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ISLAMIC AZAD UNIVERSITY SCIENCE AND RESEARCH BRANCH

NUMERICAL SOLUTION OF FUZZY SYSTEMS

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To my dears:

my honorable mother and father

and my wife

Contents

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Abstract

Solving fuzzy linear equations by direct method and fuzzy nonlinear equations by numerical methods as Newton's method and Homotopy method are considered in this research. In this dissertation first, we state elementary definitions and results in fuzzy logic. In chapter two we solve fuzzy system of linear equation by direct method. In chapter three solving fuzzy nonlinear equations and a fuzzy system of nonlinear equations are considered by numerical methods and corresponding algorithms for these methods are given. Chapter four is considered for fuzzy polynomials and determining a symmetric solution.

Originality

The following sections and chapter are proposed for the first time:

Chapter 2: In this chapter solving fuzzy system of linear equation is considered by direct method.

Chapter 3: In this chapter we apply numerical methods as Newton's method and Homotopy method for solving fuzzy nonlinear equations by using parametric form of fuzzy numbers.

Chapter 4: In this chapter fuzzy polynomials is considered. Extremal solutions and a symmetric solution of fuzzy polynomials is given by using parametric form of fuzzy numbers.

Articles

The following papers are extracted from the current thesis:

1) Conjugate gradient method for fuzzy symmetric positive definite system of linear equations, *Journal of Applied Mathematics and Computation*, **171** (2005) 1184-1191 (**ISI**).

2) Lu decomposition method for solving fuzzy system of linear equations, *Journal of Applied Mathematics and Computation*, **172** (2006) 633-643 (**ISI**).

3) Newton's method for solving a system of fuzzy nonlinear equations, *Journal of Applied Mathematics and Computation*, **175** (2006) 1189-1199 (**ISI**).

4) Homotopy Method for solving fuzzy nonlinear equations, *Applied Science*, Vol.**8** (2006) 1-7.

5) A method for solving quadratic fuzzy polynomials, *Journal of Advances in Theoretical and Applied Mathematics*, Vol. **1**, No. **1** (2006).

6) Existence of symmetrical extrem solutions for fuzzy polynomials, *Proceedings of the* 6^{th} *Iranian Conference on Fuzzy Systems and* 1^{th} *Islamic Word Conference*

on Fuzzy Systems.

Introduction

The first publications in fuzzy set theory by Zadeh [1965] and Goguen [1967,1969] show the intention of the authors to generalize the classical notion of a set and a proposition (statement) to accommodate in the sense described in section 1.1.

Zadeh [1965, p.339] writes, "The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual frame-work which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and, potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially, such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables."

Fuzzy set theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied. It can also be considered as a modeling language well suited for situation in which fuzzy relations, criteria, and phenomena exist.

Solving fuzzy equations and system of fuzzy linear equations has long been a problem in fuzzy set theory and many works have been done on these [17, 19, 20, 21, 22, 23, 25, 26, 28, 29, 31, 37, 40, 41, 42, 44]. In [23], using classical methods based on the extension principle, the authors investigated solutions to linear and quadratic equations when the coefficients were real or complex fuzzy numbers and concluded that too often these equations do not have a solution. This result prompted the authors of [23] to investigate other solutions to fuzzy equations. In [22] they gave new solution concept and showed that the fuzzy quadratic equation, where the coefficients are all real fuzzy numbers, always has a solution as a real or complex fuzzy number. These results were generalized to systems of non-linear equations in [16]. In [21], the authors have considered solving the matrix equations $Ax = b$ where A is a fuzzy matrix and x and b are fuzzy vectors. In this paper, the authors concluded that using α-cuts and interval arithmetic in this equations will have no solutions for the given A and b.

A general model for solving an $n \times n$ system of fuzzy linear equation as $Ax = b$ whose coefficients matrix, A, is crisp and the right hand side column, b, is an arbitrary fuzzy number vector were first proposed by Friedman et al. [29] and they studied duality fuzzy linear systems in [28]. By notice to [29] we solve this system by direct method in chapter 2.

In [44], Wang et al. presented an iterative algorithm for dual linear system of the form $x = Ax + u$, where A is real $n \times n$ matrix, the unknown vector x and the constant u are all vectors of fuzzy numbers. Numerical methods as Jacobi, Gauss Sidel and successive over relaxation for solving system of fuzzy linear equations are considered in [40, 41, 42] where coefficients matrix is crisp and the right hand side column is an arbitrary fuzzy number vector. Recently, in [37], the authors have applied Newton's method for solving fuzzy equations. In chapter 3, we continue solving fuzzy nonlinear equations and a system of fuzzy nonlinear equations by numerical methods.

Chapter 1

Introduction

1.1 An introduction to fuzzy logic

Definition 1.1.1 *Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function*

$$\mu_A : X \to [0, 1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

It is clear that A is completely determined by the set of tuples

$$A = \{(x, \mu_A(x)) | x \in X\}.$$

Frequently we will write simply $A(x)$ instead of $\mu_A(x)$. The family of all fuzzy (sub)sets in X is denoted by $\Gamma(X)$. Fuzzy subsets of the real line are called *fuzzy quantities.*

If $X = \{x_1, x_2, ...x_n\}$ is a finite set and A is a fuzzy set in X then we often use the notation

$$A = \mu_1/x_1 + \cdots + \mu_1/x_n$$

where the term μ_1/x_i, $i = 1, 2, ...n$ signifies that μ_i is the grade of membership of x_i in A and the plus sign represents the union.

Definition 1.1.2 *(support) Let A be a fuzzy subset of X. The support of A, denoted supp(A), is the crisp subset of X whose elements all have nonzero membership grades in A*

$$supp(A) = \{x \in X|A(x) > 0\}.$$

Definition 1.1.3 *(normal fuzzy set) A fuzzy subset A of a classical set X is called normal if there exists an $x \in X$ such that $A(x) = 1$. Otherwise A is subnormal.*

Definition 1.1.4 *($\alpha - cut$) An $\alpha -$ level set of a fuzzy set A of X is a non-fuzzy set denoted by $[A]^\alpha$ and is defined by*

$$[A]^\alpha = \begin{cases} \{t \in X|A(t) > 0\} & if \quad \alpha > 0, \\ cl(suppA) & if \quad \alpha = 0, \end{cases}$$

where $cl(suppA)$ denotes the clouser of support of A.

Definition 1.1.5 *(convex fuzzy set) A fuzzy set A of X is called convex if $[A]^\alpha$ is a convex subset of X for all $\alpha \in [0, 1]$.*

Definition 1.1.6 *Let A and B are fuzzy subsets of a classical set X. We say that A is a subset of B if $A(x) \leq B(x)$, for all $x \in X$.*

Definition 1.1.7 *(equality of fuzzy sets) Let A and B are fuzzy subsets of a classical set X. A and B are said to be equal, denoted $A = B$, if $A \subset B$ and $B \subset A$. We note $A = B$ if and only if $A(x) = B(x)$ for $x \in X$.*

Definition 1.1.8 *(empty fuzzy set) The empty fuzzy subset of X is defined as the fuzzy subset $\emptyset$ of X such that $\emptyset(x) = 0$ for each $x \in X$*

It is easy to see that $\emptyset \subset A$ holds for any fuzzy subset A of X.

In many situations people are only able to characterize numeric information imprecisely. For example, people use terms such as, about 5000, near zero, or essentially bigger than 5000. These are examples of what are called fuzzy numbers. Using the theory of fuzzy subsets we can represent these fuzzy numbers as fuzzy subsets of the set of real numbers. More exactly,

Definition 1.1.9 *(fuzzy number) A fuzzy number x is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support. The family of fuzzy numbers will be denoted by E^1.*

Definition 1.1.10 *(quasi fuzzy number) A quasi fuzzy number x is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function satisfying the limit conditions*

$$\lim_{t \to \infty} x(t) = 0 \qquad \lim_{t \to -\infty} x(t) = 0.$$

Let x be a fuzzy number. Then $[x]^\alpha$ is a closed convex (compact) subset of R for all

$\alpha \in [0,1]$. Let us introduce the notations

$$a_1(\alpha) = \min[x]^\alpha, \qquad a_2(\alpha) = \max[x]^\alpha.$$

In other words, $a_1(\alpha)$ denotes the left-hand side and $a_2(\alpha)$ denotes the right-hand side of the $\alpha - cut$. It is easy to see that for $x \in \mathrm{E}^1$

$$if\ \alpha \leq \beta\ \ then\ \ [x]^\alpha \supset [x]^\beta.$$

Furthermore, the left-hand side function

$$a_1 : [0,1] \to R$$

is monotone increasing and lower semicontinuous, and the right-hand side function

$$a_2 : [0,1] \to R$$

is monotone decreasing and upper semicontinuous. We shall use the notation

$$[x]^\alpha = [a_1(\alpha), a_2(\alpha)].$$

Definition 1.1.11 *(triangular fuzzy number) A fuzzy set x is called triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form*

$$x(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & if\ \ a - \alpha \leq t \leq a, \\[2ex] 1 - \dfrac{t-a}{\beta} & if\ \ a \leq t \leq a + \beta, \\[2ex] 0 & \quad otherwise, \end{cases}$$

and we use notation $x = (a, \alpha, \beta)$. *It can easily be verified that*

$$[x]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta], \quad for\ all\ \gamma \ \in\ [0,1].$$

If in triangular fuzzy number $x = (a, \alpha, \beta)$, $\alpha = \beta$ *then* x *is called symmetric triangular fuzzy number and we use notation* $x = (a, \alpha)$.

Definition 1.1.12 *(trapezoidal fuzzy number) A fuzzy set* x *is called trapezoidal fuzzy number with tolerance interval* $[a, b]$, *left width* α *and right width* β *if its membership function has the following form*

$$x(t) = \begin{cases} 1 - \dfrac{a-t}{\alpha} & if \ \ a - \alpha \le t \le a, \\ 1 & if \ \ a \le t \le b, \\ 1 - \dfrac{t-b}{\beta} & if \ \ a \le t \le b + \beta, \\ 0 & otherwise, \end{cases}$$

and we use notation $x = (a, b, \alpha, \beta)$. *It can easily be verified that*

$$[x]^\gamma = [a - (1-\gamma)\alpha, b + (1-\gamma)\beta],$$

for all $\gamma \in [0,1]$. *If in trapezoidal fuzzy number* $x = (a, b, \alpha, \beta)$, $\alpha = \beta$ *then* x *is called symmetric trapezoidal fuzzy number and we use notation* $x = (a, b, \alpha)$.

Definition 1.1.13 *(LR-representation of fuzzy numbers) Any fuzzy number* $x \in E^1$

can be described as

$$x(t)=\begin{cases} L(\dfrac{a-t}{\alpha}) & if \;\; a-\alpha\leq t\leq a, \\ \\ 1 & if \;\; a\leq t\leq b, \\ \\ R(\dfrac{t-b}{\beta}) & if \;\; a\leq t\leq b+\beta, \\ 0 & otherwise, \end{cases}$$

where $[a,b]$ *is the peak or core of* x*,*

$$L:[0,1]\rightarrow[0,1], \qquad\qquad R:[0,1]\rightarrow[0,1]$$

are continuous and non-increasing shape functions with $L(0)=R(0)=1$ *and* $L(1)=R(1)=0$. *We call this fuzzy interval of LR-type and refer to it by* $x=(a,b,\alpha,\beta)_{LR}$.

Definition 1.1.14 *(quasi fuzzy number of type LR) Any quasi fuzzy number* $x\in E^1$ *can be described as*

$$x(t)=\begin{cases} L(\dfrac{a-t}{\alpha}) & if \;\; a-\alpha\leq t\leq a, \\ \\ 1 & if \;\; a\leq t\leq b, \\ \\ R(\dfrac{t-b}{\beta}) & if \;\; a\leq t\leq b+\beta, \\ o, & otherwise, \end{cases}$$

where $[a,b]$ *is the peak or core of* x*,*

$$L:[0,\infty)\rightarrow[0,1], \qquad\qquad R:[0,\infty)\rightarrow[0,1]$$

are continuous and non-increasing shape functions with $L(0)=R(0)=1$ *and*

$$\lim_{t\rightarrow\infty}L(t)=0 \qquad\qquad \lim_{t\rightarrow-\infty}R(t)=0.$$

For example, if $L(t) = R(t) = 1 - t$ then instead of $x = (a, b, \alpha, \beta)_{LR}$ we simply write $x = (a, b, \alpha, \beta)$.

Definition 1.1.15 *(fuzzy point) Let X be a fuzzy number. If $supp(x) = t_0$ then x is called a fuzzy point.*

Let x be a fuzzy point. it is easy to see that $[x]^\alpha = [t_0, t_0] = t_0$, for all $\alpha \in [0, 1]$.

1.2 The extension principle

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to *add*, *subtract*, *multiply* and *divide* with fuzzy quantities. The process of doing these operations is called *fuzzy arithmetic*.

We shall first introduce an important concept from fuzzy set theory called the *extension principle*. We then use it to provide for these arithmetic operations on fuzzy numbers.

In general the extension principle plays a fundamental role in enabling us to extend any point operations to operations involving fuzzy sets. In following we define this principle.

Definition 1.2.1 *(extension principle) Assume X and Y are crisp sets and let f be a mapping from X to Y,*

$$f : X \to Y$$

such that for each $x \in X$, $f(x) = y \ \in \ Y$. *Assume* A *is a fuzzy subset of* X, *using extension principle, we can define* $f(A)$ *as a fuzzy subset of* Y *such that*

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & if \quad f^{-1}(y) \neq \emptyset, \\ 0 & otherwise, \end{cases}$$

where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Definition 1.2.2 *(sup-min extension n-place functions) Let* X_1, X_2,...,X_n *and* Y *be a family of sets. Assume* f *is a mapping from the Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ *in to* Y. *Let* A_1, A_2,...,A_n *be fuzzy subsets of* X_1, X_2,...,X_n, *respectively; then we use the extension principle for the evaluation of* $f(A_1, A_2, ..., A_n)$. $f(A_1, A_2, ..., A_n)$ *is a fuzzy set such that*

$$f(A_1, A_2, ..., A_n)(y) = \begin{cases} \sup\{\min\{A_1(x_1), A_2(x_2), \ldots, A_n(x_n)\} \mid x \in f^{-1}(y)\} \\ \hfill if \quad f^{-1}(y) \neq \emptyset, \\ 0 \hfill otherwise, \end{cases}$$

where $x = (x_1, x_2, ...x_n)$.

