue of theorem 2.3.1 the solution X provides a fuzzy solution. □

Example 2.3.1 Consider the 2×2 dual fuzzy linear system

$$\begin{cases} x_1 + 3x_2 = 3x_1 + 2x_2 + (1+\alpha, 4-2\alpha), \\ 2x_1 + x_2 = x_1 + 5x_2 + (3+\alpha, 5-\alpha). \end{cases}$$

with

$$S - T = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix},$$

and

$$Y = (1+\alpha, 3+\alpha, -4+2\alpha, -5+\alpha)^T$$

and

$$(S-T)^{-1} = \begin{bmatrix} -0.5714 & -0.1429 & 0 & 0 \\ -0.1429 & -0.2857 & 0 & 0 \\ 0 & 0 & 0.5714 & 0.1429 \\ 0 & 0 & 0.1429 & 0.2857 \end{bmatrix},$$

The solution vector of $(S-T)X = Y$ is

$$X = \begin{bmatrix} -1.1667 - 0.8333\alpha \\ -1.3333 - 0.6667\alpha \\ 3.0001 - 1.2857\alpha \\ 2.0001 - 0.5715\alpha \end{bmatrix},$$

i.e.

$$x_1 = (-1.1667 - 0.8333\alpha, -3.0001 + 1.2857\alpha),$$

$$x_2 = (-1.3333 - 0.6667\alpha, -2.0001 + 0.5715\alpha).$$

The fact that x_1 and x_2 are not fuzzy numbers is obvious. The fuzzy solution in this case is a weak solution given by

$$u_1 = (-3.0001 + 1.2857\alpha, -1.1667 - 0.8333\alpha),$$

$$u_2 = (-2.0001 + 0.5715\alpha, -1.3333 - 0.6667\alpha).$$

Chapter 3

General dual fuzzy linear systems

3.1 Introduction

Systems of fuzzy linear equations arise in a large number of areas, both directly in modeling physical situations and indirectly in the numerical solution of other mathematical models. These applications occur in virtually all areas of the physical, biological and social sciences. Because of the widespread importance of fuzzy linear systems, we do much research on numerical solution of these systems and some of these are defined, analyzed and illustrated in this chapter.

Friedman *et al.* [17] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficients matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied duality in fuzzy linear systems $Ax = Bx + y$ where A , B are real $n \times n$ matrices the unknown vector x is a vector consisting of n fuzzy numbers and the constant y is a vector consisting of n fuzzy numbers, in [18]. In [3, 4, 8, 7, 11] the authors presented conjugate gradient, LU decomposition method for solving general fuzzy linear systems or symmetric fuzzy linear systems. Also, Wang *et al.* [32] presented an iterative algorithm for solving dual linear system of the form $x = Ax + u$, where A is real $n \times n$ matrix, the unknown vector x and the constant u are all vectors consisting of fuzzy numbers.

In this chapter, a numerical method for finding minimal solution of a $m \times n$ $Ax + f = Bx + c$ general duality fuzzy linear system based on pseudo inverse calculation, is given when the matrix of coefficients is row full rank or column full rank, and where A, B are real $m \times n$ matrices, the unknown vector x is a vector consisting of n fuzzy numbers and the constants f and c are vectors consisting of m fuzzy numbers.

3.2 A general fuzzy linear system

The minimal solution of an arbitrary linear system is formally defined such that:

1. If the system is consistent and has a unique solution, then this solution is also the minimal solution.

2. If the system is consistent and has a set of solutions, then the minimal solution is a member of this set that has the least Euclidean norm.

3. If the system is inconsistent and has a unique least squares solution, then this solution is also the minimal solution.

4. If the system is inconsistent and has a least squares set solution, then the minimal solution is a member of this set that has the least Euclidean norm.

Let n be a fuzzy number with membership function $\mu(x|n)$. The membership function is partially specified by (n_1, n_2, n_3, n_4) where: (1) $n_1 < n_2 < n_3 < n_4$; (2) $\mu(x|n) = 0$ outside (n_1, n_4) and equals one at (n_2, n_3); (3) $\mu(x|n)$ is continuous and monotonically increasing from zero to one on $[n_1, n_2]$; and (4) $\mu(x|n)$ is continuous and monotonically decreasing from one to zero on $[n_3, n_4]$. If $\mu(x|n)$ consists of straight line segments on (n_1, n_2) and on (n_3, n_4) we will write $n = (n_1/n_2/n_3/n_4)$.

Let $f_1(\alpha|n)$ be the inverse of $\mu(x|n)$ on $[n_1, n_2]$ and let $f_2(\alpha|n)$ be the inverse of $\mu(x|n)$ on $[n_3, n_4]$. Therefore, $n_1 = f_1(0|n)$, $n_2 = f_1(1|n)$, $n_3 = f_2(1|n)$, and $n_4 = f_2(0|n)$. If $n = (n_1/n_2/n_3/n_4)$, then $f_1(\alpha|n) = (n_2 - n_1)\alpha + n_1$ and $f_2(\alpha|n) = (n_3 - n_4)\alpha + n_4$. The inverse notation is very handy for performing multiplication and addition of fuzzy numbers [23].

The $\alpha-$cut of a fuzzy number n is

$$n^\alpha = \{x|\mu(x|n) \geq \alpha\}$$

for $0 < \alpha \leq 1$. We see that $n^\alpha = [f_1(\alpha|n), f_2(\alpha|n)]$ which we write as $[n_1^\alpha, n_2^\alpha]$, for $0 < \alpha \leq 1$. We will use the notation $\dot{n}_1^\alpha$ and $\dot{n}_2^\alpha$ for the derivative of n_1^α and n_2^α, respectively, with respect to α.

Theorem 3.2.1. Let a and c be fuzzy numbers. The equation $a + x = c$ has a solution x if and only if $c_1 - a_1 < c_2 - a_2 < c_3 - a_3 < c_4 - a_4$.

Proof. Taking α-cuts we obtain $a_i^\alpha + x_i^\alpha = c_i^\alpha, i = 1, 2$. Then

$$x_1 < x_2 < x_3 < x_4,$$

and $\dot{x}_1^\alpha > 0, \dot{x}_2^\alpha < 0$ if and only if $c_1 - a_1 < c_2 - a_2 < c_3 - a_3 < c_4 - a_4$.

$\square$

Definition 3.2.1. The $m \times n$ linear system

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & y_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & y_2, \\ \vdots & & & & & & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & y_m, \end{cases} \tag{3.1}$$

where the given matrix of coefficients $A = (a_{ij}), 1 \le i \le m$ and $1 \le j \le n$ is a real $m \times n$ matrix, the given $y_i \in E$, $1 \le i \le m$, with the unknowns $x_j \in E$, $1 \le j \le n$ is called a fuzzy linear system (FLS). The operations in (3.1) is described in next section. In this chapter, we assume the matrix A is full rank, i.e., $rank(A) = m$ (for $m \le n$) or $rank(A) = n$ (for $n < m$).

Here, a numerical method for finding minimal solution (defined later) of a fuzzy $m \times n$ linear system based on pseudo-inverse calculation by singular value decomposition, is given when matrix of coefficients has row full rank or column full rank.

Definition 3.2.2. [17] A fuzzy number vector $(x_1, x_2, ..., x_n)^t$ given by

$$x_j = (\underline{x}_j(r), \overline{x}_j(r)); \quad 1 \le j \le n, \; 0 \le r \le 1,$$

is called a solution of the fuzzy linear system (3.1) if

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y}_i, \\ \\ \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y}_i. \end{cases}$$

If, for a particular i, $a_{ij} > 0$, for all j, we simply get

$$\sum_{j=1}^{n} a_{ij}\underline{x}_j = \underline{y}_i, \qquad \sum_{j=1}^{n} a_{ij}\overline{x}_j = \overline{y}_i.$$

In general, however, an arbitrary equation for either $\underline{y}_i$ or $\overline{y}_i$ may include a linear combination of $\underline{x}_j$'s and $\overline{x}_j$'s. Consequently, in order to solve the system given by (3.1) one must solve a crisp $(2m) \times (2n)$ linear system where the right-hand side column is the vector $(\underline{y}_1, \underline{y}_2, ..., \underline{y}_m, -\overline{y}_1, -\overline{y}_2, ..., -\overline{y}_m)^t$. We get the $(2m) \times (2n)$ linear system

$$\begin{cases} s_{11}\underline{x}_1 + \cdots + s_{1n}\underline{x}_n + s_{1,n+1}(-\overline{x}_1) + \cdots + s_{1,2n}(-\overline{x}_n) = \underline{y}_1, \\ \vdots \\ s_{m,1}\underline{x}_1 + \cdots + s_{mn}\underline{x}_n + s_{m,n+1}(-\overline{x}_1) + \cdots + s_{m,2n}(-\overline{x}_n) = \underline{y}_m, \\ s_{m+1,1}\underline{x}_1 + \cdots + s_{m+1,n}\underline{x}_n + s_{m+1,n+1}(-\overline{x}_1) + \cdots + s_{m+1,2n}(-\overline{x}_n) = -\overline{y}_1, \\ \vdots \\ s_{2m,1}\underline{x}_1 + \cdots + s_{2m,n}\underline{x}_n + s_{2m,n+1}(-\overline{x}_1) + \cdots + s_{2m,2n}(-\overline{x}_n) = -\overline{y}_m, \end{cases} \tag{3.2}$$

where s_{ij} are determined as follows:

$$\begin{aligned} a_{ij} \geq 0 &\Longrightarrow s_{ij} = a_{ij}, \; s_{i+m,j+n} = a_{ij}, \\ a_{ij} < 0 &\Longrightarrow s_{i,j+n} = -a_{ij}, \; s_{i+m,j} = -a_{ij}, \end{aligned} \tag{3.3}$$

and any s_{ij} which is not determined by (3.3) is zero. Using matrix notation we get

$$SX = Y, \tag{3.4}$$

where $S = (s_{ij}) \geq 0, 1 \leq i \leq 2m, 1 \leq j \leq 2n$ and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_m \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_m \end{bmatrix}.$$

The structure of S implies that $s_{ij} \geq 0, 1 \leq i \leq 2m, 1 \leq j \leq 2n$ and that

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

where B contains the positive entries of A, and C contains the absolute values of the negative entries of A, i.e., $A = B - C$.