Example 1.2.1 *Let* $f : X \times X \to X$ *be defined as*

$$f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2.$$

Suppose A_1 *and* A_2 *are fuzzy subsets of* X. *Then using the extension principle we get*

$$f(A_1, A_2)(y) = \sup_{\lambda_1 x_1 + \lambda_2 x_2 = y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = \lambda_1 A_1 + \lambda_2 A_2$.

Definition 1.2.3 *Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be a function from $E^1(X)$ to $E^1(Y)$. Then f is called a fuzzy function (or mapping) and we use the notation*

$$f : E^1(X) \rightarrow E^1(Y).$$

Let $A = (a_1, a_2, \alpha_1, \alpha_2)_{LR}$ and $B = (b_1, b_2, \beta_1, \beta_2)_{LR}$ be fuzzy numbers of LR-type. Using the (sup-min) extension principle we can verify the following rules for addition and subtraction of fuzzy numbers of LR-type:

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_{LR}$$

$$A - B = (a_1 - b_1, a_2 - b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_{LR}$$

furthermore, if $\lambda \in R$ is a real number then λA can be represented as

$$\lambda A = \begin{cases} (\lambda a_1, \lambda a_2, \lambda\alpha_1, \lambda\alpha_2)_{LR} & if \quad \lambda \geq 0, \\ (\lambda a_2, \lambda a_1, \mid \lambda \mid \alpha_2, \mid \lambda \mid \alpha_1)_{LR} & if \quad \lambda < 0. \end{cases}$$

In particular if $A = (a_1, a_2, \alpha_1, \alpha_2)$ and $B = (b_1, b_2, \beta_1, \beta_2)$ are fuzzy numbers of trapezoidal form, then

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$A - B = (a_1 - b_1, a_2 - b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2).$$

If $A = (a, \alpha_1, \alpha_2)$ and $B = (b, \beta_1, \beta_2)$ are fuzzy numbers of triangular form, then

$$A + B = (a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$A - B = (a - b, \alpha_1 + \beta_2, \alpha_2 + \beta_1),$$

and if $A = (a, \alpha)$ and $B = (b, \beta)$ are fuzzy numbers of symmetrical triangular form, then

$$A + B = (a + b, \alpha + \beta)$$

$$A - B = (a - b, \alpha + \beta),$$

$$\lambda A = (\lambda a, \mid \lambda \mid \alpha).$$

The above results can be generalized to linear combination of fuzzy numbers.

Let A and B be fuzzy numbers with $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$. Then it can easily be shown that

$$[A + B]^\alpha = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)],$$

$$[-A]^\alpha = [-a_2(\alpha), -a_1(\alpha)],$$

$$[A - B]^\alpha = [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)],$$

$$[\lambda A]^\alpha = [\lambda a_1(\alpha), \lambda a_2(\alpha)] \quad if \quad \lambda \geq 0,$$

$$[\lambda A]^\alpha = [\lambda a_2(\alpha), \lambda a_1(\alpha)] \quad if \quad \lambda < 0,$$

for all $\alpha \in [0, 1]$, i.e. any α-level set of the extended sum of two fuzzy numbers is equal to the sum of their α-level sets. The following two theorems show that this property is valid for any continuous function.

Theorem 1.2.1 *[7] Let $f : X \to X$ be a continuous function and let A be fuzzy number. Then*

$$[f(A)]^\alpha = f([A]^\alpha)$$

where $f(A)$ is defined by the extension principle and

$$f([A]^\alpha) = \{f(x) \mid x \in [A]^\alpha\}.$$

If $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and f is continuous and monotone increasing then from the above theorem we get

$$[f(A)]^\alpha = f([A]^\alpha) = f([a_1(\alpha), a_2(\alpha)]) = [f(a_1(\alpha)), f(a_2(\alpha))].$$

Theorem 1.2.2 *[7] Let $f : X \times X \to X$ be a continuous function and let A and B be fuzzy numbers. Then*

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha)$$

where

$$f([A]^\alpha, [B]^\alpha) = \{f(x_1, x_2) \mid x_1 \in [A]^\alpha, x_2 \in [B]^\alpha\}.$$

Let $f(x, y) = xy$ and let $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$, $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ be two fuzzy numbers. Applying above theorem we get

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) = [A]^\alpha[B]^\alpha$$

The equation

$$[AB]^\alpha = [A]^\alpha[B]^\alpha = [a_1(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)]$$

holds if and only if A and B are both nonnegative, i.e. $A(x) = B(x) = 0$ for $x \le 0$. If B is nonnegative then we have

$$[A]^\alpha[B]^\alpha = [\min\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha)\}, \max\{a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}].$$

In general case we obtain a very complicated expression for the α-level sets of the product AB

$$[A]^{\alpha}[B]^{\alpha} = [\min\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\},$$

$$\max\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}].$$

1.3 Metrics for fuzzy numbers

Let A and B are fuzzy numbers with $[A]^{\alpha} = [a_1(\alpha), a_2(\alpha)]$ and $[B]^{\alpha} = [b_1(\alpha), b_2(\alpha)]$. In this section we metricize the set of fuzzy numbers by following metric.

Definition 1.3.1 *(Hausdorff distance) The Hausdorff distance between two (nonempty) sets $X,\ Y \subseteq R$ is given as*

$$d_H(X, Y) = \max\{\beta(X, Y), \beta(Y, X)\},$$

where $\beta(X, Y) = \sup_{x\in X} \rho(x, Y)$ and $\rho(x, Y) = \inf_{y\in Y} \mid x - y \mid$. The generalization

$$d_H(A, B) \quad = \quad \sup_{\alpha\in(0,1]} d_H([A]^{\alpha}, [B]^{\alpha}) \quad \forall\ A, B \in E^1,$$

defines a distance measure. It is clear that

$$d_H(A, B) = \sup_{\alpha\in[0,1]} \max\{\mid a_1(\alpha) - b_1(\alpha) \mid, \mid a_2(\alpha) - b_2(\alpha) \mid\},$$

i.e. $d_H(A, B)$ is the maximal distance between α-level sets of A and B.

Chapter 2

Direct method for solving system of fuzzy linear equations

In this chapter LU Decomposition method, for solving system of fuzzy linear equations is considered. We consider the method in special case when the coefficient matrix is symmetric positive definite. The method in detail is discussed and followed by convergence theorem and illustrated by solving some numerical examples.

2.1 Introduction

System of linear equations play major role in science such as mathematics, physics, statistics, engineering and so on. Usually in many applications at least some of the system's parameters are represented by fuzzy rather than crisp numbers, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them. A general model for solving an $n \times n$ SFLE (system of fuzzy linear equation) whose coefficients matrix is crisp and the right hand side column is an arbitrary fuzzy number vector were first proposed by Friedman et al.[29] and they studied duality fuzzy linear systems in [28]. In this chapter, by according to [29], we solve SFLE with direct method. We start by briefly of results of fuzzy numbers and definitions.

We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements, [43],

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$,
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, 0 \leq r \leq 1$. Using the extension principle, [4], the addition and the scalar multiplication of fuzzy numbers are defined by

$$(u + v)(x) = \sup_{x=s+t} min\{u(s), v(t)\},$$

$$(ku)(x) = u(xk);\ \ k \neq 0,$$

for $u, v \in E^1,\ k \in R$. Equivalently, for arbitrary $u = (\underline{u}, \overline{u}),\ v = (\underline{v}, \overline{v})$ and $k \in R$, we may define the addition and the scaler multiplication as

$$(\underline{u + v})(r) = \underline{u}(r) + \underline{v}(r), (\overline{u + v})(r) = \overline{u}(r) + \overline{v}(r),$$

$$(\underline{ku})(r) = k\underline{u}(r), \quad (\overline{ku})(r) = k\overline{u}(r), \quad k \geq 0,$$

$$(\underline{ku})(r) = k\overline{u}(r), \quad (\overline{ku})(r) = k\underline{u}(r), \quad k \leq 0.$$

Definition 2.1.1 *The $n \times n$ linear system of equations*

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = y_1, \\ &a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = y_2, \\ &\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = y_n, \end{aligned} \tag{2.1}$$

where the coefficients matrix $A = (a_{ij}), 1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and $y_i \in E^1, 1 \leq i \leq n$ is called a system of fuzzy linear equations(SFLE).

Definition 2.1.2 *A fuzzy number vector $(x_1, x_2, \ldots, x_n)^T$ given by $x_i = (\underline{x_i}(r), \overline{x_i}(r)), 1 \leq i \leq n, 0 \leq r \leq 1$, is called a solution of (2.1) if*

$$\underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y_i}, \tag{2.2}$$

$$\overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y_i}. \tag{2.3}$$

From (2.2) and (2.3) we have a $2n \times 2n$ crisp linear system as follows:

$$SX = Y, \tag{2.4}$$

or

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{X} \\ \overline{X} \end{pmatrix} = \begin{pmatrix} \underline{Y} \\ \overline{Y} \end{pmatrix},$$

where $X = (\underline{x}_1, \ldots \underline{x}_n, \overline{x}_1, \ldots \overline{x}_n)^T$ and $Y = (\underline{y}_1, \ldots \underline{y}_n, \overline{y}_1, \ldots \overline{y}_n)^T$ and s_{ij} are determined as:

$$a_{ij} \geq 0 \implies s_{ij} = s_{i+n,j+n} = a_{ij},$$

$$a_{ij} \leq 0 \implies s_{i+n,j} = s_{i,j+n} = a_{ij},$$

and any s_{ij} which is not determined is zero such that:

$$A = S_1 + S_2.$$

Theorem 2.1.1 *The matrix S is nonsingular if and only if the matrix $A = S_1 + S_2$ and $S_1 - S_2$ are both nonsingular, [29].*

Definition 2.1.3 *Let $X = \{(\underline{x}_i(r), \overline{x}_i(r)), 1 \le i \le n\}$ denotes the unique solution of $SX = Y$, if $\underline{y}_i(r), \overline{y}_i(r)$ are linear functions of r, then the fuzzy number vector $U = \{(\underline{u}_i(r), \overline{u}_i(r)), 1 \le i \le n\}$ defined by*

$$\underline{u}_i(r) = min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\},$$

$$\overline{u}_i(r) = max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\},$$

is called the fuzzy solution of $SX = Y$. If $(\underline{x}_i(r), \overline{x}_i(r)), 1 \le i \le n$, are all fuzzy numbers then $\underline{u}_i(r) = \underline{x}_i(r), \overline{u}_i(r) = \overline{x}_i(r)$, and then U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

Theorem 2.1.2 *Let S be nonsingular, the unique solution of (2.4) is always a fuzzy number vector for arbitrary vector Y, if and only if S^{-1} is nonnegative, [29].*

2.2 LU decomposition method for solving fuzzy system of linear equations

Theorem 2.2.1 *An arbitrary matrix is positive definite if and only if all its eigenvalues are positive, [2].*

Theorem 2.2.2 *If $S_1 + S_2$ and $S_1 - S_2$ are symmetric positive definite then S is symmetric positive definite.*

Proof. *Obviously S is symmetric. Let λ be the eigenvalue of*

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix},$$

and X is the corresponding eigenvector. Hence

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \lambda \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

and then

$$\begin{cases} S_1X_1 + S_2X_2 = \lambda X_1, \\ S_2X_1 + S_1X_2 = \lambda X_2. \end{cases}$$

By addition and subtraction of upper relations we have

$$\begin{cases} (S_1 + S_2)(X_1 + X_2) = \lambda(X_1 + X_2), \\ (S_1 - S_2)(X_1 - X_2) = \lambda(X_1 - X_2). \end{cases}$$

The above relations show that λ is the eigenvalue of $S_1 + S_2$ and $S_1 - S_2$, which concludes the proof.

Theorem 2.2.3 *If $A = S_1 + S_2$ is a tridiagonal symmetric positive definite matrix then $S_1 - S_2$ is symmetric positive definite.*

Proof. *Let*

$$S_1 + S_2 = \begin{pmatrix} g_1 & f_2 & & & \\ f_2 & g_2 & f_3 & & \\ & \ddots & \ddots & \ddots & \\ & & f_{n-1} & g_{n-1} & f_n \\ & & & f_n & g_n \end{pmatrix},$$

where $g_1, g_2, \ldots, g_n$ are positive. Now we have

$$\det(A - \lambda I) = (g_n - \lambda)P_{n-1}(\lambda) - f_n^2 P_{n-2}(\lambda) = \det(S_1 - S_2 - \lambda I),$$

where $P_k(\lambda)$ is the characteristic polynomial of $k \times k$ submatrix of A, which concludes the proof.

Theorem 2.2.4 *Let A be an $n \times n$ matrix with all nonzero leading principal minors. Then A has a unique factorization:*

$$A = LU,$$

where L is unit lower triangular and U is upper triangular, [2].

In order to decomposition of S, we must find matrices L and U such that $S = LU$, which

$$L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}.$$

L_{11} and L_{22} is lower triangular matrix, U_{11} and U_{22} is upper triangular matrix.

Now we suppose that $A = S_1 + S_2$ has LU decomposition. We have

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix},$$

then

$$S_1 = L_{11}U_{11}, \tag{2.5}$$

$$S_2 = L_{11}U_{12} \Rightarrow U_{12} = L_{11}^{-1}S_2,$$

$$S_2 = L_{21}U_{11} \Rightarrow L_{21} = S_2U_{11}^{-1},$$

$$S_1 = L_{21}U_{12} + L_{22}U_{22}.$$

Now we can write

$$S_1 - S_2S_1^{-1}S_2 = L_{22}U_{22}. \tag{2.6}$$

From (2.5) and (2.6) if S_1 and $S_1 - S_2S_1^{-1}S_2$ both have LU decomposition, then S has LU decomposition.

Theorem 2.2.5 *Let A be an $n \times n$ symmetric positive definite matrix then there exists a unique lower triangular matrix L with positive diagonal entries such that $A = LL^T$, [6].*

So if S be a symmetric positive definite matrix we have

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{pmatrix},$$

then

$$S_1 = L_{11}L_{11}^T, \tag{2.7}$$

$$S_2 = L_{11}L_{21}^T \Rightarrow L_{21}^T = L_{11}^{-1}S_2,$$

$$S_2 = L_{21}L_{11}^T \Rightarrow L_{21} = S_2(L_{11}^T)^{-1},$$

$$S_1 = L_{21}U_{12} + L_{22}U_{22},$$

and hence

$$S_1 - S_2 S_1^{-1} S_2 = L_{22} L_{22}^T. \tag{2.8}$$

For using this method the matrices S_1 and $S_1 - S_2 S_1^{-1} S_2$ should be symmetric positive definite.

2.3 Numerical Examples

Example 2.3.1 *Consider the* 2×2 *fuzzy system*

$$x_1 - x_2 = (-7 + 2r, -3 - 2r),$$

$$x_1 + 3x_2 = (19 + 4r, 27 - 4r).$$

The extended 4×4 matrix is

$$S = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix},$$

and

$$S_1 = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},$$

$$S_1 - S_2 S_1^{-1} S_2 = \begin{pmatrix} 1 & 0.3333 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0.3333 \\ 0 & 2.6666 \end{pmatrix},$$

and

$$S = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -0.3333 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0.3333 \\ 0 & 0 & 0 & 2.6666 \end{pmatrix}.$$

Now the exact solution is

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (1+r, 3-r),$$
$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (6+r, 8-r).$$

The exact and obtained solutions with *LU* decomposition are plotted and compared in Fig. 2.1.

Example 2.3.2 *Consider the* 3×3 *fuzzy system*

$$2x_1 + x_2 + 3x_3 = (11+8r, 27-8r),$$

$$4x_1 + x_2 - x_3 = (-23+10r, -5-8r),$$

$$-x_1 + 3x_2 + x_3 = (10+5r, 27-12r).$$

The extended 6×6 matrix is

$$S = \begin{pmatrix} 2 & 1 & 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & -1 \\ 0 & 3 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & -1 & 4 & 1 & 0 \\ -1 & 0 & 0 & 0 & 3 & 1 \end{pmatrix},$$

and

$$S_1 = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & -17 \end{pmatrix},$$

$$S_1 - S_2 S_1^{-1} S_2 = \begin{pmatrix} 2.0000 & 1.0000 & 3.0000 \\ 4.0000 & 1.0000 & -0.3592 \\ -0.3592 & 3.0000 & 1.0000 \end{pmatrix},$$

$$= \begin{pmatrix} 1.0000 & 0 & 0 \\ 0.5000 & 1.0000 & 0 \\ -0.1764 & -6.3592 & 1.0000 \end{pmatrix} \begin{pmatrix} 2.0000 & 1.0000 & 3.0000 \\ 0 & -0.5000 & -1.6529 \\ 0 & 0 & -10.2422 \end{pmatrix},$$

and hence $S = LU$ that

$$L = \begin{pmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 2.0000 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & -3.0000 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & -0.3529 & 0.5000 & 1.0000 & 0 \\ 0 & 0 & 0 & -0.1764 & -6.3529 & 1.0000 \end{pmatrix},$$

$$U = \begin{pmatrix} 2.0000 & 1.0000 & 3.0000 & 0 & 0 & 0 \\ 0 & -1.0000 & -6.0000 & 0 & 0 & 0 \\ 0 & 0 & -17.0000 & 6 & 0 & 0 \\ 0 & 0 & 0 & 2.0000 & 1.0000 & 3.0000 \\ 0 & 0 & 0 & 0 & -0.5000 & -1.8529 \\ 0 & 0 & 0 & 0 & 0 & -10.2422 \end{pmatrix}.$$

Now the exact solution is

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (-4 + 2r, -1 - r),$$
$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (1 + r, 5 - 3r),$$
$$x_3 = (\underline{x}_3(r), \overline{x}_3(r)) = (6 + r, 8 - r).$$

The exact and obtained solutions with LU decomposition are plotted and compared in Fig. 2.2.

Example 2.3.3 *Consider the* 3×3 *fuzzy system*

$$4x_1 + 2x_2 - x_3 = (-27 + 7r, -7 - 13r),$$

$$2x_1 + 7x_2 + 6x_3 = (1 + 15r, 40 - 24r),$$

$$-x_1 + 6x_2 + 10x_3 = (26 + 18r, 47 - 33r).$$

The extended 6×6 matrix is

$$S = \begin{pmatrix} 4 & 2 & 0 & 0 & 0 & -1 \\ 2 & 7 & 6 & 0 & 0 & 0 \\ 0 & 6 & 10 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 2 & 7 & 6 \\ -1 & 0 & 0 & 0 & 6 & 10 \end{pmatrix},$$

and

$$S_1 = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 7 & 6 \\ 0 & 6 & 10 \end{pmatrix} = \begin{pmatrix} 2.0000 & 0 & 0 \\ 1.0000 & 2.4495 & 0 \\ 0 & 2.4495 & 2.0000 \end{pmatrix} \begin{pmatrix} 2.0000 & 1.0000 & 0 \\ 0 & 2.4495 & 2.4495 \\ 0 & 0 & 2.0000 \end{pmatrix},$$

$$S_1 - S_2 S_1^{-1} S_2 = \begin{pmatrix} 3.7500 & 2.0000 & -0.1250 \\ 2.0000 & 7.0000 & 6.0000 \\ -0.1250 & 6.0000 & 9.6458 \end{pmatrix}$$

$$= \begin{pmatrix} 1.9365 & 0 & 0 \\ 1.0328 & 2.4358 & 0 \\ -0.0645 & 2.4906 & 1.8544 \end{pmatrix} \begin{pmatrix} 1.9365 & 1.0328 & -0.0645 \\ 0 & 2.4358 & 2.4906 \\ 0 & 0 & 1.8544 \end{pmatrix},$$

and hence $S = LL^T$ that

$$L = \begin{pmatrix} 2.0000 & 0 & 0 & 0 & 0 & 0 \\ 1.0000 & 2.4495 & 0 & 0 & 0 & 0 \\ 0 & 2.4495 & 2.0000 & 0 & 0 & 0 \\ 0 & 0 & -0.5000 & 1.9365 & 0 & 0 \\ 0 & 0 & 0 & 1.0328 & 2.4358 & 0 \\ -0.5000 & 0.2041 & -0.2500 & -0.0645 & 2.4956 & 1.8544 \end{pmatrix}.$$

Now the exact solution is

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (-5 + r, -2 - 2r),$$
$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (-1 + r, 2 - 2r),$$
$$x_3 = (\underline{x}_3(r), \overline{x}_3(r)) = (3 + r, 5 - r).$$

The exact and obtained solutions with LL^T decomposition are plotted and compared in Fig. 2.3.

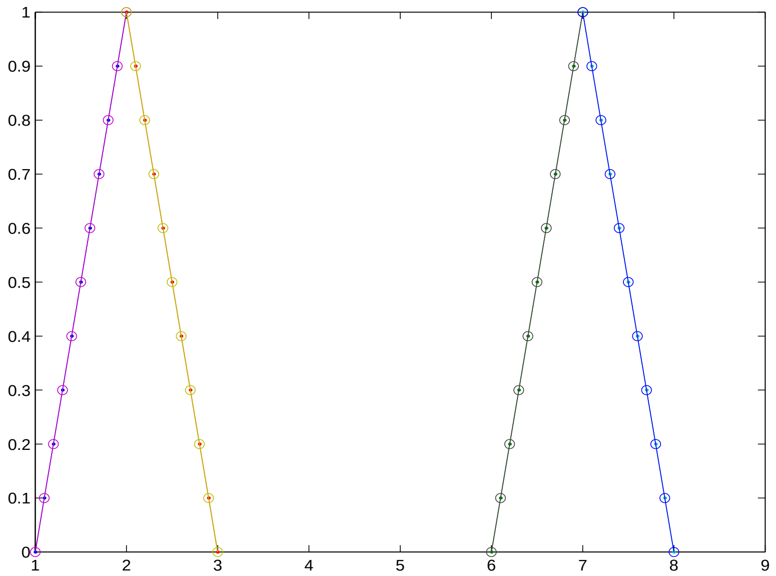

Figure 2.1. The Hausdrff distance of solutions is 1.7764e-015.

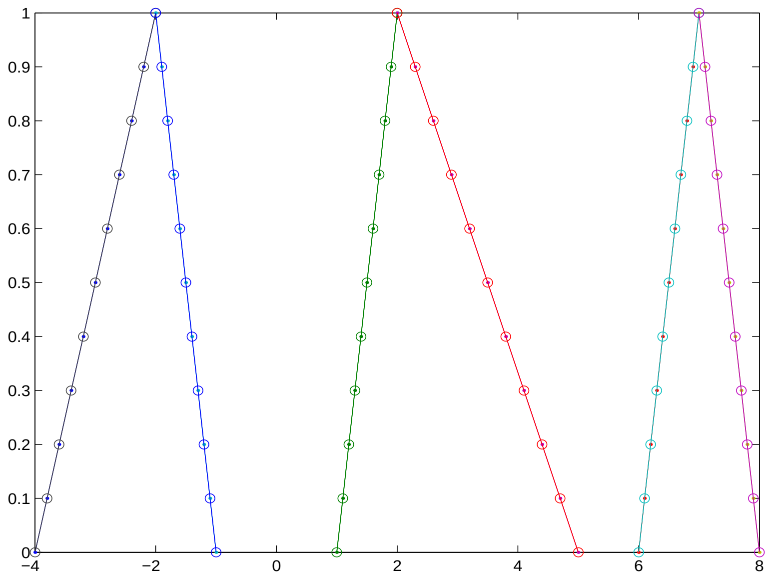

Figure 2.2. The Hausdrff distance of solutions is 3.5527e-015.

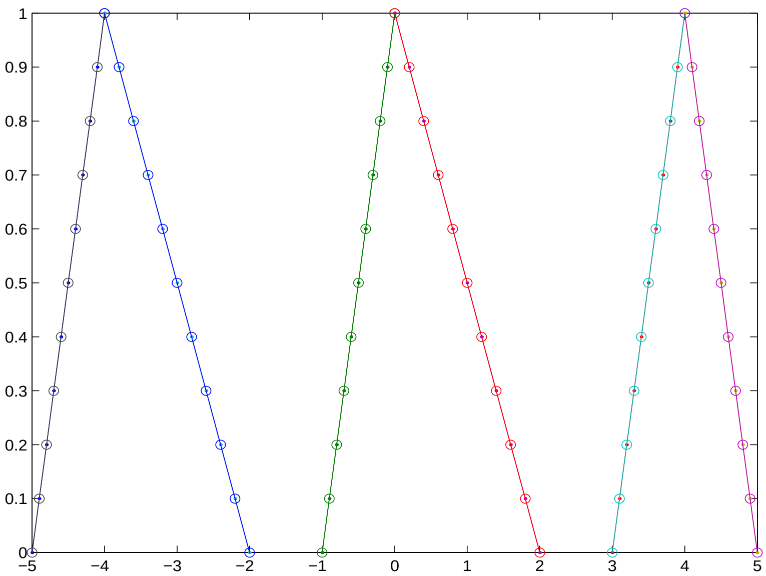

Figure 2.3. The Hausdrff distance of solutions is 2.6645e-015.

Because of limited storage capacity a real number may or may not be represented exactly on a computer. Thus when using computer to find solutions we have to deal with approximations of the real numbers, so we have Floating-Point Errors in our computations.

Chapter 3

Iterative methods for solving fuzzy nonlinear equations

In this chapter,our main aim is developing numerical methods to solving a system of fuzzy nonlinear equations and fuzzy nonlinear equations. For this purpose we consider Homotopy and Newton's method. The fuzzy quantities are presented in parametric form. Some numerical illustrations are given to show the efficiency of algorithms.

3.1 Introduction

In the second chapter we solved system of fuzzy linear equation by direct method. In this chapter we are interested in finding numerical solutions to fuzzy nonlinear equations by Homotopy and Newton's methods and a system of fuzzy nonlinear equations by Newton's method. One of the major applications of fuzzy number arithmetic is nonlinear systems whose parameters are all or partially represented by fuzzy numbers [45, 11, 24].

Standard analytical techniques like Buckley and Qu method, [20, 21, 22, 23], can not suitable for solving the systems such as

$$\begin{cases} a_1x^5 + b_1x^4 + a_2y^3 + b_2y^4 + cx^3 + dx - e = f, \\ \\ x - \cos(y) = g, \\ \end{cases}$$

and equations such as

$$(i) \quad a_1x^5 + b_1x^4 + cx^3 + dx - e = f, \qquad (ii) \quad x - \sin(x) = g,$$

where $x, y, a_1, b_1, a_2, b_2, c, d, e, f$ and g are fuzzy numbers. According to this fact that systems of simultaneous nonlinear system play a major role in various areas such as mathematics, statistics, engineering and social sciences, therefore we need to develop the numerical methods to find the roots of such equations. I remind that the authors in [37] are applied Newton's method for solving fuzzy nonlinear equation first time.