Theorem 3.2.2. Let T be $p \times q$ real column full rank or row full rank. There exists a $p \times p$ orthogonal matrix U, a $q \times q$ orthogonal matrix V, and a $p \times q$ diagonal matrix Σ with $\langle \Sigma \rangle_{ij} = 0$ for $i \neq j$ and $\langle \Sigma \rangle_{ii} = \sigma_i > 0$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_s > 0$, where $s = \min\{p, q\}$, such that the singular value decomposition

$$T = U \Sigma V^t,$$

is valid and if Σ^+ is that $q \times p$ matrix whose only nonzero entries are $\langle \Sigma^+ \rangle_{ii} = 1/\sigma_i$ for $1 \leq i \leq s$, then $T^+ = V\Sigma^+ U^t$ is the unique pseudo-inverse of T.

We refer the reader to [14] for more information on finding pseudo-inverse of an arbitrary matrix, and when we work with full rank matrices, there are not any problem and all calculations are stable and well-posed.

Theorem 3.2.3. The matrix S is row full rank (for $m \leq n$) or column full rank (for $n < m$) if and only if the matrices $A = B - C$ and $B + C$ are both row full rank (for $m \leq n$) or column full rank (for $n < m$).

In order to solve the linear fuzzy system (3.1), we must calculate S^+. The next result is taken from the theory of block matrices and provides the structure of S^+. For finding S^+, we must find the pseudo-inverse of two real full rank $m \times n$ matrices by following theorem.

Theorem 3.2.4. The pseudo-inverse of non-negative full rank matrix

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$$

is

$$S^+ = \begin{pmatrix} D & E \\ E & D \end{pmatrix}, \tag{3.5}$$

where

$$D = \frac{1}{2}[(B+C)^+ + (B-C)^+], \qquad E = \frac{1}{2}[(B+C)^+ - (B-C)^+].$$

Proof. Let S^+ be the pseudo-inverse of S, it is unique. Without loss of generality, suppose that

$$S^+ = \begin{pmatrix} D & E \\ E & D \end{pmatrix}.$$

We know

$$SS^+S = S,$$

hence

$$\begin{pmatrix} B & C \\ C & B \end{pmatrix} \begin{pmatrix} D & E \\ E & D \end{pmatrix} \begin{pmatrix} B & C \\ C & B \end{pmatrix} = \begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

and get

$$BDB + BEC + CDC + CEB = B, \qquad BDC + BEB + CDB + CEC = C. \quad (3.6)$$

By adding and then by subtracting the two parts of (3.6), we obtain

$$(B+C)(D+E)(B+C) = (B+C), \qquad (B-C)(D-E)(B-C) = (B-C),$$

also, we can show

$$(D+E)(B+C)(D+E) = (D+E), \qquad (D-E)(B-C)(D-E) = (D-E),$$

$$[(B+C)(D+E)]^+ = (B+C)(D+E), \qquad [(B-C)(D-E)]^+ = (B-C)(D-E),$$

$$[(D+E)(B+C)]^+ = (D+E)(B+C), \qquad [(D-E)(B-C)]^+ = (D-E)(B-C).$$

Thus S^+ must have the structure given by (3.5). In order to calculate E and D in (3.5), we have

$$(B+C)^+ = D+E, \qquad (B-C)^+ = D-E,$$

and consequently,

$$D = \frac{1}{2}[(B+C)^{+} + (B-C)^{+}], \qquad E = \frac{1}{2}[(B+C)^{+} - (B-C)^{+}].$$

$\square$

Corollary 3.2.1. [24] The minimal solution of (3.4) is obtained by

$$X = S^{+}Y. \tag{3.7}$$

The following result provides necessary and sufficient condition for existing a fuzzy vector solution.

Theorem 3.2.5. The solution X of (3.7) is given a fuzzy vector for an arbitrary vector Y if and only if S^{+} is nonnegative, i.e.

$$(S^{+})_{ij} \geq 0, \qquad 1 \leq i \leq 2m, \qquad 1 \leq j \leq 2n.$$

Proof. The same as the proof of theorem 3 in [17].

$\square$

Having calculated X which solves (3.4) we now define the fuzzy solution to the original system (3.1).

Definition 3.2.3. Let $x = \{(\underline{x}_j(r), \overline{x}_j(r)),\ 1 \leq j \leq n\}$ denotes the minimal solution of (3.4), if $\underline{y}_i(r), \overline{y}_i(r),\ 1 \leq i \leq m$ are linear functions of r and $\underline{y}_i(1) = \overline{y}_i(1)$, then the fuzzy number vector $u = \{(\underline{u}_j(r), \overline{u}_j(r)),\ 1 \leq j \leq n\}$ defined by

$$\underline{u}_j(r) = min\{\underline{x}_j(r), \overline{x}_j(r), \underline{x}_j(1)\},$$

$$\overline{u}_j(r) = max\{\underline{x}_j(r), \overline{x}_j(r), \underline{x}_j(1)\},$$

is called the minimal fuzzy solution of (3.1). If $(\underline{x}_j(r), \overline{x}_j(r)),\ 1 \le j \le n$, are all fuzzy numbers then $\underline{u}_j(r) = \underline{x}_j(r), \overline{u}_j(r) = \overline{x}_j(r)$, and then u is called a strong minimal fuzzy solution. Otherwise, u is a weak minimal fuzzy solution.

3.3 A general dual fuzzy linear system

Usually, there is no inverse element for an arbitrary fuzzy number $u \in E^1$, i.e., there exists no element $v \in E^1$ such that

$$u + v = 0.$$

Actually, for all non-crisp fuzzy numbers $u \in E^1$ we have

$$u + (-u) \neq 0.$$

Therefore, the fuzzy linear equation system

$$Ax + f = Bx + c$$

cannot be equivalently replaced by the fuzzy linear equation system

$$(A - B)x = c - f$$

which had been investigated. In the sequel, we will call the fuzzy linear system

$$Ax + f = Bx + c \tag{3.8}$$

where $A = (a_{ij}), B = (b_{ij}), 1 \le i \le m, 1 \le j \le n$ are crisp coefficients matrices and c, f fuzzy number vectors, a general dual fuzzy linear system.

Theorem 3.3.1. Let $A = ((a_{ij}), B = (b_{ij}), 1 \leq i \leq m, 1 \leq j \leq n$, be nonnegative matrices. The general dual fuzzy linear system (3.8) has a minimal fuzzy solution if and only if the pseudo inverse matrix of $A - B$ has only nonnegative entries and $c_{i1} - f_{i1} < c_{i2} - f_{i2} < c_{i3} - f_{i3} < c_{i4} - f_{i4}$ for $i = 1, \ldots, m$.

Proof. The general dual fuzzy linear system

$$\sum_{j=1}^{n} a_{ij}x_j + f_i = \sum_{j=1}^{n} b_{ij}x_j + c_i$$

is equivalent to (since $a_{ij} \geq 0$ and $b_{ij} \geq 0$ for all i,j)

$$\sum_{j=1}^{n} a_{ij}\underline{x}_j + \underline{f}_i = \sum_{j=1}^{n} b_{ij}\underline{x}_j + \underline{c}_i, \qquad \sum_{j=1}^{n} a_{ij}\overline{x}_j + \overline{f}_i = \sum_{j=1}^{n} b_{ij}\overline{x}_j + \overline{c}_i.$$

It follows that

$$\sum_{j=1}^{n}(a_{ij} - b_{ij})\underline{x}_j = \underline{c}_i - \underline{f}_i, \qquad i = 1, \ldots, m, \tag{3.9}$$

$$\sum_{j=1}^{n}(a_{ij} - b_{ij})\overline{x}_j = \overline{c}_i - \overline{f}_i, \qquad i = 1, \ldots, m. \tag{3.10}$$

Eqs. (3.9) and (3.10) have minimal solutions $(\underline{x}_j)_1^n, (\overline{x}_j)_1^n$ if and only if $(A-B)^{+}_{i,j} \geq 0$ for all i, j and $c_{i1} - f_{i1} < c_{i2} - f_{i2} < c_{i3} - f_{i3} < c_{i4} - f_{i4}$ for $i = 1, \ldots, m$, i.e., $(\underline{x}_j, \overline{x}_j)_1^n$ is a minimal fuzzy number vector.

□

The following theorem guarantees the existence of a minimal fuzzy solution for a general case. Consider the general dual fuzzy linear system (3.8), and transform its $m \times n$ coefficient matrix A and B into $(2m) \times (2n)$ matrices as in the Eqs.(3.2)-(3.3).

Define matrices $S=(s_{i,j}), T=(t_{i,j}); 1\le i\le m, 1\le j\le n$ by

$$a_{ij}\ge 0 \Longrightarrow s_{ij}=a_{ij},\ s_{i+m,j+n}=a_{ij},$$

$$a_{ij}< 0 \Longrightarrow s_{i,j+n}=-a_{ij},\ s_{i+m,j}=-a_{ij},$$

$$b_{ij}\ge 0 \Longrightarrow t_{ij}=b_{ij},\ t_{i+m,j+n}=b_{ij},$$

$$b_{ij}< 0 \Longrightarrow t_{i,j+n}=-b_{ij},\ t_{i+m,j}=-b_{ij},$$

while all the remaining s_{ij}, t_{ij} are taken zero.

Theorem 3.3.2. The general dual fuzzy linear equation system (3.8) has a minimal fuzzy solution vector for arbitrary fuzzy vectors f, c if and only if $(S-T)^{+}_{i,j}\ge 0,\ \ 1\le i\le 2m,\ \ 1\le j\le 2n$ and $c_{i1}-f_{i1}<c_{i2}-f_{i2}<c_{i3}-f_{i3}<c_{i4}-f_{i4}$ for $i=1,\dots,m$.

Proof. Using the form of Eq.(3.2), we obtain that the system (3.8) is equivalent to the function equation system

$$Sx+f'=Tx+c',$$

where

$$x=\begin{bmatrix}\underline{x}_1\\ \vdots\\ \underline{x}_n\\ -\overline{x}_1\\ \vdots\\ -\overline{x}_n\end{bmatrix},\quad c'=\begin{bmatrix}\underline{c}_1\\ \vdots\\ \underline{c}_m\\ -\overline{c}_1\\ \vdots\\ -\overline{c}_m\end{bmatrix},\quad f'=\begin{bmatrix}\underline{f}_1\\ \vdots\\ \underline{f}_m\\ -\overline{f}_1\\ \vdots\\ -\overline{f}_m\end{bmatrix}.$$

Consequently,

$$(S-T)x=c'-f',$$

and a minimal fuzzy solution vector exists if and only if $(S-T)^+$ is nonnegative and $c_{i1}-f_{i1}<c_{i2}-f_{i2}<c_{i3}-f_{i3}<c_{i4}-f_{i4}$ for $i=1,\ldots,m$.

$\square$

Let, we define

$$S-T=\begin{pmatrix} B_1 & C_1 \\ C_1 & B_1 \end{pmatrix}.$$

Example 3.3.1. Consider the 2×3 fuzzy linear system

$$\begin{cases} 2x_1+x_3+(2r,3-r)=x_1+x_3+(3r+1,6-2r), \\ x_2+x_3+(r+1,3-r)=-x_2+(2r+3,7-2r). \end{cases}$$

By simple calculation

$$(B_1+C_1)^+=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\times\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\times\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$(B_1-C_1)^+=\begin{bmatrix} 0 & 0.8944 & 0.4472 \\ 1 & 0 & 0 \\ 0 & -0.4472 & 0.8944 \end{bmatrix}\times\begin{bmatrix} 0.4472 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\times\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have

$$x = \begin{bmatrix} \underline{x}_1(r) \\ \underline{x}_2(r) \\ \underline{x}_3(r) \\ -\overline{x}_1(r) \\ -\overline{x}_2(r) \\ -\overline{x}_3(r) \end{bmatrix} = (S-T)^+(c'-f') = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & -0.2 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & -0.2 & 0 & 0.2 \\ 0 & 0.4 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} r+1 \\ r+2 \\ r-3 \\ r-4 \end{bmatrix}.$$

Therefore, the minimal solution of the extended system is

$$\begin{aligned} \underline{x}_1(r) &= r+1, & \overline{x}_1(r) &= 3-r, \\ \underline{x}_2(r) &= 1.2, & \overline{x}_2(r) &= 1.2, \\ \underline{x}_3(r) &= r-0.4, & \overline{x}_3(r) &= 1.6-r. \end{aligned}$$

Here $\underline{x}_1 \leq \overline{x}_1, \underline{x}_2 \leq \overline{x}_2, \underline{x}_3 \leq \overline{x}_3$; $\underline{x}_1, \underline{x}_2, \underline{x}_3$ are monotonic increasing functions and $\overline{x}_1, \overline{x}_2, \overline{x}_3$ are monotonic decreasing functions. Therefore the fuzzy solution $x_1 = (\underline{x}_1, \overline{x}_1), x_2 = (\underline{x}_2, \overline{x}_2), x_3 = (\underline{x}_3, \overline{x}_3)$ is a strong fuzzy solution.

Example 3.3.2. Consider the 3×2 fuzzy linear system

$$\begin{cases} 2x_1 + x_2 + (r, 2-r) = x_1 + (2r+1, 5-2r), \\ x_1 - x_2 + (1+r, 3-r) = x_1 + x_2 + (2r+2, 6-2r), \\ x_2 + (2+r, 4-r) = x_1 + (2r+3, 7-2r). \end{cases}$$

By simple calculation

$$(B_1 + C_1)^+ = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 0.7071 & 0 & 0 \\ 0 & 0.7071 & 0 \end{bmatrix} \times \begin{bmatrix} -0.7071 & 0 & 0.7071 \\ -0.7071 & 0 & -0.7071 \\ 0 & -1 & 0 \end{bmatrix},$$

and

$$(B_1 - C_1)^+ = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0.4083 & 0 & 0 \\ 0 & 0.7071 & 0 \end{bmatrix} \times \begin{bmatrix} 0.4082 & -0.8165 & 0.4082 \\ -0.7071 & 0 & 0.7071 \\ 0.5774 & 0.5774 & 0.5774 \end{bmatrix}.$$

We have

$$x = \begin{bmatrix} \underline{x}_1(r) \\ \underline{x}_2(r) \\ -\overline{x}_1(r) \\ -\overline{x}_2(r) \end{bmatrix} = (S - T)^+(c' - f') =$$

$$\begin{bmatrix} 0.5 & 0 & -0.5 & 0 & 0 & 0 \\ 0.333 & -0.1667 & 0.333 & 0.1667 & 0.1667 & 0.1667 \\ 0 & 0 & 0 & 0.5 & 0 & -0.5 \\ 0.1667 & 0.1667 & 0.1667 & 0.333 & -0.1667 & 0.333 \end{bmatrix} \begin{bmatrix} r+1 \\ r+1 \\ r+1 \\ r-3 \\ r-3 \\ r-3 \end{bmatrix}.$$

Therefore, the minimal solution of the extended system is

$$\underline{x}_1(r) = 0, \qquad \overline{x}_1(r) = 0,$$

$$\underline{x}_2(r) = 0.9994r - 1.001, \qquad \overline{x}_2(r) = 0.9978 - 0.9994r.$$

Here $\underline{x}_1 \leq \overline{x}_1, \underline{x}_2 \leq \overline{x}_2$; $\underline{x}_1, \underline{x}_2$ are monotonic increasing functions and $\overline{x}_1, \overline{x}_2$ are monotonic decreasing functions. Therefore the minimal fuzzy solution $x_1 = (\underline{x}_1, \overline{x}_1), x_2 = (\underline{x}_2, \overline{x}_2)$ is a strong minimal fuzzy solution.

3.4 Fully fuzzy general dual linear system

Consider the $n \times n$ general dual fully fuzzy linear system of equations:

$$\begin{cases} a_{11}\underline{x}_1 + a_{12}\underline{x}_2 + \cdots + a_{1n}\underline{x}_n + b_1 = c_{11}\underline{x}_1 + c_{12}\underline{x}_2 + \cdots + c_{1n}\underline{x}_n + d_1, \\ a_{21}\underline{x}_1 + a_{22}\underline{x}_2 + \cdots + a_{2n}\underline{x}_n + b_2 = c_{21}\underline{x}_1 + c_{22}\underline{x}_2 + \cdots + c_{2n}\underline{x}_n + d_2, \\ \vdots \\ a_{n1}\underline{x}_1 + a_{n2}\underline{x}_2 + \cdots + a_{nn}\underline{x}_n + b_n = c_{n1}\underline{x}_1 + c_{n2}\underline{x}_n + \cdots + c_{nn}\underline{x}_n + d_n, \end{cases} \tag{3.11}$$

the matrix form of above equation is

$$Ax + b = Cx + d,$$

where the coefficient matrices $A = (a_{ij})$ and $C = (c_{ij})$, $1 \leq i, j \leq n$ are $n \times n$ fuzzy matrices, $b = (b_i)$, $d = (d_i)$ and $x = (x_i)$, $1 \leq i \leq n$ are fuzzy vectors.