3.2 Newton's method for solving a system of fuzzy nonlinear equations

Now our aim is to obtain a solution for fuzzy nonlinear system

$$\begin{cases} F(x, y) = c, \\ G(x, y) = d, \end{cases} \tag{3.1}$$

where x, y, c and d are fuzzy numbers. The parametric form is as follows:

$$\begin{cases} F_1(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r) = \underline{c}(r), \\ F_2(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r) = \overline{c}(r), \\ & \forall r \in [0, 1] \\ G_1(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r) = \overline{d}(r), \\ G_2(\underline{x}, \overline{x}, \underline{y}, \overline{y}, r) = \overline{d}(r). \end{cases} \tag{3.2}$$

Definition 3.2.1 *The solution of (3.2)* $\forall r \in [0,1]$, *is called analytical solution of (3.1).*

Suppose that $x = (\underline{\alpha}, \overline{\alpha})$and $y = (\underline{\beta}, \overline{\beta})$ are the solutions to above the system, i.e.

$$\begin{cases} F_1(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, r) = \underline{c}(r), \\ F_2(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, r) = \overline{c}(r), \\ \qquad\qquad\qquad\qquad \forall r \in [0,1] \\ G_1(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, r) = \underline{d}(r), \\ G_2(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, r) = \overline{d}(r). \end{cases}$$

Therefore, if $x_0 = (\underline{x}_0, \overline{x}_0)$ and $y_0 = (\underline{y}_0, \overline{y}_0)$ are approximation solutions for this system, then $\forall r \in [0,1]$, there are $h_i(r), k_i(r)$ $i = 1,2$, such that

$$\begin{cases} \underline{\alpha}(r) = \underline{x}_0(r) + h_1(r), \\ \overline{\alpha}(r) = \overline{x}_0(r) + k_1(r), \\ \underline{\beta}(r) = \underline{y}_0(r) + h_2(r), \\ \overline{\beta}(r) = \overline{y}_0(r) + k_2(r). \end{cases}$$

Now if we use the Taylor series of F_1, F_2, G_1, G_2 about $(\underline{x}_0, \overline{x}_0, \underline{y}_0, \overline{y}_0)$, then $\forall r \in [0,1]$,

$$\begin{cases} F_1(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, r) = F_1(\Delta) + h_1 F_{1_{\underline{x}}}(\Delta) + k_1 F_{1_{\overline{x}}}(\Delta) + \\ h_2 F_{1_{\underline{y}}}(\Delta) + k_2 F_{1_{\overline{y}}}(\Delta) + O(\Gamma) = \underline{c}(r), \\ \\ F_2(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, r) = F_2(\Delta) + h_1 F_{2_{\underline{x}}}(\Delta) + k_1 F_{2_{\overline{x}}}(\Delta) + h_2 F_{2_{\underline{y}}}(\Delta) + \\ k_2 F_{2_{\overline{y}}}(\Delta) + O(\Gamma) = \overline{c}(r), \\ \\ G_1(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, r) = G_1(\Delta) + h_1 G_{1_{\underline{x}}}(\Delta) + k_1 G_{1_{\overline{x}}}(\Delta) + h_2 G_{1_{\underline{y}}}(\Delta) + \\ k_2 G_{1_{\overline{y}}}(\Delta) + O(\Gamma) = \underline{d}(r), \\ \\ G_2(\underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta}, r) = G_2(\Delta) + h_1 G_{2_{\underline{x}}}(\Delta) + k_1 G_{2_{\overline{x}}}(\Delta) + h_2 G_{2_{\underline{y}}}(\Delta) + \\ k_2 G_{2_{\overline{y}}}(\Delta) + O(\Gamma) = \overline{d}(r), \end{cases}$$

where $\Delta = (\underline{x}_0, \overline{x}_0, \underline{y}_0, \overline{y}_0, r)$, $\Gamma = h_1^2 + h_2^2 + h_1 k_1 + h_2 k_2 + h_1 k_2 + h_2 k_1 + k_1^2 + k_2^2$ and if $\underline{x}_0$, $\overline{x}_0$, $\underline{y}_0$ and $\overline{y}_0$ are near to $\underline{\alpha}$, $\overline{\alpha}$, $\underline{\beta}$ and $\overline{\beta}$, respectively, then $h_i(r)$ and $k_i(r)$;

$i = 1, 2$ are small. We assume, of course, that all needed partial derivatives exist and bounded. Therefore for enough small $h_i(r)$ and $k_i(r)$; $i = 1, 2$, we have $\forall r \in [0, 1]$,

$$\begin{cases} F_1(\Delta) + h_1 F_{1_{\underline{x}}}(\Delta) + k_1 F_{1_{\overline{x}}}(\Delta) + h_2 F_{1_{\underline{y}}}(\Delta) + k_2 F_{1_{\overline{y}}}(\Delta) \simeq \underline{c}(r), \\ F_2(\Delta) + h_1 F_{2_{\underline{x}}}(\Delta) + k_1 F_{2_{\overline{x}}}(\Delta) + h_2 F_{2_{\underline{y}}}(\Delta) + k_2 F_{2_{\overline{y}}}(\Delta) \simeq \overline{c}(r), \\ G_1(\Delta) + h_1 G_{1_{\underline{x}}}(\Delta) + k_1 G_{1_{\overline{x}}}(\Delta) + h_2 G_{1_{\underline{y}}}(\Delta) + k_2 G_{1_{\overline{y}}}(\Delta) \simeq \underline{d}(r), \\ G_2(\Delta) + h_1 G_{2_{\underline{x}}}(\Delta) + k_1 G_{2_{\overline{x}}}(\Delta) + h_2 G_{2_{\underline{y}}}(\Delta) + k_2 G_{2_{\overline{y}}}(\Delta) \simeq \overline{d}_2(r), \end{cases}$$

and hence $h_i(r)$ and $k_i(r)$; $i = 1, 2$ are unknown quantities which can be obtained by solving the following equations, $\forall r \in [0, 1]$,

$$\mathrm{J}(\Delta) \begin{bmatrix} h_1(r) \\ k_1(r) \\ h_2(r) \\ k_2(r) \end{bmatrix} = \begin{bmatrix} \underline{c}(r) - F_1(\Delta) \\ \overline{c}(r) - F_2(\Delta) \\ \underline{d}(r) - G_1(\Delta) \\ \overline{d}(r) - G_2(\Delta) \end{bmatrix},$$

where

$$\mathrm{J}(\Delta) = \begin{bmatrix} F_{1_{\underline{x}}} & F_{1_{\overline{x}}} & F_{1_{\underline{y}}} & F_{1_{\overline{y}}} \\ F_{2_{\underline{x}}} & F_{2_{\overline{x}}} & F_{2_{\underline{y}}} & F_{2_{\overline{y}}} \\ G_{1_{\underline{x}}} & G_{1_{\overline{x}}} & G_{1_{\underline{y}}} & G_{1_{\overline{y}}} \\ G_{2_{\underline{x}}} & G_{2_{\overline{x}}} & G_{2_{\underline{y}}} & G_{2_{\overline{y}}} \end{bmatrix}_{(\Delta)}.$$

Hence, the next approximations for $\underline{x}(r)$, $\overline{x}(r)$, $\underline{y}(r)$ and $\overline{y}(r)$ are as follows

$$\begin{cases} \underline{x}_1(r) = \underline{x}_0(r) + h_1(r), \\ \overline{x}_1(r) = \overline{x}_0(r) + k_1(r), \\ \underline{y}_1(r) = \underline{y}_0(r) + h_2(r), \\ \overline{y}_1(r) = \overline{y}_0(r) + k_2(r), \end{cases}$$

for all $r \in [0, 1]$.

We can obtain approximated solution, $\forall r \in [0, 1]$, by using the recursive scheme

$$\begin{cases} \underline{x}_n(r) = \underline{x}_{n-1}(r) + h_{1,n-1}(r), \\ \overline{x}_n(r) = \overline{x}_{n-1}(r) + k_{1,n-1}(r), \\ \underline{y}_n(r) = \underline{y}_{n-1}(r) + h_{2,n-1}(r), \\ \overline{y}_n(r) = \overline{x}_{n-1}(r) + k_{2,n-1}(r), \end{cases}$$

where $n = 1, 2, \dots$, $h_{i,0}(r) = h_i(r)$ and $k_{i,0}(r) = k_i(r)$; $i = 1, 2$. For initial guess, one can use the fuzzy number

$$\begin{cases} x_0 = (\underline{x}(1), \overline{x}(1), \underline{x}(1) - \underline{x}(0), \overline{x}(0) - \overline{x}(1)), \\ y_0 = (\underline{y}(1), \overline{y}(1), \underline{y}(1) - \underline{y}(0), \overline{y}(0) - \overline{y}(1)), \end{cases}$$

and in parametric form

$$\underline{x}_0(r) = \underline{x}(1) + (\underline{x}(1) - \underline{x}(0))(r-1), \quad \overline{x}_0(r) = \overline{x}(1) + (\overline{x}(0) - \overline{x}(1))(1-r),$$

$$\underline{y}_0(r) = \underline{y}(1) + (\underline{y}(1) - \underline{y}(0))(r-1), \quad \overline{y}_0(r) = \overline{y}(1) + (\overline{y}(0) - \overline{y}(1))(1-r).$$

Remark 3.2.1 *Sequence* $\{(\underline{x}_n, \overline{x}_n)\}_{n=0}^{\infty}$ *and* $\{(\underline{y}_n, \overline{y}_n)\}_{n=0}^{\infty}$ *convergent to* $(\underline{\alpha}, \overline{\alpha})$ *and* $(\underline{\beta}, \overline{\beta})$*, respectively, iff for all* $r \in [0, 1]$

$$\lim_{n\to\infty} \underline{x}_n(r) = \underline{\alpha}(r), \qquad \lim_{n\to\infty} \overline{x}_n(r) = \overline{\alpha}(r),$$

$$\lim_{n\to\infty} \underline{y}_n(r) = \underline{\beta}(r) \text{ and } \lim_{n\to\infty} \overline{y}_n(r) = \overline{\beta}(r).$$

Lemma 3.2.1 *Let*

$$\begin{cases} F(\underline{\alpha}, \overline{\alpha}) = (\underline{c}, \overline{c}), \\ G(\underline{\beta}, \overline{\beta}) = (\underline{d}, \overline{d}), \end{cases}$$

and if the sequence of $\{(\underline{x}_n, \overline{x}_n)\}_{n=0}^{\infty}$ *and* $\{(\underline{y}_n, \overline{y}_n)\}_{n=0}^{\infty}$ *convergent to* $(\underline{\alpha}, \overline{\alpha})$ *and* $(\underline{\beta}, \overline{\beta})$*, respectively, according to Newton's method, then*

$$\lim_{n\to\infty} P_n = 0,$$

where

$$P_n = \sup_{0 \le r \le 1} \max\{h_{1,n}(r), k_{1,n}(r), h_{2,n}(r), k_{2,n}(r)\}.$$

Proof. *It is obviously, because for all $r \in [0,1]$ in convergent case*

$$\lim_{n\to\infty} h_{i,n}(r) = \lim_{n\to\infty} k_{i,n}(r) = 0; i = 1, 2.$$

The following numerical examples show the efficiency of this algorithm.

Example 3.2.1 *Consider system of fuzzy nonlinear equations*

$$\begin{cases} x^2 + y^2 = (5, 5, 0.6, 2), \\ x^2 + y^3 + (2, 2, 1, 1) = (11, 11, 2.4, 6.2). \end{cases}$$

Without any loss of generality, assume that x and y are positive, then the parametric form of this system is as follows

$$\begin{cases} \underline{x}^2(r) + \underline{y}^2(r) = (4.4 + 0.6r), \\ \overline{x}^2(r) + \overline{y}^2(r) = (7 - 0.2r), \\ \underline{x}^2(r) + \underline{y}^3(r) + (1 + r) = (8.6 + 2.4r), \\ \overline{x}^2(r) + \overline{y}^3(r) + (3 - r) = (17.2 - 6.2r). \end{cases}$$

To obtain initial guess we use above system for $r = 0$ and $r = 1$, therefore

$$\begin{cases} \underline{x}^2(0) & + & \underline{y}^2(0) & = & 4.4, \\ \overline{x}^2(0) & + & \overline{y}^2(0) & = & 7, \\ \underline{x}^2(0) & + & \underline{y}^3(0) & = & 7.6, \\ \overline{x}^2(0) & + & \overline{y}^3(0) & = & 14.2, \end{cases} \quad and \quad \begin{cases} \underline{x}^2(1) & + & \underline{y}^2(1) & = & 5, \\ \overline{x}^2(1) & + & \overline{y}^2(1) & = & 5, \\ \underline{x}^2(1) & + & \underline{y}^3(1) & = & 9, \\ \overline{x}^2(1) & + & \overline{y}^3(1) & = & 9. \end{cases}$$

Consequently $\underline{x}(0) = 0.9036$, $\overline{x}(0) = 1.2567$, $\underline{y}(0) = 1.893$, $\overline{y}(0) = 2.32824$, $\underline{x}(1) = \overline{x}(1) = 1$ *and* $\underline{x}(1) = \overline{x}(1) = 2$. *Therefore initial guess is* $x_0 = (1, 1, 0.09036, 0.2567)$ *and* $y_0 = (2, 2, 0.107, 0.32824)$. *After 2 iterations, we obtain the solutions of* x *and* y *which the maximum error would be about* 10^{-2} *and* 9×10^{-3}, *respectively. For more details see Figs. 3.1 and 3.2.*