Definition 3.4.1. [25] $h \leq g$ if and only if $h(r) \leq g(r)$ for all $r \in [0, 1]$. It is easy to show that

$$\begin{aligned} &[h \vee g](r) = max\{h(r), g(r)\}, \\ &[h \wedge g](r) = min\{h(r), g(r)\}. \end{aligned} \tag{3.12}$$

Definition 3.4.2. [25] For arbitrary fuzzy number $u = (\underline{u}, \overline{u})$, the number $u_0 = \frac{1}{2}(\underline{u}(1) + \overline{u}(1)))$ is said to be a location index number of u, and two non-decreasing left continuous functions

$$\begin{aligned} u_* &= u_0 - \underline{u}, \\ u^* &= \overline{u} - u_0, \end{aligned} \tag{3.13}$$

are called the left fuzziness index function and the right fuzziness index function, respectively.

According to definition 3.4.2, every fuzzy number can be represented by (u_0, u_*, u^*). It is obvious that a fuzzy number u is symmetric if and only if $u_* = u^*$.

Definition 3.4.3. [25] For arbitrary fuzzy numbers $u = (u_0, u_*, u^*)$ and $v = (v_0, v_*, v^*)$ the four arithmetic operations are defined by

$$u \bigodot v = (u_0 \bigodot v_0, u_* \vee v_*, u^* \vee v^*),$$

where $u \bigodot v$ is either of $u+v, u-v, u.v$ and u/v.

Theorem 3.4.1. For all fuzzy numbers u, v and w, we have

1. $u + v = v + u$,
2. $(u + v) + w = u + (v + w)$,
3. $uv = vu$,
4. $(uv)w = u(vw)$,
5. There exists $1 \in E$ such that $u.1 = u$ for all $u \in E$,
6. $u(v + w) = uv + uw$.

Proof. It is trivial by definition 3.4.3. □

Definition 3.4.4. A matrix $A = (a_{ij})$ is called a fuzzy matrix, if each element of A is a fuzzy number, we represent $A = (a_{ij})$ by $a_{ij} = ((a_{ij})_0, (a_{ij})_*, (a_{ij})^*)$ where $(a_{ij})_0$ is location index of a_{ij} and $(a_{ij})_*, (a_{ij})^*$ are left fuzziness index and right fuzziness index.

Definition 3.4.5. A vector $b = (b_i)$ is called a fuzzy vector, if each element of b is a fuzzy number, with new notation $b = ((b_i)_0, (b_i)_*, (b_i)^*)$, where $(b_i)_0$ is location index of b_i, $(b_i)_*$ and $(b_i)^*$ are left fuzziness index and right fuzziness index.

Let x_i be a solution of (3.11), that is

$$\sum_{j=1}^{n} a_{ij}x_j + b_i = \sum_{j=1}^{n} c_{ij}x_j + d_i, \quad i = 1, 2, \ldots, n, \tag{3.14}$$

therefore, we have with definition 3.4.2 and Eq. (3.14)

$$\begin{aligned}&\textstyle\sum_{j=1}^{n}((a_{ij})_0, (a_{ij})_*, (a_{ij})^*).((x_j)_0, (x_j)_*, (x_j)^*) + ((b_i)_0, (b_i)_*, (b_i)^*) =\\ &\textstyle\sum_{j=1}^{n}((c_{ij})_0, (c_{ij})_*, (c_{ij})^*).((x_j)_0, (x_j)_*, (x_j)^*) + ((d_i)_0, (d_i)_*, (d_i)^*),\\ &\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, n,\end{aligned} \tag{3.15}$$

this implies for $i = 1, 2, \ldots, n$

$$\begin{aligned}&\textstyle\sum_{j=1}^{n}((a_{ij})_0(x_j)_0, max\{(a_{ij})_*, (x_j)_*\}, max\{(a_{ij})^*, (x_j)^*\}) + ((b_i)_0, (b_i)_*, (b_i)^*) =\\ &\textstyle\sum_{j=1}^{n}((c_{ij})_0(x_j)_0, max\{(c_{ij})_*, (x_j)_*\}, max\{(c_{ij})^*, (x_j)^*\}) + ((d_i)_0, (d_i)_*, (d_i)^*),\end{aligned} \tag{3.16}$$

let

$$
\begin{aligned}
&max\{(a_{ij})_*, (x_j)_*\} = (m_{ij})_*,\\
&max\{(a_{ij})^*, (x_j)^*\} = (m_{ij})^*,\\
&max\{(c_{ij})_*, (x_j)_*\} = (n_{ij})_*,\\
&max\{(c_{ij})^*, (x_j)^*\} = (n_{ij})^*,
\end{aligned} \tag{3.17}
$$

then, we have for $i = 1, 2, \ldots, n$:

$$
\begin{aligned}
&\textstyle\sum_{j=1}^{n}((a_{ij})_0(x_j)_0 + (b_i)_0, max\{(m_{ij})_*, (b_i)_*\}, max\{(m_{ij})^*, (b_i)^*\}) =\\
&\textstyle\sum_{j=1}^{n}((c_{ij})_0(x_j)_0 + (d_i)_0, max\{(n_{ij})_*, (d_i)_*\}, max\{(n_{ij})^*, (d_i)^*\}).
\end{aligned} \tag{3.18}
$$

Theorem 3.4.2. If $(x_j)_0$ for $j = 1, 2, \ldots, n$ are the solution of the crisp linear system

$$\textstyle\sum_{j=1}^{n}(a_{ij})_0(x_j)_0 + (b_i)_0 = \sum_{j=1}^{n}(c_{ij})_0(x_j)_0 + (d_i)_0,$$

$$i = 1, 2, \ldots, n,$$

and $(x_j)_*$, $(x_j)^*$ for $j = 1, 2, \ldots, n$ are obtained by

$$(x_j)_* = \max_{1\le i\le n}\{(a_{ij})_*, (b_i)_*, (c_{ij})_*, (d_i)_*\}, \tag{3.19}$$

$$(x_j)^* = \max\nolimits_{1\le i\le n}\{(a_{ij})^*, (b_i)^*, (c_{ij})^*, (d_i)^*\}. \tag{3.20}$$

Then the fuzzy vector $x = (x_j) = (\underline{x}_j, \overline{x}_j)$ obtained by

$$
\begin{aligned}
\underline{x_j} &= (x_j)_0 + (x_j)_*,\\
\overline{x_j} &= (x_j)_0 + (x_j)^*,
\end{aligned}
$$

for $j = 1, 2, \ldots, n$ is a solution of (3.11).

Proof. By attention of the results, it is clear. $\square$

Example 3.4.1. Consider the 2×2 fully fuzzy linear system

$$\begin{cases} (1,2,3)x_1 + (4,6,9)x_2 + (1,3,4) = (0,1,3)x_1 + (5,6,8)x_2 + (0,1,7), \\ (5,6,8)x_1 + (3,5,6)x_2 + (0,7,8) = (1,4,5)x_1 + (2,3,4)x_2 + (8,9,12). \end{cases}$$

By simple calculations of the new arithmetic, we have the following system for finding location index number of x_1 and x_2:

$$\begin{cases} 2(x_1)_0 + 6(x_2)_0 + 3 = (x_1)_0 + 6(x_2)_0 + 1, \\ 6(x_1)_0 + 5(x_2)_0 + 7 = 4(x_1)_0 + 3(x_2)_0 + 9, \end{cases}$$

therefore, we have

$$(x_1)_0 = -2, \qquad (x_2)_0 = 3.$$

We obtain left fuzziness index and right fuzziness index function of x_1 and x_2 by Eq.(3.19) and Eq. (3.20):

$$(x_1)_* = 1 - r, \quad (x_2)_* = 2 - 2r,$$

$$(x_1)^* = 1 - r, \quad (x_2)^* = 1 - r.$$

The parametric form of x_1 and x_2 are

$$\underline{x}_1(r) = r - 3, \qquad \underline{x}_2(r) = 2r + 1,$$

$$\overline{x}_1(r) = -1 - r, \quad \overline{x}_2(r) = 4 - r.$$

Example 3.4.2. For production of a high quality chemical compound, we need about 0.4 ((0.3,0.4,0.5)) kg poly ethylene high density (PEHD) and about 0.3 ((0.1,0.3,0.45)) kg poly ethylene low density (PELD) and from poly propylen (PP), we need exactly 0.267 kg which its price is about 3 ((1.217,3,4.775)) dollar. Now from the same chemical compound with lower quality with the same cost so the products would have higher expansion. We need from the PEHD about 0.5 ((0.3,0.5,0.8)) kg and from PELD about 0.4 ((0.2,0.4,0.5)) kg and from PP exactly 0.1 kg which its price is about 3 ((1,3,3.5)) dollar and for production of second high quality chemical compound, we need about 0.2 ((0.15,0.2,0.3)) kg PEHD and about 0.7 ((0.6,0.7,0.95)) kg PELD and from poly estyrene (PE), we need exactly 0.1 kg which its price is about 5 ((2.625,5,6.75)) dollar. Now from the same chemical compound with lower quality with the same cost so the products would have higher expansion. We need from the PEHD about 0.3 ((0.2,0.3,0.5)) kg and from PELD about 0.3 ((0.15,0.3,0.4)) kg and from PE exactly 0.3 kg which its price is about 5 ((4,5,7)) dollar. For obtaining these two chemical compounds with different qualities how much would be about the price of PEHD and PELD ?

Let x_1 and x_2 be the price of PEHD and PELD. Together, these equations form a fully fuzzy linear system as follows:

$$\begin{cases} (0.3, 0.4, 0.5)x_1 + (0.1, 0.3, 0.45)x_2 + (0.325, 0.8, 1.275) = \\ \qquad\qquad (0.3, 0.5, 0.8)x_1 + (0.2, 0.4, 0.5)x_2 + (0.1, 0.3, 0.35), \\ (0.15, 0.2, 0.3)x_1 + (0.6, 0.7, 0.95)x_2 + (0.2625, 0.5, 0.675) = \\ \qquad\qquad (0.2, 0.3, 0.5)x_1 + (0.15, 0.3, 0.4)x_2 + (1.2, 1.5, 2.1). \end{cases}$$

By simple calculations of the new arithmetic, we have the following system for finding location index number of x_1 and x_2:

$$\begin{cases} 0.4(x_1)_0 + 0.3(x_2)_0 + 0.8 = 0.5(x_1)_0 + 0.4(x_2)_0 + 0.3, \\ 0.2(x_1)_0 + 0.7(x_2)_0 + 0.5 = 0.3(x_1)_0 + 0.3(x_2)_0 + 1.5, \end{cases}$$

therefore, we have

$$(x_1)_0 = 2, \qquad (x_2)_0 = 3.$$

We obtain left fuzziness index and right fuzziness index function x_1 and x_2 by Eq.(3.19) and Eq. (3.20):

$$(x_1)_* = 0.5 - 0.5r, \quad (x_2)_* = 0.75 - 0.75r,$$

$$(x_1)^* = 0.5 - 0.5r, \quad (x_2)^* = 0.5 - 0.5r.$$

The parametric form of x_1 and x_2 are the following form:

$$\underline{x}_1(r) = 1.5 + 0.5r, \qquad \underline{x}_2(r) = 2.25 + 0.75r,$$

$$\overline{x}_1(r) = 2.5 - 0.5r, \qquad \overline{x}_2(r) = 3.5 - 0.5r.$$

$$x_1 = (1.5 + 0.5r, 2.5 - 0.5r), \quad x_2 = (2.25 + 0.75r, 3.5 - 0.5r).$$

Chapter 4

System of fuzzy polynomial equations

4.1 Introduction

System of polynomials play a major role in various areas such as pure and applied mathematics, engineering and social science. Standard analytic techniques like Buckley and Qu method [15], may be not suitable for solving the systems such as

$$\begin{cases} ax^5 + by^4 + cx^3 + dy^3 + exy - f = r, \\ ax - y = k, \end{cases}$$

where $x, y, a, b, c, d, e, f, r$ and k are fuzzy numbers and x, y are unknowns, because they found solutions linear equations like $ax + b = c$ and quadratic equations like $ax^2 + bx = c$. In this chapter, we are interested in finding the solution of system of fuzzy polynomials like

$$\begin{cases} F(x, y) = c, \\ G(x, y) = d, \end{cases}$$

where x, y, c and d are fuzzy numbers and F, G are fuzzy polynomials of x, y and also we propose an efficient extension of Newton's method by modified Adomian decomposition method for solving (if exists) fuzzy polynomial equations, $\sum_{i=1}^{n} a_i x^i = c$ where x, c are fuzzy numbers and all coefficients are fuzzy numbers.

4.2 The fixed point method

Now our aim is to obtain a solution for system of fuzzy polynomial equations

$$\begin{cases} F(x,y) = c, \\ G(x,y) = d, \end{cases} \tag{4.1}$$

where x, y, c and d are fuzzy numbers. The parametric form, for any $r \in [0,1]$, is as follows:

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \underline{c}(r), \\ \overline{F}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \overline{c}(r), \\ \underline{G}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \underline{d}(r), \\ \overline{G}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \overline{d}(r). \end{cases} \tag{4.2}$$

Let $\underline{F}, \overline{F}, \underline{G}, \overline{G}$, be continuously differentiable with respect to $\underline{x}, \overline{x}, \underline{y}, \overline{y}$, for any $r \in [0,1]$ also one of the cases of fixed point method is the following form, that the problem of finding the roots of (4.2) has been reformulated in an equivalent form as:

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) - \underline{c}(r) + \underline{x}(r) = \underline{x}(r), \\ \overline{F}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) - \overline{c}(r) + \overline{x}(r) = \overline{x}(r), \\ \underline{G}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) - \underline{d}(r) + \underline{y}(r) = \underline{y}(r), \\ \overline{G}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) - \overline{d}(r) + \overline{y}(r) = \overline{y}(r). \end{cases} \tag{4.3}$$

Therefore, the problem (4.2) has been reformulated in an equivalent form as

$$\begin{cases} \underline{H}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \underline{x}(r), \\ \overline{H}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \overline{x}(r), \\ \underline{K}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \underline{y}(r), \\ \overline{K}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \overline{y}(r), \end{cases} \tag{4.4}$$

where

$$\begin{cases} \underline{H}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \underline{F}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) - \underline{c}(r) + \underline{x}(r), \\ \overline{H}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \overline{F}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) - \overline{c}(r) + \overline{x}(r), \\ \underline{K}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \underline{G}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) - \underline{d}(r) + \underline{y}(r), \\ \overline{K}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) = \overline{G}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) - \overline{d}(r) + \overline{y}(r). \end{cases} \tag{4.5}$$

The solution of (4.4) for any $r \in [0, 1]$, is called analytical solution of (4.1).

Suppose that $x = (\underline{\alpha}, \overline{\alpha})$ and $y = (\underline{\beta}, \overline{\beta})$, are solutions of (4.4), i.e., for any $r \in [0, 1]$

$$\begin{cases} \underline{H}(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}; r) = \underline{\alpha}(r), \\ \overline{H}(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}; r) = \overline{\alpha}(r), \\ \underline{K}(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}; r) = \underline{\beta}(r), \\ \overline{K}(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}; r) = \overline{\beta}(r). \end{cases} \tag{4.6}$$

Therefore, if $x_0 = (\underline{x}_0(r), \overline{x}_0(r))$ and $y_0 = (\underline{y}_0(r), \overline{y}_0(r))$ are approximate solutions for this system, then we use the fixed point iteration

$$\begin{cases} \underline{x}_{n+1}(r) = \underline{H}(\underline{x}_n, \overline{x}_n, \underline{y}_n, \overline{y}_n; r), \\ \overline{x}_{n+1}(r) = \overline{H}(\underline{x}_n, \overline{x}_n, \underline{y}_n, \overline{y}_n; r), \\ \underline{y}_{n+1}(r) = \underline{K}(\underline{x}_n, \overline{x}_n, \underline{y}_n, \overline{y}_n; r), \\ \overline{y}_{n+1}(r) = \overline{K}(\underline{x}_n, \overline{x}_n, \underline{y}_n, \overline{y}_n; r). \end{cases} \tag{4.7}$$

Using vector notation, we write Eq.(4.7) as

$$z_{n+1} = T(z_n), \quad \cdot \tag{4.8}$$

with

$$z_n = \begin{bmatrix} \underline{x}_n(r) \\ \overline{x}_n(r) \\ \underline{y}_n(r) \\ \overline{y}_n(r) \end{bmatrix}, T(z) = \begin{bmatrix} \underline{H}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) \\ \overline{H}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) \\ \underline{K}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) \\ \overline{K}(\underline{x}, \overline{x}, \underline{y}, \overline{y}; r) \end{bmatrix}. \tag{4.9}$$

Now we use Taylor expansion $\underline{H}, \overline{H}, \underline{K}, \overline{K}$, and for any $r \in [0,1]$, we put:

$$\begin{bmatrix} \underline{\alpha}(r) - \underline{x}_{n+1}(r) \\ \overline{\alpha}(r) - \overline{x}_{n+1}(r) \\ \underline{\beta}(r) - \underline{y}_{n+1}(r) \\ \overline{\beta}(r) - \overline{y}_{n+1}(r) \end{bmatrix} = \begin{bmatrix} \frac{\partial \underline{H}(\xi_n^1)}{\partial \underline{x}} & \frac{\partial \underline{H}(\xi_n^1)}{\partial \overline{x}} & \frac{\partial \underline{H}(\xi_n^1)}{\partial \underline{y}} & \frac{\partial \underline{H}(\xi_n^1)}{\partial \overline{y}} \\ \frac{\partial \overline{H}(\xi_n^2)}{\partial \underline{x}} & \frac{\partial \overline{H}(\xi_n^2)}{\partial \overline{x}} & \frac{\partial \overline{H}(\xi_n^2)}{\partial \underline{y}} & \frac{\partial \overline{H}(\xi_n^2)}{\partial \overline{y}} \\ \frac{\partial \underline{K}(\xi_n^3)}{\partial \underline{x}} & \frac{\partial \underline{K}(\xi_n^3)}{\partial \overline{x}} & \frac{\partial \underline{K}(\xi_n^3)}{\partial \underline{y}} & \frac{\partial \underline{K}(\xi_n^3)}{\partial \overline{y}} \\ \frac{\partial \overline{K}(\xi_n^4)}{\partial \underline{x}} & \frac{\partial \overline{K}(\xi_n^4)}{\partial \overline{x}} & \frac{\partial \overline{K}(\xi_n^4)}{\partial \underline{y}} & \frac{\partial \overline{K}(\xi_n^4)}{\partial \overline{y}} \end{bmatrix} \begin{bmatrix} \underline{\alpha}(r) - \underline{x}_n(r) \\ \overline{\alpha}(r) - \overline{x}_n(r) \\ \underline{\beta}(r) - \underline{y}_n(r) \\ \overline{\beta}(r) - \overline{y}_n(r) \end{bmatrix}, \tag{4.10}$$

where $\xi_n^i = (\underline{\xi}_{1,n}^i, \overline{\xi}_{1,n}^i, \underline{\xi}_{2,n}^i, \overline{\xi}_{2,n}^i)$, for $i = 1, \ldots, 4$.