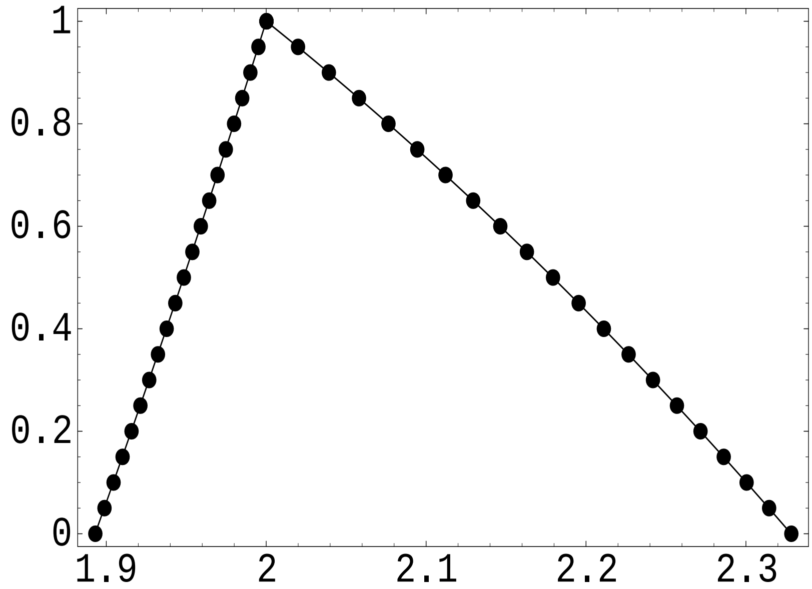

Fig. 3.1. The exact (solid line) and approximate solution for x

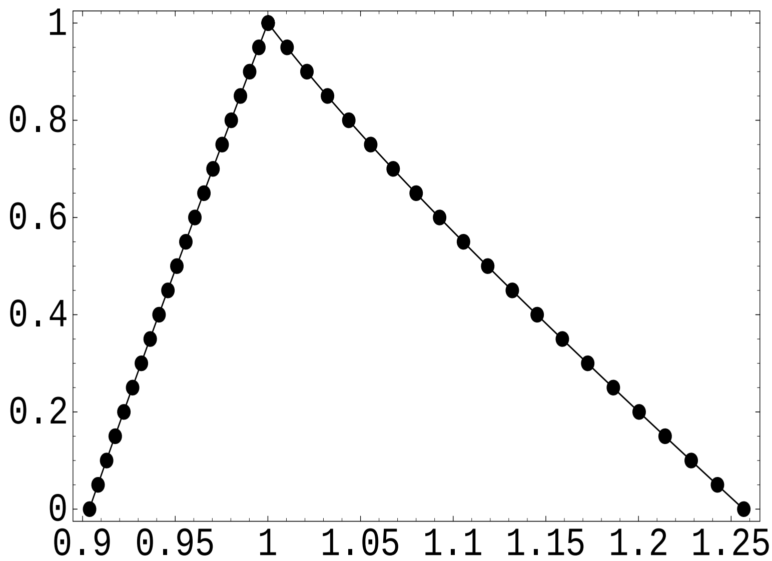

Fig. 3.2. The exact (solid line) and approximate solution for y

Example 3.2.2 *Consider system of fuzzy nonlinear equations*

$$\begin{cases} x^2 + y^2 = (13, 13, 1, 4.6) \\ x^2 - \frac{1}{4}y^2 = (1.75, 1.75, 1.15, 1.75) \end{cases}$$

Without any loss of generality, assume that x and y are positive, then parametric form of this system is as follows

$$\begin{cases} \underline{x}^2(r) & + & \underline{y}^2(r) & = & (12 + r), \\ \overline{x}^2(r) & + & \overline{y}^2(r) & = & (17.6 - 4.6r), \\ \underline{x}^2(r) & - & \frac{1}{4}\overline{y}^2(r) & = & (0.6 + 1.15r), \\ \overline{x}^2(r) & - & \frac{1}{4}\underline{y}^2(r) & = & (3.5 - 1.75r). \end{cases}$$

By solving the above system for $r = 0$ and $r = 1$, we obtain the initial guess $x_0 = (2, 2, 0.10263, 0.3664)$ and $y_0 = (3, 3, 0.10172, 0.4641)$. If we apply two iterations from Newton's method, the maximum error would be less than 10^{-2}, Figs. 3.3 and 3.4.

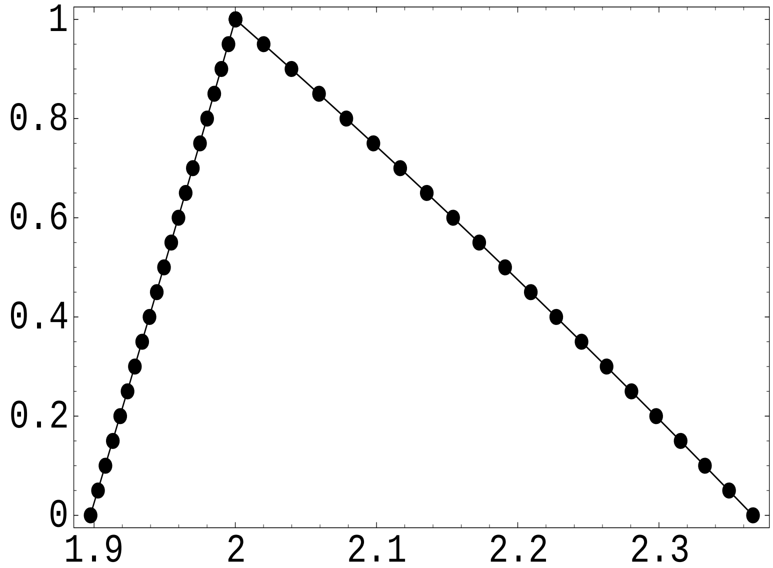

Fig. 3.3.The exact (solid line) and approximate solution for x

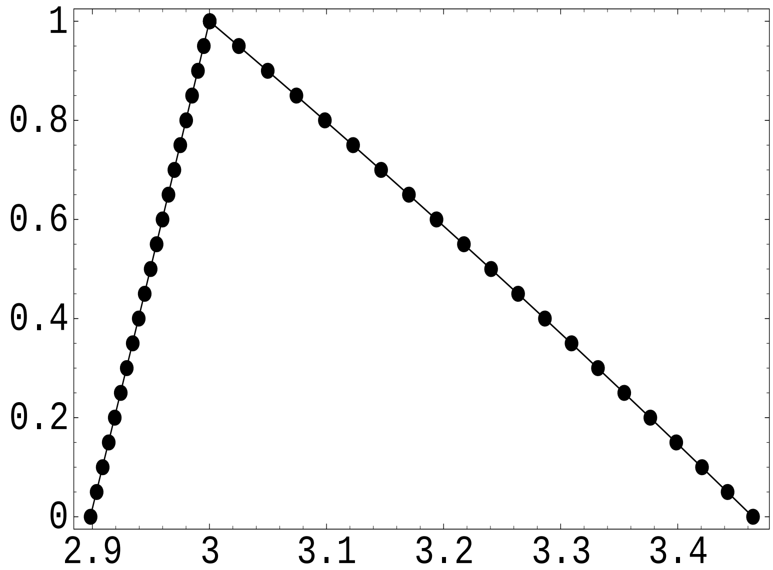

Fig. 3.4. The exact (solid line) and approximate solution for y

3.3 Newton's method for solving quadratic fuzzy polynomials

Since quadratic fuzzy polynomials have many applications in economics and physics, therefore individuals such as Buckley and Qu, [17, 23], have considered solving these polynomials and have given analytical techniques. In this section we give Newton's method for numerical solution of quadratic fuzzy polynomials as

$$Ex^2 + Fx + G = x, \tag{3.3}$$

for $x \in E^1$. I remind that, usually, there is no inverse element for an arbitrary fuzzy number $x \in E^1$, i.e. there exists no element $y \in E^1$ such that $x + y = 0$. Actually, for all non-crisp fuzzy number $x \in E^1$ we have $x + (-x) \neq 0$. Therefore the fuzzy polynomial (3.3) cannot be equivalently replaced by the fuzzy polynomial

$$Ex^2 + (F-1)x + G = 0.$$

Before giving this method, we need some definitions and results.

Definition 3.3.1 *For each fuzzy number $x \in E^1$, we define the functions*

$$x_L : [0,1] \to R, \; x_R : [0,1] \to R$$

given by $x_L(\alpha) = x_{\alpha l}$ and $x_R(\alpha) = x_{\alpha r}$, for each $\alpha \in [0,1]$. It is clear that if $(\underline{x}(\alpha), \overline{x}(\alpha))$ be parametric form of $x \in E^1$, then $x_L(\alpha) = \underline{x}(\alpha)$ and $x_R(\alpha) = \overline{x}(\alpha)$.

We consider a partial ordering $\leq$ in E^1 for all $\alpha \in (0,1]$, given by

$$x, y \in E^1, \; x \leq y \;\Leftrightarrow\; (x_{\alpha l} \leq y_{\alpha l} \; and \; x_{\alpha r} \leq y_{\alpha r}, \; \forall \alpha \in [0,1]).$$

Using a fixed point theorem in complete lattices for non-decreasing mapping, the authors proved in [**?**] the following existence results for (3.3).

Theorem 3.3.1 *([**?**], Theorem 12) Let E, F, G be fuzzy numbers such that*

$$E, F, G \geq \chi_{\{0\}},$$

and suppose that exists $p > 0$ such that

$$E_L(1)p^2 + F_L(1)p + G_L(1) \leq p,$$

and

$$E_R(0)p^2 + F_R(0)p + G_R(0) \leq p.$$

Then (3.3) has extremal solutions in the interval

$$[\chi_{\{0\}}, \chi_{\{p\}}] := \{x \in E^1 \quad | \quad \chi_{\{0\}} \leq x \leq \chi_{\{p\}}\}.$$

Now our aim is to obtain a solution for quadratic fuzzy polynomials (3.3), i.e.

$$Ex^2 + Fx + G = x,$$

where E, F, G, x are non-negative fuzzy numbers. The parametric form for all $\alpha \in [0, 1]$ is as follows:

$$\begin{cases} \underline{E}(\alpha)\underline{x}^2(\alpha) + \underline{F}(\alpha)\underline{x}(\alpha) + \underline{G}(\alpha) = \underline{x}(\alpha), \\ \overline{E}(\alpha)\overline{x}^2(\alpha) + \overline{F}(\alpha)\overline{x}(\alpha) + \overline{G}(\alpha) = \overline{x}(\alpha). \end{cases} \tag{3.4}$$

The solutions of (3.4) for all $\alpha \in [0, 1]$, gives a solution of (3.3). Let $x = (\underline{z}, \overline{z})$ be the solution of (3.4), i.e., for all $\alpha \in [0,1]$ we have to solve

$$\begin{cases} \underline{E}(\alpha)\underline{z}^2(\alpha) + (\underline{F}(\alpha) - 1)\underline{z}(\alpha) + \underline{G}(\alpha) = 0, \\ \overline{E}(\alpha)\overline{z}^2(\alpha) + (\overline{F}(\alpha) - 1)\overline{z}(\alpha) + \overline{G}(\alpha) = 0. \end{cases} \tag{3.5}$$

Obviously, these equations may have no real solutions. Now define the functions $\underline{H}$ and $\overline{H}$ are defined for all $\alpha \in [0,1]$ as follows:

$$\begin{cases} \underline{H}(\underline{z},\alpha) = \underline{E}(\alpha)\underline{z}^2(\alpha) + (\underline{F}(\alpha) - 1)\underline{z}(\alpha), \\ \overline{H}(\overline{z},\alpha) = \overline{E}(\alpha)\overline{z}^2(\alpha) + (\overline{F}(\alpha) - 1)\overline{z}(\alpha). \end{cases}$$

Therefore, if $x_0 = (\underline{x}_0, \overline{x}_0)$ is an approximate solutions for system (3.5)

$$\begin{cases} \underline{z}(\alpha) = \underline{x}_0(\alpha) + h_1(\alpha), \\ \overline{z}(\alpha) = \overline{x}_0(\alpha) + k_1(\alpha), \end{cases}$$

where $h_1(\alpha)$ and $k_1(\alpha)$ give us the error. Now if we use the Taylor series of $\underline{H}, \overline{H}$ about $(\underline{x}_0, \overline{x}_0)$, we have

$$\begin{cases} \underline{H}(\underline{z},\alpha) = \underline{H}(\underline{x}_0,\alpha) + h_1(\alpha)\frac{\partial \underline{H}}{\partial \underline{x}}(\underline{x}_0,\alpha) + O(h_1^2(\alpha)) = -\underline{G}(\alpha), \\ \overline{H}(\overline{z},\alpha) = \overline{H}(\overline{x}_0,\alpha) + k_1(\alpha)\frac{\partial \overline{H}}{\partial \overline{x}}(\overline{x}_0,\alpha) + O(k_1^2(\alpha)) = -\overline{G}(\alpha). \end{cases}$$

If $\underline{x}_0$ and $\overline{x}_0$ are near to $\underline{z}$ and $\overline{z}$, respectively, then $h_1(\alpha)$ and $k_1(\alpha)$ are small. We assume, of course, that all needed partial derivatives exist and are bounded. Therefore for enough small $h_1(\alpha)$ and $k_1(\alpha)$ we have

$$\begin{cases} \underline{H}(\underline{x}_0,\alpha) + h_1(\alpha)\underline{H}_{\underline{x}}(\underline{x}_0,\alpha) \simeq -\underline{G}(\alpha), \\ \overline{H}(\overline{x}_0,\alpha) + k_1(\alpha)\overline{H}_{\overline{x}}(\overline{x}_0,\alpha) \simeq -\overline{G}(\alpha). \end{cases}$$

Hence $h_1(\alpha)$ and $k_1(\alpha)$ are unknown quantities which can be obtained by solving the following equations:

$$J(\underline{x}_0, \overline{x}_0, \alpha) \begin{bmatrix} h_1(\alpha) \\ k_1(\alpha) \end{bmatrix} = \begin{bmatrix} -\underline{G}(\alpha) - \underline{H}(\underline{x}_0,\alpha) \\ -\overline{G}(\alpha) - \overline{H}(\overline{x}_0,\alpha) \end{bmatrix},$$

where

$$J(\underline{x}_0, \overline{x}_0, \alpha) = \begin{bmatrix} \frac{\partial \underline{H}}{\partial \underline{x}}(\underline{x}_0, \alpha) & 0 \\ 0 & \frac{\partial \overline{H}}{\partial \overline{x}}(\overline{x}_0, \alpha) \end{bmatrix}.$$

To obtain another approximation for $\underline{x}(\alpha)$ and $\overline{x}(\alpha)$ we have

$$\begin{cases} \underline{x}_1(\alpha) = \underline{x}_0(\alpha) + h_1(\alpha), \\ \overline{x}_1(\alpha) = \overline{x}_0(\alpha) + k_1(\alpha), \end{cases}$$

for all $\alpha \in [0, 1]$.