Let J_n denote the matrix in (4.10). Then we can rewrite this equation as matrix form

$$\lambda - z_{n+1} = J_n(\lambda - z_n), \tag{4.11}$$

where $\lambda = (\underline{\alpha}(r), \overline{\alpha}(r), \underline{\beta}(r), \overline{\beta}(r))$ and $z_{n+1} = (\underline{x}_{n+1}(r), \overline{x}_{n+1}(r), \underline{y}_{n+1}(r), \overline{y}_{n+1}(r))$.

It is convenient to introduce the Jacobian matrix for the functions $\underline{H}, \overline{H}, \underline{K}$ and $\overline{K}$:

$$J(x) = \begin{bmatrix} \frac{\partial \underline{H}(x)}{\partial \underline{x}} & \frac{\partial \underline{H}(x)}{\partial \overline{x}} & \frac{\partial \underline{H}(x)}{\partial \underline{y}} & \frac{\partial \underline{H}(x)}{\partial \overline{y}} \\ \frac{\partial \overline{H}(x)}{\partial \underline{x}} & \frac{\partial \overline{H}(x)}{\partial \overline{x}} & \frac{\partial \overline{H}(x)}{\partial \underline{y}} & \frac{\partial \overline{H}(x)}{\partial \overline{y}} \\ \frac{\partial \underline{K}(x)}{\partial \underline{x}} & \frac{\partial \underline{K}(x)}{\partial \overline{x}} & \frac{\partial \underline{K}(x)}{\partial \underline{y}} & \frac{\partial \underline{K}(x)}{\partial \overline{y}} \\ \frac{\partial \overline{K}(x)}{\partial \underline{x}} & \frac{\partial \overline{K}(x)}{\partial \overline{x}} & \frac{\partial \overline{K}(x)}{\partial \underline{y}} & \frac{\partial \overline{K}(x)}{\partial \overline{y}} \end{bmatrix}. \tag{4.12}$$

We can obtain approximate solutions, for any r belong to $[0,1]$ by using the recursive scheme (4.7) where $n = 0, 1, \ldots$ for initial guess, one can use the fuzzy number

$$\begin{cases} x_0 = (\underline{x}(1), \overline{x}(1), \underline{x}(1) - \underline{x}(0), \overline{x}(0) - \overline{x}(1)), \\ y_0 = (\underline{y}(1), \overline{y}(1), \underline{y}(1) - \underline{y}(0), \overline{y}(0) - \overline{y}(1)), \end{cases} \tag{4.13}$$

and in parametric form

$$\begin{cases} \underline{x}_0(r) = \underline{x}(1) + (\underline{x}(1) - \underline{x}(0))(r-1), \\ \overline{x}_0(r) = \overline{x}(1) + (\overline{x}(0) - \overline{x}(1))(1-r), \\ \underline{y}_0(r) = \underline{y}(1) + (\underline{y}(1) - \underline{y}(0))(r-1), \\ \overline{y}_0(r) = \overline{y}(1) + (\overline{y}(0) - \overline{y}(1))(1-r). \end{cases} \tag{4.14}$$

Remark 4.2.1. Sequences $(\underline{x}_n, \overline{x}_n)_{n=0}^{\infty}$ and $(\underline{y}_n, \overline{y}_n)_{n=0}^{\infty}$ convergent to $(\underline{\alpha}, \overline{\alpha})$ and $(\underline{\beta}, \overline{\beta})$, respectively, if and only if for any $r \in [0,1]$,

$$\lim_{n \longrightarrow \infty} \underline{x}_n(r) = \underline{\alpha}(r), \lim_{n \longrightarrow \infty} \overline{x}_n(r) = \overline{\alpha}(r), \lim_{n \longrightarrow \infty} \underline{y}_n(r) = \underline{\beta}(r), \lim_{n \longrightarrow \infty} \overline{y}_n(r) = \overline{\beta}(r).$$

Now we are going to present convergence theorem.

Theorem 4.2.1. Let for each r belong to $[0,1]$, D be a closed, bounded and convex set in the plane. Also let for each r belong to $[0,1]$, the $\underline{H}, \underline{K}, \overline{H}$ and $\overline{K}$ are continuously differentiable with respect to $\underline{x}, \underline{y}, \overline{x}$ and $\overline{y}$, and further assume for each r belong to $[0,1]$, $T(D) \subset D$. Assume that there exist $(\underline{\alpha}, \overline{\alpha}) \in R^2$ and $(\underline{\beta}, \overline{\beta}) \in R^2$ and a $0 < l < 1$ such that, for each r belong to $[0,1]$, $Max_D \|J(x)\|_\infty < l$, then the Fixed point method converges to $(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta})$, and obeys

$$\begin{aligned} &\|\lambda - z_{n+1}\|_\infty \leq (\|J(\lambda)\|_\infty + \varepsilon_n)\|\lambda - z_n\|_\infty \\ &with \quad \varepsilon_n \longrightarrow 0 \;\; as \;\; n \longrightarrow \infty. \end{aligned} \tag{4.15}$$

Proof. From (4.11), we have

$$\|\lambda - z_{n+1}\|_\infty \leq \|J_n\|_\infty . \|\lambda - z_n\|_\infty, \qquad \forall r \in [0,1]. \tag{4.16}$$

As $n \longrightarrow \infty$, the points ξ_n^i used in evaluating J_n will all tend to λ, since they are on the line segment joining z_n and λ. Hence

$$\|J_n\|_\infty \longrightarrow \|J(\lambda)\|_\infty \;\; as \;\; n \longrightarrow \infty.$$

Result (4.15) follows from (4.16) by letting $\varepsilon_n = \|J_n\|_\infty - \|J(\lambda)\|_\infty$.

Example 4.2.1. Consider the fuzzy nonlinear system

$$\begin{cases} x^2 + y^2 = (5, 0.6, 2) \\ x^2 + y^3 + (2, 1, 1) = (11, 2.4, 6.2), \end{cases}$$

Assume that x and y are positive, then the parametric form of this system is as follows:

$$\begin{cases} \underline{x}^2(r) + \underline{y}^2(r) = 4.4 + 0.6r, \\ \overline{x}^2(r) + \overline{y}^2(r) = 7 - 2r, \\ \underline{x}^2(r) + \underline{y}^3(r) + (1 + r) = 8.6 + 2.4r, \\ \overline{x}^2(r) + \overline{y}^3(r) + (3 - r) = 17.2 - 6.2r. \end{cases}$$

Initial guess is obtained from the above system by using $r = 0$ and $r = 1$, therefore

$$\begin{cases} \underline{x}^2(0) + \underline{y}^2(0) = 4.4, \\ \overline{x}^2(0) + \overline{y}^2(0) = 7, \\ \underline{x}^2(0) + \underline{y}^3(0) + 1 = 8.6, \\ \overline{x}^2(0) + \overline{y}^3(0) + 3 = 17.2, \end{cases} \qquad \begin{cases} \underline{x}^2(1) + \underline{y}^2(1) = 5, \\ \overline{x}^2(1) + \overline{y}^2(1) = 5, \\ \underline{x}^2(1) + \underline{y}^3(1) + 2 = 11, \\ \overline{x}^2(1) + \overline{y}^3(1) + 2 = 11. \end{cases}$$

Consequently $\underline{x}(0) = 0.9036, \overline{x}(0) = 1.2567, \underline{y}(0) = 1.893, \overline{y}(0) = 2.32824, \underline{x}(1) = \overline{x}(1) = 1$ and $\underline{y}(1) = \overline{y}(1) = 2$. Therefore initial guess is

$$x_0 = (1, 0.0964, 0.2567) \qquad y_0 = (2, 0.107, 0.32824).$$

After three iterations, we obtain the approximate solutions of x and y with the maximum error about 10^{-2} and 2×10^{-3}, respectively. For more details see Figs. 4.1 and

4.2.

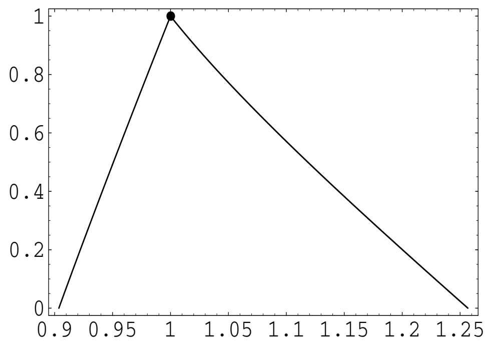

Figure. 4.1. Standard analytical solution (solid line) and approximate solution (•) for x.

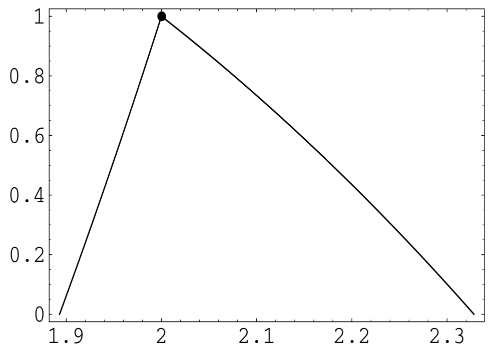

Figure. 4.2. Standard analytical solution (solid line) and approximate solution (•) for y.

Now suppose x and y are negative, we have

$$\begin{cases} \overline{x}^2(r) + \overline{y}^2(r) = 4.4 + 0.6r, \\ \underline{x}^2(r) + \underline{y}^2(r) = 7 - 2r, \\ \overline{x}^2(r) + \underline{y}^3(r) + (1 + r) = 8.6 + 2.4r, \\ \underline{x}^2(r) + \overline{y}^3(r) + (3 - r) = 17.2 - 6.2r. \end{cases}$$

For $r = 0$, we have $\underline{y}(0) = 2.01886$ and $\overline{y}(0) = 2.24241$, therefore negative roots do not exits.

4.3 The Adomian decomposition method

Consider the fuzzy polynomial equation

$$P_n(x) = \sum_{i=1}^{n} a_i x^i = c, \tag{4.17}$$

where x, c and all coefficients are fuzzy numbers. The parametric form for any $r \in [0,1]$, is as follows:

$$\begin{cases} \underline{P}_n(\underline{x}, \overline{x}; r) = \underline{c}(r), \\ \overline{P}_n(\underline{x}, \overline{x}; r) = \overline{c}(r). \end{cases} \tag{4.18}$$

The problem (4.18) has been reformulated in an equivalent form as

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}; r) = 0, \\ \overline{F}(\underline{x}, \overline{x}; r) = 0, \end{cases} \tag{4.19}$$

where

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}; r) = \underline{P}_n(\underline{x}, \overline{x}; r) - \underline{c}(r), \\ \overline{F}(\underline{x}, \overline{x}; r) = \overline{P}_n(\underline{x}, \overline{x}; r) - \overline{c}(r). \end{cases}$$

Suppose that (α, β) is the solution of (4.19), i.e.,

$$\begin{cases} \underline{F}(\alpha, \beta; r) = 0, \\ \overline{F}(\alpha, \beta; r) = 0. \end{cases}$$

Now if we use the Taylor series of $\underline{F}, \overline{F}$ about $(\underline{x}, \overline{x})$, then for each $r \in [0,1]$,

$$\begin{cases} \underline{F}(\underline{x}-h, \overline{x}-k; r) = \underline{F}(\underline{x}, \overline{x}; r) - h\underline{F}_{\underline{x}}(\underline{x}, \overline{x}; r) - k\underline{F}_{\overline{x}}(\underline{x}, \overline{x}; r) + O(h^2+k^2+hk) = 0, \\ \overline{F}(\underline{x}-h, \overline{x}-k; r) = \overline{F}(\underline{x}, \overline{x}; r) - h\overline{F}_{\underline{x}}(\underline{x}, \overline{x}; r) - k\overline{F}_{\overline{x}}(\underline{x}, \overline{x}; r) + O(h^2+k^2+hk) = 0, \end{cases}$$

we assume, of course, that all needed partial derivatives exist and bounded. Therefore for enough small $h(r)$ and $k(r)$, for each $r \in [0,1]$,

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}; r) - h\underline{F}_{\underline{x}}(\underline{x}, \overline{x}; r) - k\underline{F}_{\overline{x}}(\underline{x}, \overline{x}; r) \simeq 0, \\ \overline{F}(\underline{x}, \overline{x}; r) - h\overline{F}_{\underline{x}}(\underline{x}, \overline{x}; r) - k\overline{F}_{\overline{x}}(\underline{x}, \overline{x}; r) \simeq 0, \end{cases}$$

and hence $h = h(r)$ and $k = k(r)$ are unknown quantities that can be obtained by solving the following equations, for each $r \in [0,1]$,

$$J(\underline{x}, \overline{x}; r) \begin{bmatrix} h(r) \\ k(r) \end{bmatrix} = \begin{bmatrix} \underline{F}(\underline{x}, \overline{x}; r) \\ \overline{F}(\underline{x}, \overline{x}; r) \end{bmatrix},$$

where,

$$J(\underline{x}, \overline{x}; r) = \begin{bmatrix} \underline{F}_{\underline{x}}(\underline{x}, \overline{x}; r) & \underline{F}_{\overline{x}}(\underline{x}, \overline{x}; r) \\ \overline{F}_{\underline{x}}(\underline{x}, \overline{x}; r) & \overline{F}_{\overline{x}}(\underline{x}, \overline{x}; r) \end{bmatrix},$$

that $\underline{F}_{\underline{x}}$ means, the derivative of $\underline{F}$ with respect to $\underline{x}$, and so on.

The Newton's method is given by

$$\begin{cases} \underline{x}_{n+1}(r) = \underline{x}_n(r) + h_n(r), \\ \overline{x}_{n+1}(r) = \overline{x}_n(r) + k_n(r), \end{cases} \tag{4.20}$$

where $n = 0, 1, 2, \ldots$. For initial guess, one can use the fuzzy number

$$x_0 \simeq (\underline{x}(1), \overline{x}(1), \underline{x}(1) - \underline{x}(0), \overline{x}(0) - \overline{x}(1)),$$

and in parametric form

$$\underline{x}_0(r) \simeq \underline{x}(1) + (\underline{x}(1) - \underline{x}(0))(r-1),$$

$$\overline{x}_0(r) \simeq \overline{x}(1) + (\overline{x}(0) - \overline{x}(1))(1-r).$$

The iteration (4.20) will converge to (α, β) if the starting point $(\underline{x}_0(r), \overline{x}_0(r))$ is close enough to (α, β) for $0 \leq r \leq 1$, local convergence property.

If we consider the Taylor's expansion of $\underline{F}(\underline{x}, \overline{x}; r)$, $\overline{F}(\underline{x}, \overline{x}; r)$ to a higher order and we are looking for $h(r)$ and $k(r)$ such that:

$$[\underline{F} - h\underline{F}_{\underline{x}} - k\underline{F}_{\overline{x}} + \frac{1}{2}(h^2\underline{F}_{\underline{x}\ \underline{x}} + 2hk\underline{F}_{\underline{x}\overline{x}} + k^2\underline{F}_{\overline{x}\ \overline{x}})](\underline{x}, \overline{x}; r) \simeq 0,$$

$$[\overline{F} - h\overline{F}_{\underline{x}} - k\overline{F}_{\overline{x}} + \frac{1}{2}(h^2\overline{F}_{\underline{x}\ \underline{x}} + 2hk\overline{F}_{\underline{x}\overline{x}} + k^2\overline{F}_{\overline{x}\ \overline{x}})](\underline{x}, \overline{x}; r) \simeq 0,$$

given

$$h(r) = [(\underline{F} - k\underline{F}_{\overline{x}} + \frac{1}{2}(h^2\underline{F}_{\underline{x}\ \underline{x}} + 2hk\underline{F}_{\underline{x}\overline{x}} + k^2\underline{F}_{\overline{x}\ \overline{x}}))/\underline{F}_{\underline{x}}](\underline{x}, \overline{x}; r),$$

$$k(r) = [(\overline{F} - h\overline{F}_{\underline{x}} + \frac{1}{2}(h^2\overline{F}_{\underline{x}\ \underline{x}} + 2hk\overline{F}_{\underline{x}\overline{x}} + k^2\overline{F}_{\overline{x}\ \overline{x}}))/\overline{F}_{\overline{x}}](\underline{x}, \overline{x}; r),$$

or

$$\begin{bmatrix} h(r) \\ k(r) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + N\left(\begin{bmatrix} h(r) \\ k(r) \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} N_1(h,k) \\ N_2(h,k) \end{bmatrix}, \tag{4.21}$$

where $c_1 = \frac{\underline{F}}{\underline{F}_{\underline{x}}}(\underline{x}, \overline{x}; r)$ and $c_2 = \frac{\overline{F}}{\overline{F}_{\overline{x}}}(\underline{x}, \overline{x}; r)$ are constants and N is a vector quadratic polynomial and for approximating $h(r)$ and $k(r)$, we can apply the multivariable Adomian decomposition method [1].