We can obtain approximated solution by using the recursive scheme

$$\begin{cases} \underline{x}_n(\alpha) = \underline{x}_{n-1}(\alpha) + h_{1,n-1}(\alpha), \\ \overline{x}_n(\alpha) = \overline{x}_{n-1}(\alpha) + k_{1,n-1}(\alpha), \end{cases}$$

where $h_{1,0}(\alpha) = h_1(\alpha)$ and $k_{1,0}(\alpha) = k_1(\alpha)$ for $n = 1, 2, \ldots$. For initial guess, one can use Theorem (3.3.1). If

$$\varphi(p) = E_R(0)p^2 + F_R(0)p + G_R(0)$$

and the discriminant is positive, there exist two zeros for φ and, if $F_R(0) \leq 1$, we can take

$$p = \frac{(1 - F_R(0)) \pm \sqrt{(F_R(0) - 1)^2 - 4E_R(0)G_R(0)}}{2E_R(0)} > 0.$$

Let

$$p_1 = \frac{(1 - F_R(0)) - \sqrt{(F_R(0) - 1)^2 - 4E_R(0)G_R(0)}}{2E_R(0)},$$
$$p_2 = \frac{(1 - F_R(0)) + \sqrt{(F_R(0) - 1)^2 - 4E_R(0)G_R(0)}}{2E_R(0)}.$$

It is clear that $\forall q \in [p_1, p_2]$, we have $\varphi(q) \leq 0$. Therefore in Theorem (3.3.1) we can take $p = p_1$. Now for initial guess we can use the triangular fuzzy number x_0 such that $0 \leq \underline{x}_0 \leq \overline{x}_0 \leq p_1$.

Remark 3.3.1 *Sequence* $\{(\underline{x}_n, \overline{x}_n)\}_{n=0}^{\infty}$ *convergent to* $(\underline{z}, \overline{z})$ *if and only if* $\forall \alpha \in [0,1]$, $\lim_{n\to\infty} \underline{x}_n(\alpha) = \underline{z}(\alpha)$ *and* $\lim_{n\to\infty} \overline{x}_n(\alpha) = \overline{z}(\alpha)$.

Here we present two examples to illustrate the Newton's method introduced in this section for fuzzy quadratic polynomials.

Example 3.3.1 *Consider the fuzzy equation*

$$(0,1,2)x^2 + (0,.1,.2)x + (0, \frac{.28}{16}, \frac{.28}{8}) = x.$$

Suppose that x is positive, therefore the parametric form of this equation is as follows

$$\begin{cases} .5\alpha\underline{x}^2(\alpha) + (.1\alpha - 1)\underline{x}(\alpha) + \frac{.28}{16}\alpha = 0, \\ \\ (2-\alpha)\overline{x}^2(\alpha) + (.2 - .1\alpha - 1)\overline{x}(\alpha) + (\frac{.28}{8} - \frac{.28}{16}\alpha) = 0. \end{cases}$$

Since $p_1 = .05$, therefore we choose initial guess as follows:

$$x_0 = (0, .02, .05).$$

After 2 iterations, we obtain an approximate solution of x with an error less than 10^{-2} *. For more details see Fig. 3.5.*

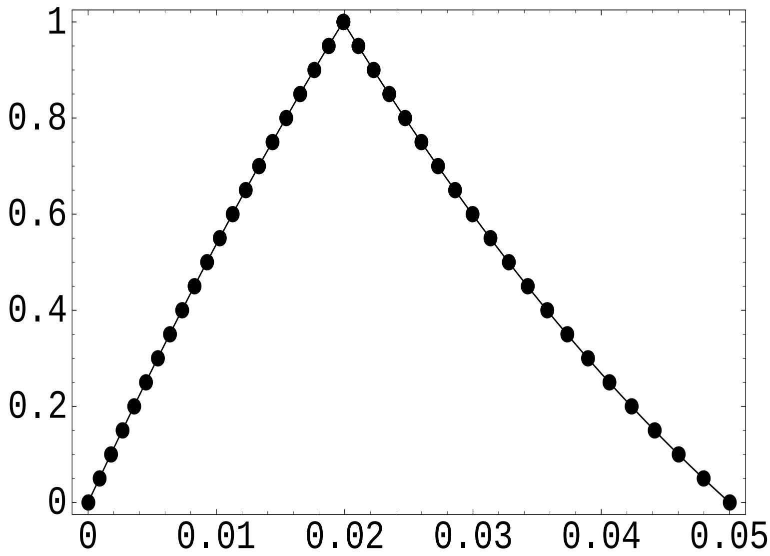

Fig. 3.5. The exact (solid line) and approximate solution of Example 3.2.1

Example 3.3.2 *Consider quadratic fuzzy equation*

$$(0,.03,.09)x^2 + (0,.25,.5)x + (0,\frac{1}{8},\frac{1}{4}) = x,$$

with positive solution. The parametric form of this equation is as follows

$$\begin{cases} .03\alpha\underline{x}^2(\alpha) + (.25-1)\underline{x}(\alpha) + \frac{1}{8}\alpha = 0, \\ \\ (.09-.06\alpha)\overline{x}^2(\alpha) + (.5-.25\alpha-1)\overline{x}(\alpha) + (\frac{1}{4}-\frac{1}{8}\alpha) = 0. \end{cases}$$

Since $p_1 = .56$, therefore we choose initial guess as follows:

$$x_0 = (0,.28,.56).$$

If we apply two iterations from Newton's method, the maximum error is less than 10^{-2}. See Figure 3.6.

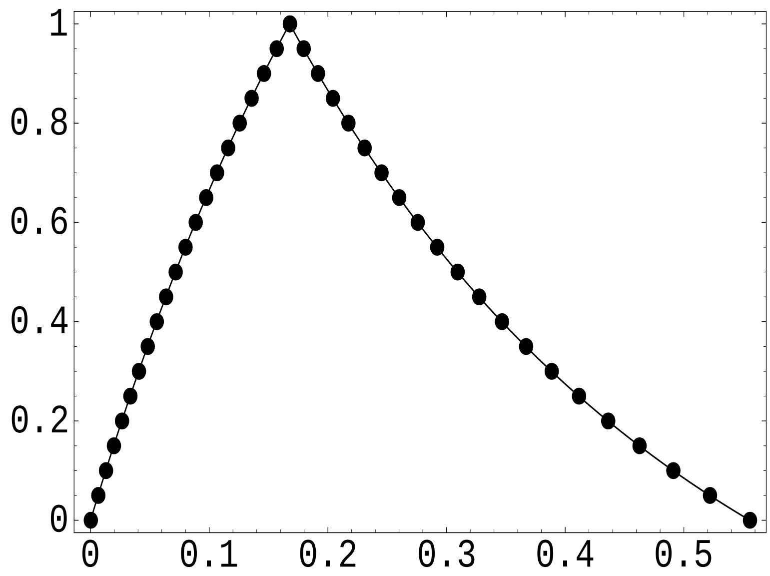

Fig. 3.6. The exact (solid line) and approximate solution of Example 3.2.2

3.4 Homotopy or continuation method

Now our aim is to obtain a solution for fuzzy nonlinear equation $F(x) = 0$. The parametric form is as follows:

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}, r) = 0, \\ & \forall r \in [0,1], \\ \overline{F}(\underline{x}, \overline{x}, r) = 0. \end{cases} \tag{3.6}$$

In homotopy method (3.6) is embedded in a one-parameter family of problems using a parameter $\lambda \in [0, 1]$. The original problem (3.6) corresponds to $\lambda = 1$ and a problem with a known solution corresponds to $\lambda = 0$. For example for $x_0 \in E$, the set of problems

$$G(\lambda, x) = \lambda F(x) + (1 - \lambda)[F(x) - F(x_0)] = 0, \quad 0 \le \lambda \le 1,$$

or in parametric form $\forall r \in [0,1]$

$$\begin{cases} \underline{G}(\lambda, \underline{x}, \overline{x}, r) = \underline{F}(\underline{x}, \overline{x}, r) + (\lambda - 1)\underline{F}(\underline{x}_0, \overline{x}_0, r) = 0, \\ \overline{G}(\lambda, \underline{x}, \overline{x}, r) = \overline{F}(\underline{x}, \overline{x}, r) + (\lambda - 1)\overline{F}(\underline{x}_0, \overline{x}_0, r) = 0, \end{cases} \tag{3.7}$$

where $x_0 = (\underline{x_0}, \overline{x_0})$ is an initial approximation of (3.6). It is obvious that

$$G(0,x) = F(x) - F(x_0) = 0, \qquad G(1,x) = F(x) = 0,$$

the changing process of λ from zero to unity is just that of $G(\lambda, x)$ from $F(x) - F(x_0)$ to $F(x)$. In topology, this called deformation, $F(x) - F(x_0)$ and $F(x)$ are called homotopic.

Homotopy or continuation method attempts to determine $x^* = (\underline{x}_1, \overline{x}_1)$ (for $\lambda = 1$) by solving the sequence of problems according to $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_m = 1$. The initial approximation to the solution of

$$\begin{cases} \underline{G}(\lambda_i, \underline{x}, \overline{x}, r) = \underline{F}(\underline{x}, \overline{x}, r) + (\lambda_i - 1)\underline{F}(\underline{x}_0, \overline{x}_0, r) = 0, \\ \overline{G}(\lambda_i, \underline{x}, \overline{x}, r) = \overline{F}(\underline{x}, \overline{x}, r) + (\lambda_i - 1)\overline{F}(\underline{x}_0, \overline{x}_0, r) = 0, \end{cases} \tag{3.8}$$

would be the solution $x_{\lambda_{i-1}} = (\underline{x_{\lambda_{i-1}}}, \overline{x_{\lambda_{i-1}}})$ to the problem

$$\begin{cases} \underline{G}(\lambda_{i-1}, \underline{x}, \overline{x}, r) = \underline{F}(\underline{x}, \overline{x}, r) + (\lambda_{i-1} - 1)\underline{F}(\underline{x}_0, \overline{x}_0, r) = 0, \\ \overline{G}(\lambda_{i-1}, \underline{x}, \overline{x}, r) = \overline{F}(\underline{x}, \overline{x}, r) + (\lambda_{i-1} - 1)\overline{F}(\underline{x}_0, \overline{x}_0, r) = 0. \end{cases} \tag{3.9}$$

In this work $\forall r \in [0,1]$ and a fixed $\lambda \in [0,1]$, we use Newton's method for solving (3.8) and (3.9). Newton's method to (3.8) generates a sequence

$$x_{\lambda_i}^{(k)} = x_{\lambda_i}^{(k-1)} - J(x_{\lambda_i}^{(k-1)})^{-1} G(\lambda_i, x_{\lambda_i}^{(k-1)}), \tag{3.10}$$

which converges rapidly to a solution x_{λ_i} if $x_{\lambda_i}^{(0)} (= x_{\lambda_{i-1}})$ is sufficiently close to x_{λ_i}, where

$$J(x_\lambda) = \begin{bmatrix} \underline{G}_{\underline{x}}(\lambda, \underline{x}_\lambda, \overline{x}_\lambda, r) & \underline{G}_{\overline{x}}(\lambda, \underline{x}_\lambda, \overline{x}_\lambda, r) \\ \\ \overline{G}_{\underline{x}}(\lambda, \underline{x_\lambda}, \overline{x_\lambda}, r) & \overline{G}_{\overline{x}}(\lambda, \underline{x_\lambda}, \overline{x_\lambda}, r) \end{bmatrix},$$

and $\|J(x_\lambda)^{-1}\| \leq M$ for a constant M, [**?**].

Computing (3.10) is performed in a two step manner. First, a vector y is found that will satisfy

$$J(x_{\lambda_i}^{(k-1)})y = -G(\lambda_i, x_{\lambda_i}^{(k-1)}),$$

second, the new approximation is obtained by adding y to $x_{\lambda_i}^{(k)}$.

For initial guess, we can use the fuzzy number

$$x_0 = (\underline{x}(1), \overline{x}(1), \underline{x}(1) - \underline{x}(0), \overline{x}(0) - \overline{x}(1)),$$

and in parametric form

$$\underline{x}_0(r) = \underline{x}(1) + (\underline{x}(1) - \underline{x}(0))(r-1),$$

$$\overline{x}_0(r) = \overline{x}(1) + (\overline{x}(0) - \overline{x}(1))(1-r)$$

where $(\underline{x}(0), \overline{x}(0))$ and $(\underline{x}(1), \overline{x}(1))$ are the solutions

$$\begin{cases} \underline{F}(\underline{x(r)}, \overline{x}(r)) = 0, \\ & r = 0, 1. \\ \overline{F}(\underline{x(r)}, \overline{x}(r)) = 0. \end{cases}$$

Here we consider two examples to illustrating the homotopy method for fuzzy nonlinear equations from Buckley and Qu [23].

Example 3.4.1 *Consider the fuzzy nonlinear equation*

$$(3,4,5)x^2 + (1,2,3)x = (1,2,3).$$

Without any loss of generality, assume that x is positive, then the parametric form of this equation is as follows

$$\begin{cases} (3+r)\underline{x}^2(r) & + & (1+r)\underline{x}(r) & = & (1+r), \\ (5-r)\overline{x}^2(r) & + & (3-r)\overline{x}(r) & = & (3-r). \end{cases}$$

To obtain initial guess we use above system for $r = 0$ and $r = 1$, therefore

$$\begin{cases} 4\underline{x}^2(1) & + & 2\underline{x}(1) & = & 2, \\ 4\overline{x}^2(1) & + & 2\overline{x}(1) & = & 2, \end{cases} \qquad \begin{cases} 3\underline{x}^2(0) & + & \underline{x}(0) & = & 1, \\ 5\overline{x}^2(0) & + & 3\overline{x}(0) & = & 3. \end{cases}$$

Consequently $\underline{x}(0) = 0.4343$, $\overline{x}(0) = 0.5307$ and $\underline{x}(1) = \overline{x}(1) = \frac{1}{2}$. Therefore initial guess is $x_0 = (0.4343, 0.5, 0.5307)$ and hence $x_0 = (\underline{x_0}, \overline{x_0}) = (0.435 + 0.065r, 0.531 - 0.031r)$. The Jacobian matrix is

$$\begin{bmatrix} 2(3+r)\underline{x_\lambda}(r) + (1+r) & 0 \\ 0 & 2(5-r)\overline{x_\lambda}(r) + (3-r) \end{bmatrix}.$$

By $\lambda_i = \lambda_{i-1} + 0.25$ for $i = 1, 2, 3, 4$, we obtain the solution which the maximum error would be less than 10^{-3}, Figures 3.7 and 3.8. Now suppose x is negative, we have

$$\begin{cases} (3+r)\overline{x}^2(r) & + & (3-r)\underline{x}(r) & = & (1+r), \\ (5-r)\underline{x}^2(r) & + & (1+r)\overline{x}(r) & = & (3-r). \end{cases}$$

For $r = 0$, we have, $\underline{x}(0) \simeq -0.629$ and $\overline{x}(0) \simeq -0.98$, hence $\underline{x}(0) > \overline{x}(0)$, therefore negative root does not exist.