The Adomian decomposition technique considers the representation of the solution

(4.21) as a series

$$h = \sum_{n=0}^{\infty} h_n, \qquad k = \sum_{n=0}^{\infty} k_n \tag{4.22}$$

and the nonlinear functions are decomposed as

$$N_i(h,k) = \sum_{n=0}^{\infty} A_{in}(h_0, h_1, \ldots, h_n, k_0, k_1, \ldots, k_n), \qquad i = 1,2. \tag{4.23}$$

where the A_{in}'s are Adomian's polynomials given by [2],

$$A_{in} = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N_i(\sum_{j=0}^{\infty} \lambda^j h_j, \sum_{j=0}^{\infty} \lambda^j k_j)]_{\lambda=0} \qquad i = 1,2, j = 0,1,\ldots.$$

Upon substituting (4.22), (4.23) in the (4.21) yields

$$h_0 = c_1, \quad h_{n+1} = A_{1n}, \quad k_0 = c_2, \quad k_{n+1} = A_{2n},$$

for $n = 0,1,\ldots$, the multivariable polynomials A_{in} are generated by practical formulate presented in [1], for $i = 1,2$, we have

$$A_{i0} = N_i(h_0, k_0),$$
$$A_{in} = \sum_{\varphi} \frac{h_1^{p_1}}{p_1!} \cdots \frac{h_n^{p_n}}{p_n!} . \frac{k_1^{q_1}}{q_1!} \cdots \frac{k_n^{q_n}}{q_n!} . \frac{\partial^{\varphi_1+\varphi_2}}{\partial h^{\varphi_1} \partial k^{\varphi_2}} N_i(h_0, k_0), \quad n \neq 0,$$

where φ stands for $(p_1 + 2p_2 + \ldots + np_n) + (q_1 + 2q_2 + \ldots + nq_n) = n$, and $\varphi_1 = p_1 + p_2 + \ldots + p_n$, $\varphi_2 = q_1 + q_2 + \ldots + q_n$.

In practice, of course, the sum of the infinite series has to be truncated at some order M. The quantities $\sum_{n=0}^{M} h_n$ and $\sum_{n=0}^{M} k_n$, can thus be reasonable approximations of the exact solution of (4.19), provided M is sufficiently large. As $M \longrightarrow \infty$, the series converge smoothly toward the exact solution for $0 \leq r \leq 1$.

Let

$$H_M = h_0 + h_1 + \ldots + h_M = h_0 + A_{10} + A_{11} + \ldots + A_{1M-1},$$
$$K_M = k_0 + k_1 + \ldots + k_M = k_0 + A_{20} + A_{21} + \ldots + A_{2M-1}, \tag{4.24}$$

denote the $(M+1)$-term approximations of h and k, respectively. Since the series converge very rapidly, then (4.24) can serves as a practical solution in each iteration.

Remark 4.3.1. The problem (4.17) with uncertainty functions $P_n(x)$ where the coefficients are fuzzy numbers, is a mapping from $E \longrightarrow E$.

4.3.1 Comparison with other methods

This study would not be completed without comparing it with other existing methods. Some comparisons are as follows:

- In [5] and [6] researchers used the Newton's method for solving fuzzy nonlinear equations and systems of fuzzy nonlinear equations and in [12] Fixed point method for solving fuzzy nonlinear equations. The Adomian decomposition method for $M = 0$ is the Newton's method. See examples 4.3.1, 4.3.2 for more details.

- In [9, 10] a FNN_2 equivalent to the fuzzy polynomial equation and system of fuzzy polynomials F of s fuzzy polynomial equations such as

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= A_{10}, \\ &\vdots \\ f_l(x_1, x_2, \ldots, x_n) &= A_{l0}, \\ &\vdots \\ f_s(x_1, x_2, \ldots, x_n) &= A_{s0}, \end{aligned} \tag{4.25}$$

where $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and all coefficients are fuzzy numbers were built. In [9], a method for solving fuzzy polynomials such as

$$a_1x + a_2x^2 + \ldots + a_nx^n = c$$

where x, c and all coefficients are fuzzy numbers was proposed.

Example 4.3.1. Consider the fuzzy nonlinear equation

$$(4,1,1)x^2 + (2,1,1)x = (2,1,1),$$

assume that x is positive, then the parametric form of this equation is as follows:

$$\begin{cases} (3+r)\underline{x}^2(r) + (1+r)\underline{x}(r) = 1+r, \\ (5-r)\overline{x}^2(r) + (3-r)\overline{x}(r) = (3-r). \end{cases}$$

Initial guess is $(0.3, 0.05, 0.05)$.

By Adomian decomposition method, we obtain the numerical results for $M = 0, 1, 2$, see figure 4.3 and table 4.1 for more details.

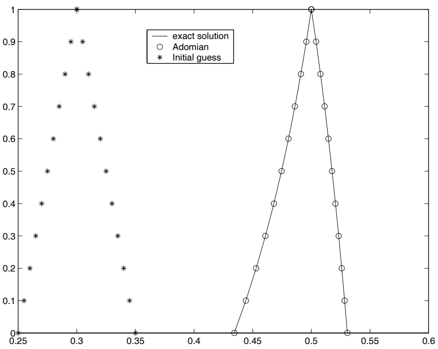

Figure 4.3. Approximate and analytical solution of example 1.

M	Iteration 1	Iteration 2	Iteration 3	Iteration 4
0	4.159×10^{-2}	1.293×10^{-3}	1.389×10^{-5}	1.605×10^{-12}
1	2.007×10^{-2}	1.197×10^{-5}	2.331×10^{-14}	
2	1.280×10^{-2}	7.160×10^{-8}	2.220×10^{-16}	

Table 4.1. The error of Adomian decomposition method.

Example 4.3.2. [15] Suppose a corporation wishes to set aside around one million dollars ($A = (1, 0.2, 0.2)$) to be invested at interest rate R so that after one year they may withdraw approximately 250000 dollars ($S_1 = (0.25, 0.05, 0.05)$ and after two years the amount that is left will accumulate to about 900000 dollars ($S_2 = (0.9, 0.3, 0.3)$). Find R so that A will be sufficient to cover both S_1 and S_2. R will be a fuzzy number whose support lies in $[0, 1]$.

After one year the amount in the account will be

$$A + AR.$$

After withdrawing S_1 the amount to start of the second year is

$$(A - S_1) + AR.$$

At the end of the second year the accumulated total is

$$[(A - S_1) + AR] + [(A - S_1 + AR]R,$$

or

$$AR^2 + BR + D$$

where $B = 2A - S_1$ and $D = A - S_1$, since multiplication distributes over addition for positive fuzzy numbers. Therefore, we must solve

$$AR^2 + BR + D = S_2,$$

or

$$(1, 0.2, 0.2)R^2 + (1.75, 0.45, 0.45)R + (0.75, 0.25, 0.25) = (0.9, 0.3, 0.3)$$

Without any loss of generality, assume that R is positive, then the parametric form of this equation is as follows:

$$\begin{cases} (0.2r + 0.8)\underline{R}^2(r) + (0.45r + 1.3)\underline{R}(r) + (0.25r + 0.5) = 0.3r + 0.6, \\ (1.2 - 0.2r)\overline{R}^2(r) + (2.2 - 0.45r)\overline{R}(r) + (1 - 0.25r) = 1.2 - 0.3r. \end{cases}$$

Initial guess is $(3, 0.5, 0.5)$.

By Adomian decomposition method, we obtain the numerical results for $M = 0, 1, \ldots, 4$, see figure 4.4 and table 4.2 for more details.

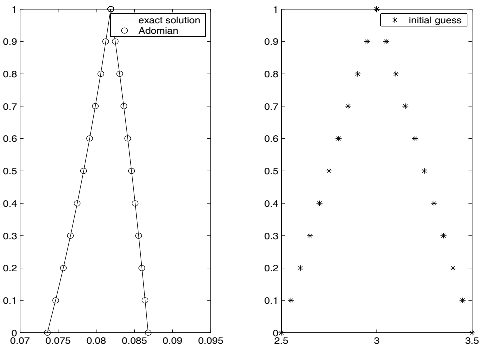

Figure 4.4. Approximate and analytical solution of example 2.

M	Iteration 1	Iteration 2	Iteration 3	Iteration 4	Iteration 5
0	1.319	3.745×10^{-1}	5.089×10^{-2}	1.228×10^{-3}	7.506×10^{-7}
1	8.223×10^{-1}	7.401×10^{-2}	1.009×10^{-3}	5.589×10^{-10}	
2	5.898×10^{-1}	1.460×10^{-2}	2.668×10^{-8}	6.939×10^{-17}	
3	4.476×10^{-1}	1.031×10^{-2}	7.809×10^{-12}		
4	2.116×10^{-1}	2.001×10^{-3}	3.467×10^{-16}		

Table 4.2. The error of Adomian decomposition method.

Conclusions

In this work, we propose a general model for solving a system of m fuzzy linear equations with n variables. The original system with coefficients matrices A and B are replaced by a $(2m) \times (2n)$ crisp linear system with coefficients matrices S and T which may not be full rank even if A and B are full rank. For finding the pseudo-inverse of S, we find the pseudo-inverse of two $m \times n$ matrices. Also, a condition for the existence of a fuzzy solution to the fuzzy general linear system, is presented and also, we suggested numerical method for solving fuzzy nonlinear system instead of standard analytic techniques. Initially we wrote system of fuzzy polynomial equations as parametric form and then solved it by Fixed point method and for finding the real root (if exists) for fuzzy polynomial equation, we applied the Adomian decomposition method.

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