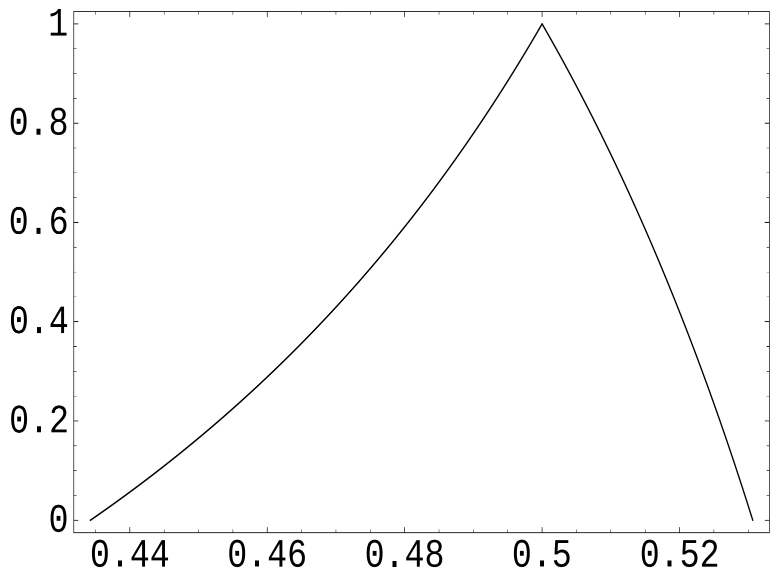

Figure 3.7. Standard Analytical Solution

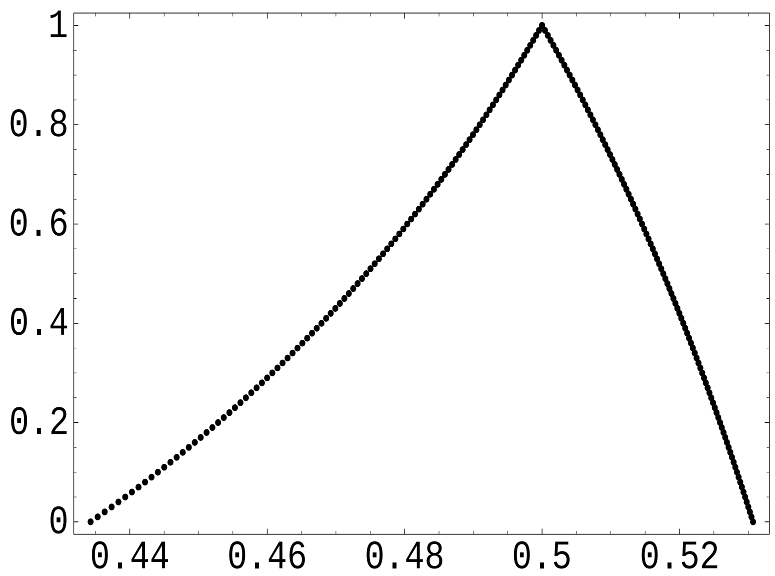

Figure 3.8. Solution of Homotopy Method

Example 3.4.2 *Consider fuzzy nonlinear equation*

$$(1,2,3)x^3 + (2,3,4)x^2 + (3,4,5) = (5,8,13).$$

Without any loss of generality, assume that x is positive, then parametric form of this equation is as follows

$$\begin{cases} (1+r)\underline{x}^3(r) + (2+r)\underline{x}^2(r) + (3+r) = (5+3r), \\ (3-r)\overline{x}^3(r) + (4-r)\overline{x}^2(r) + (5-r) = (13-5r). \end{cases}$$

By solving the above system for $r = 0$ and $r = 1$, we obtain the initial guess $x_0 = (0.76, 0.91, 1.06)$ and hence $x_0 = (\underline{x_0}, \overline{x_0}) = (0.76 + 0.15r, 1.06 - 0.15r)$. The Jacobian matrix is

$$\begin{bmatrix} 3(1+r)\underline{x_\lambda}^2(r) + 2(2+r)\underline{x_\lambda}(r) & 0 \\ 0 & 3(3-r)\overline{x_\lambda}^2(r) + 2(4-r)\overline{x_\lambda}(r) \end{bmatrix}.$$

By $\lambda_i = \lambda_{i-1} + 0.25$ for $i = 1, 2, 3, 4$, we obtain the solution which the maximum error would be less than 10^{-3}, Figures 3.9 and 3.10.

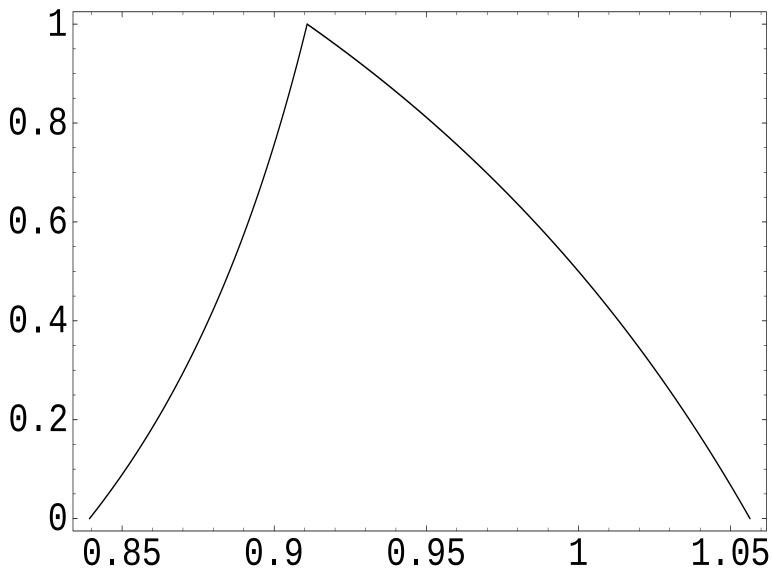

Figure 3.9. Standard Analytical Solution

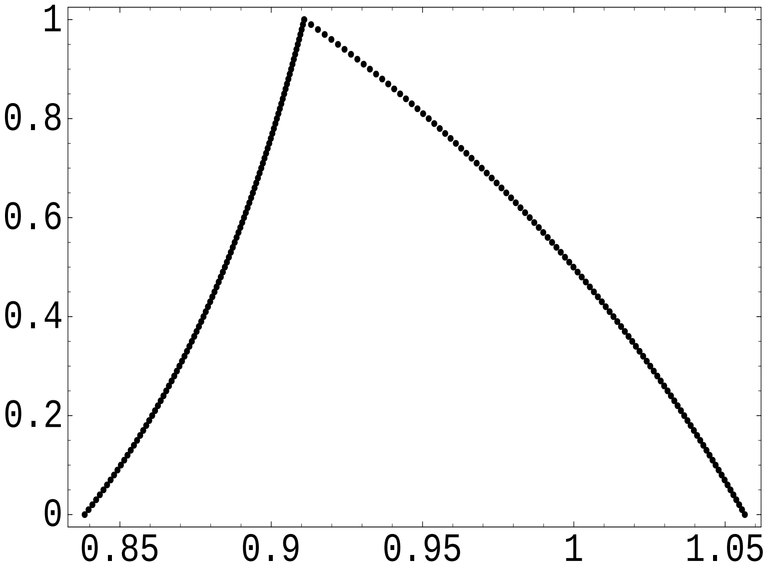

Figure 3.10. Solution of Homotopy Method

Chapter 4

Existence of symmetrical extrem solutions for fuzzy polynomials

In this chapter, we consider the existence of a solution for fuzzy polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = x,$$

where $a_i \succeq 0$, $i = 0, 1, 2, \cdots n$ and $x \succeq 0$ are fuzzy numbers satisfying certain conditions and " $\succeq$ " be the partial ordering. To this purpose, we use fixed point theory, applying results such as the well-known fixed point theorem of Tarski, presenting some results regarding the existence of extremal solutions to the above equation.

4.1 Introduction

In the previous chapter we gave iterative methods for solving fuzzy nonlinear equations and a nonlinear system. Since fuzzy polynomials have many applications on economic and finance, therefore considering of these equations do in this chapter. Some work has been done on solving quadratic fuzzy equation [17, 23] and the authors in these references have given analytical techniques on finding solutions. Recently, in [25], the authors studied the existence of extremal solutions for quadratic fuzzy equation

$$Ex^2 + Fx + G = x,$$

where E, F, G and x are positive fuzzy numbers satisfying certain conditions, and gave the interval which contains extremal solutions ([25], Theorem 12). In this chapter, by notice to results of [25], we consider fuzzy polynomials such as

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = x, \tag{4.1}$$

where x and a_i, $i = 0, 1, ...n$, are positive fuzzy numbers. Since standard analytical techniques like Buckley and Qu method, [20, 21, 22, 23], can not suitable for solving these equations, therefore we must give numerical methods for solving this polynomials. For the reason that in some numerical methods for solving these fuzzy polynomials we need initial guess, therefore give the interval which contains extremal solutions. We start by briefly fundamental results of [25].

Definition 4.1.1 *For a fuzzy number $x \in E^1$, we denote the α-level set*

$$[x]^\alpha = \{t \in R : x(t) \geq \alpha\}$$

by the interval $[x_{\alpha l}, x_{\alpha r}]$, for each $\alpha \in (0, 1]$, and

$$[x]^0 = \overline{\bigcup_{\alpha \in (0,1]} [x]^\alpha} = [x_{0l}, x_{0r}].$$

We consider the partial ordering "$\preceq$" in E^1 for all $\alpha \in (0, 1]$, given by

$$x, y \in E^1,\ x \preceq y \ \Leftrightarrow\ (y_{\alpha l} \leq x_{\alpha l} \text{ and } x_{\alpha r} \leq y_{\alpha r}).$$

By notice that definition (3.3.1) have many usage in this chapter therefore we give this definition again.

Definition 4.1.2 *For each fuzzy number $x \in E^1$, we define the functions*

$$x_L : [0,1] \to R, \ x_R : [0,1] \to R$$

given by $x_L(\alpha) = x_{\alpha l}$ and $x_R(\alpha) = x_{\alpha r}$, for each $\alpha \in [0,1]$.

Theorem 4.1.1 *([25], Theorem 8) Suppose that x and y are fuzzy numbers, then*

$$d_H(x,y) = max\{\|x_L - y_L\|_\infty \ , \ \|x_R - y_R\|_\infty\}.$$

Theorem 4.1.2 *(Tarski's Fixed Point Theorem) Let F be a monotone operator on the complete lattice L into itself. Then the set of fixed points of F is a nonempty complete lattice for the ordering of L.*

4.2 Existence results

Lemma 4.2.1 *([25], Lemma 4) If G, x, y are fuzzy numbers such that $G \succeq \chi_{\{0\}}$ and $\chi_{\{0\}} \preceq x \preceq y$, then $\chi_{\{0\}} \preceq Gx \preceq Gy$.*

Theorem 4.2.2 *Let a_i, $i = 0, 1, 2, ...n$, be fuzzy numbers such that*

$$a_i \succeq \chi_{\{0\}},$$

and suppose that there exist $p > 0$ satisfying

$$-p \leq \sum_{i=1}^{n} \min\{(a_i)_L(0), -(a_i)_R(0)\}p^i + (a_0)_L(0), \tag{4.2}$$

and

$$\sum_{i=1}^{n} \max\{-(a_i)_L(0), (a_i)_R(0)\}p^i + (a_0)_R(0) \leq p. \tag{4.3}$$

Then (4.1) has a solution in

$$[\chi_{\{0\}}, \chi_{[-p,p]}] = \{x \in E^1 \ : \ \chi_{\{0\}} \preceq x \preceq \chi_{[-p,p]}\}.$$

Proof. We define the mapping

$$A : [\chi_{\{0\}}, \chi_{[-p,p]}] \longrightarrow E^1,$$

by $Ax = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$. We show that $A([\chi_{\{0\}}, \chi_{[-p,p]}]) \subseteq [\chi_{\{0\}}, \chi_{[-p,p]}]$. It is easy to prove that

$$A\chi_{\{0\}} = a_n(\chi_{\{0\}})^n + \cdots + a_1\chi_{\{0\}} + a_0 = a_0.$$

and, for every $\alpha \in [0,1]$,

$$[A\chi_{[-p,p]}]^\alpha =$$

$$[(a_n)_L(\alpha), (a_n)_R(\alpha)][-p^n, p^n] + [(a_{n-1})_L(\alpha), (a_{n-1})_R(\alpha)][-p^{n-1}, p^{n-1}] + \cdots +$$

$$[(a_1)_L(\alpha), (a_1)_R(\alpha)][-p, p] + [(a_0)_L(\alpha), (a_0)_R(\alpha)]$$

$$= [\min\{(a_n)_L(\alpha)p^n, -(a_n)_R(\alpha)p^n\}, \max\{-(a_n)_L(\alpha)p^n, (a_n)_R(\alpha)p^n\}] +$$

$$[\min\{(a_{n-1})_L(\alpha)p^{n-1}, -(a_{n-1})_R(\alpha)p^{n-1}\},$$

$$\max\{-(a_{n-1})_L(\alpha)p^{n-1}, (a_{n-1})_R(\alpha)p^{n-1}\}] + \cdots +$$

$$[\min\{(a_1)_L(\alpha)p, -(a_1)_R(\alpha)p\}, \max\{-(a_1)_L(\alpha)p, (a_1)_R(\alpha)p\}] +$$

$$[(a_0)_L(\alpha), (a_0)_R(\alpha)],$$

so that, for $\alpha \in [0,1]$,

$$[A\chi_{[-p,p]}]_L(\alpha) =$$

$$\min\{(a_n)_L(\alpha)p^n, -(a_n)_R(\alpha)p^n\} + \min\{(a_{n-1})_L(\alpha)p^{n-1}, -(a_{n-1})_R(\alpha)p^{n-1}\}$$

$$+ \cdots + \min\{(a_1)_L(\alpha)p, -(a_1)_R(\alpha)p\} + (a_0)_L(\alpha),$$

$$[A\chi_{[-p,p]}]_R(\alpha) =$$

$$\max\{-(a_n)_L(\alpha)p^n, (a_n)_R(\alpha)p^n\} + \max\{-(a_{n-1})_L(\alpha)p^{n-1}, (a_{n-1})_R(\alpha)p^{n-1}\}$$

$$+ \cdots + \max\{-(a_1)_L(\alpha)p, (a_1)_R(\alpha)p\} + (a_0)_R(\alpha).$$

By hypotheses and using the monotonicity properties of a_i , $i = 0, 1, ..., n$, we obtain, for $\alpha \in [0,1]$,

$$-p \leq \sum_{i=1}^{n} \min\{(a_i)_L(0), -(a_i)_R(0)\}p^i + (a_0)_L(0) \leq$$

$$\sum_{i=1}^{n} \min\{(a_i)_L(\alpha), -(a_i)_R(\alpha)\}p^i + (a_0)_L(\alpha) = [A\chi_{[-p,p]}]_L(\alpha)$$

and

$$[A\chi_{[-p,p]}]_L(\alpha) = \sum_{i=1}^{n} \max\{-(a_i)_L(\alpha), (a_i)_R(\alpha)\}p^i + (a_0)_R(\alpha) \leq$$

$$\sum_{i=1}^{n} \max\{-(a_i)_L(0), (a_i)_R(0)\}p^i + (a_0)_R(0) \leq p.$$

This proves that $A\chi_{[-p,p]} \preceq \chi_{[-p,p]}$. Besides, A is a nondecreasing operator. Take $\chi_{\{0\}} \preceq x \preceq y$, if $n = 2k$ then

$$y_L(\alpha) \leq x_L \leq 0 \leq x_R(\alpha) \leq y_R(\alpha) \forall \alpha \in [0,1],$$

and

$$\{0\} \subseteq [x^n]^\alpha = [x_L(\alpha), x_R(\alpha)]^n =$$

$$[\min\{x_L(\alpha)x_R^{n-1}(\alpha), x_L^{n-1}(\alpha)x_R(\alpha)\}, \max\{x_L^n(\alpha), x_R^n(\alpha)\}], \forall\alpha \in [0,1].$$

Analogously for y. Hence, since $y_R(\alpha) \geq 0$ and $x_L(\alpha) \leq 0$,

$$y_L(\alpha)y_R^{n-1}(\alpha) \leq y_L(\alpha)x_R^{n-1}(\alpha) \leq x_L(\alpha)x_R^{n-1}(\alpha),$$

$$y_L^{n-1}(\alpha)y_R(\alpha) \leq y_L^{n-1}(\alpha)x_R(\alpha) \leq x_L^{n-1}(\alpha)x_R(\alpha),$$

and, using that $x_L^n(\alpha) \leq y_L^n(\alpha)$, $x_R^n(\alpha) \leq y_R^n(\alpha)$ we obtain

$$\min\{x_L(\alpha)x_R^{n-1}(\alpha), x_L^{n-1}(\alpha)x_R(\alpha)\} \geq \min\{y_L(\alpha)y_R^{n-1}(\alpha), y_L^{n-1}(\alpha)y_R(\alpha)\}$$

and

$$\max\{x_L^n(\alpha), x_R^n(\alpha)\} \leq \max\{y_L^n(\alpha), y_R^n(\alpha)\}, \forall\alpha \in [0,1],$$

which proves that

$$\{0\} \subseteq [x^n]^\alpha \subseteq [y^n]^\alpha, \forall\alpha \in [0,1],$$

and

$$\chi_{\{0\}} \preceq x^n \preceq y^n,$$

now if $n = 2k+1$, then

$$\{0\} \subseteq [x^n]^\alpha = [x_L\alpha, x_R(\alpha)]^n =$$

$$[\min\{x_L^n(\alpha), x_L(\alpha)x_R^{n-1}(\alpha)\}, \max\{x_L^{n-1}(\alpha)x_R(\alpha), x_R^n(\alpha)\}], \forall\alpha \in [0,1].$$

Analogously for y. Since $y_R(\alpha) \geq 0$ and $x_L(\alpha) \leq 0$, we can write

$$y_L(\alpha)y_R^{n-1}(\alpha) \leq y_L(\alpha)x_R^{n-1}(\alpha) \leq x_L(\alpha)x_R^{n-1}(\alpha),$$

$$y_L^n(\alpha) \leq x_L^n(\alpha),$$

$$x_L^{n-1}(\alpha)x_R(\alpha) \leq y_L^{n-1}(\alpha)x_R(\alpha) \leq y_L^{n-1}(\alpha)y_R(\alpha)$$

and

$$x_R^n(\alpha) \leq y_R^n(\alpha),$$

therefore we obtain

$$\min\{y_L^n(\alpha), y_L(\alpha)y_R^{n-1}(\alpha)\} \leq \min\{x_L^n(\alpha), x_L(\alpha)x_R^{n-1}(\alpha)\}$$

and

$$\max\{x_L^{n-1}(\alpha)x_R(\alpha), x_R^n(\alpha)\} \leq \max\{y_L^{n-1}(\alpha)y_R(\alpha), y_R^n(\alpha)\}, \forall\alpha \in [0,1],$$

which proves that

$$\{0\} \subseteq [x^n]^\alpha \subseteq [y^n]^\alpha, \forall\alpha \in [0,1],$$

and

$$\chi_{\{0\}} \preceq x^n \preceq y^n.$$

Using that $a_i \succeq \chi_{\{0\}}$ for $i = 0, 1, ...n$, and Lemma (4.2.1), we obtain the nondecreasing character of A, $Ax \preceq Ay$,for $\chi_{\{0\}} \preceq x \preceq y$. Tarski's Fixed Point Theorem gives the extremal fixed points for

$$A : [\chi_{\{0\}}, \chi_{[-p,p]}] \longrightarrow [\chi_{\{0\}}, \chi_{[-p,p]}],$$

in the complete lattice $[\chi_{\{0\}}, \chi_{[-p,p]}]$.

Remark 4.2.1 *In the hypotheses of Theorem* (4.2.2), *conditions (4.2) and (4.3) can be written, equivalently, as*

$$\sum_{i=1}^{n} d_H(a_i, \chi_{\{0\}})p^i + d_H(a_0, \chi_{\{0\}}) \leq p.$$

Since for $x \in E^1$, $x \succeq \chi_{\{0\}}$, *we have*

$$x_L(0) \leq x_L(\alpha) \leq 0 \leq x_R(\alpha) \leq x_R(0) \quad \forall \alpha \in [0,1]$$

for all $\alpha \in [0,1]$. *Hence*

$$d_H(x,\chi_{\{0\}}) = \sup_{\alpha\in[0,1]} d_H([x]^\alpha,[\chi_{\{0\}}]^\alpha) =$$

$$\sup_{\alpha\in[0,1]} \max\{\mid x_L(\alpha)\mid, \mid x_R(\alpha)\mid\} =$$

$$\max\{\mid x_L(0)\mid, \mid x_R(0)\mid\} = \max\{-x_L(0), x_R(0)\},$$

and

$$-d_H(x,\chi_{\{0\}}) = \min\{x_L(0), -x_R(0)\}.$$

Now $a_i \succeq \chi_{\{0\}}$, *conditions (4.2) and (4.3) are equivalent to*

$$-p \leq -\sum_{i=1}^{n} d_H(a_i,\chi_{\{0\}})p^i + (a_0)_L(0),$$

$$\sum_{i=1}^{n} d_H(a_i,\chi_{\{0\}})p^i + (a_0)_R(0) \leq p,$$

or also

$$\sum_{i=1}^{n} d_H(a_i,\chi_{\{0\}})p^i \leq p + (a_0)_L(0),$$

$$\sum_{i=1}^{n} d_H(a_i,\chi_{\{0\}})p^i \leq p - (a_0)_R(0),$$

that is,

$$\sum_{i=1}^{n} d_H(a_i,\chi_{\{0\}})p^i \leq \min\{p + (a_0)_L(0), p - (a_0)_R(0)\}$$

$$= p + \min\{(a_0)_L(0), -(a_0)_R(0\} = p - d_H(a_0,\chi_{\{0\}}).$$

Hence, we have obtained the equivalent condition

$$\sum_{i=1}^{n} d_H(a_i, \chi_{\{0\}})p^i + d_H(a_0, \chi_{\{0\}}) \leq p.$$

Lemma 4.2.3 *Let* $i \in \{0, 1, 2, ...n\}$, $j \in \{1, 2, ...n\}$ *and* a_i *fuzzy numbers such that* $a_i \succeq \chi_{\{0\}}$ *and*

1. $d_H(a_0, \chi_{\{0\}}) \leq \frac{n}{n+1}$,

2. $d_H(a_j, \chi_{\{0\}}) \leq \frac{1}{n(n+1)}$.

Then (4.1) has a solution in

$$[\chi_{\{0\}}, \chi_{[-1,1]}] = \{x \in E^1 \ : \ \chi_{\{0\}} \preceq x \preceq \chi_{[-1,1]}\}.$$

Proof. Indeed, by hypothesis we have

$$\sum_{i=1}^{n} d_H(a_i, \chi_{\{0\}}) + d_H(a_0, \chi_{\{0\}}) \leq \sum_{i=1}^{n} \frac{1}{n(n+1)} + \frac{n}{n+1} = 1.$$

Now according to $Theorem$ (4.2.2), (4.1) has extremal solutions in $[\chi_{\{0\}}, \chi_{[-1,1]}]$.

Theorem 4.2.4 *Let* a_0 *and* a_i, $i = 1, ..., n$, *be fuzzy numbers such that*

$$a_i \succeq \chi_{\{0\}},$$

and suppose that there exist b, $c \in E^1$ *with* $c \succ b \succeq \chi_{\{0\}}$ *and*

$$a_n b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 \succeq b,$$

$$a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 \preceq c,$$

Then (4.1) has extremal solutions in the interval

$$[b,c]=\{x\in E^1\ :\ b\preceq x\preceq c\}.$$

Moreover, if $b=c$, b is a solution to (4.1).

Proof. Define

$$A:[b,c]\longrightarrow E^1,$$

by $Ax=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$. We show that $A([b,c])\subseteq[b,c]$. Indeed, by hypotheses

$$Ab=a_nb^n+a_{n-1}b^{n-1}+\cdots+a_1b+a_0\succeq b,$$

$$Ac=a_nc^n+a_{n-1}c^{n-1}+\cdots+a_1c+a_0\preceq c.$$

Then $Ab\preceq Ac$. Moreover, A is nondecreasing operator. Indeed, for $\chi_{\{0\}}\preceq x\preceq y$, we have

$$Ax\preceq Ay.$$

Therefore, $A:[b,c]\longrightarrow E^1$ is nondecreasing and $[b,c]$ is a complete lattice. Tarski's Fixed Point Theorem provides the existence of extremal fixed points for A in $[b,c]$, that is, extremal solutions to (4.1) in the same interval.

Theorem 4.2.5 *Let $a_i\succeq\chi_{\{0\}}$, $i=0,1,,...,n$, are symmetric fuzzy numbers.*

i) (4.1) has symmetric solution as $x\succeq\chi_{\{0\}}$ and

$$\begin{cases} x_L=(a_n)_Lx_R^n+(a_{n-1})_Lx_R^{n-1}+\cdots+(a_1)_Lx_R+(a_0)_L \\ x_R=(a_n)_Rx_R^n+(a_{n-1})_Rx_R^{n-1}+\cdots+(a_1)_Rx_R+(a_0)_R, \end{cases} \tag{4.4}$$

where $[x]^\alpha=[x_L(\alpha),x_R(\alpha)]$ for all $\alpha\in[0,1]$.

ii) If exists $p > 0$ satisfying

$$(a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + (a_1)_R(0)p + (a_0)_R(0) \leq p, \tag{4.5}$$

then (4.1) has extremal solutions in the interval

$$[\chi_{\{0\}}, \chi_{[-p,p]}] = \{x \in E^1 \ : \ \chi_{\{0\}} \preceq x \preceq \chi_{[-p,p]}\}.$$

Proof. Let $\chi_{\{0\}} \preceq x \in E^1$ is symmetric. Therefore $x_R(\alpha) = -x_L(\alpha)$ for every $\alpha \in [0,1]$. we must prove that

$$[a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0]^\alpha = [x]^\alpha \ \ \forall \ \alpha \ \in [0,1].$$

Suppose

$$\sum_{i=1}^{n}[(a_i)_L, (a_i)_R][x_L, x_R]^i + [(a_0)_L, (a_0)_R] = [M, N]. \tag{4.6}$$

Now prove that $[M,N] = [x_L, x_R]$. By hypotheses, we can write (4.6) as follows

$$[M,N] = \sum_{i=1}^{n}[(-a_i)_R, (a_i)_R][-x_R, x_R]^i + [-(a_0)_R, (a_0)_R] =$$

$$\sum_{i=1}^{n}(a_i)_R x_R^i[-1,1][-1,1]^i + [-(a_0)_R, (a_0)_R] =$$

$$\sum_{i=1}^{n}(a_i)_R x_R^i[-1,1][-1,1] + [-(a_0)_R, (a_0)_R] =$$

$$\sum_{i=1}^{n}(a_i)_R x_R^i[-1,1] + [-(a_0)_R, (a_0)_R] =$$

$$\sum_{i=1}^{n}[-(a_i)_R, (a_i)_R]x_R^i + [-(a_0)_R, (a_0)_R] =$$

$$[-\sum_{i=1}^{n}(a_i)_R x_R^i - (a_0)_R, \sum_{i=1}^{n}(a_i)_R x_R^i + (a_0)_R] = [x_L, x_R].$$

For proof (ii), we use $Theorem$ (4.2.2) and hypotheses. Indeed, if

$$-p \leq \sum_{i=1}^{n} \min\{(a_i)_L(0), -(a_i)_R(0)\}p^i + (a_0)_L(0) = \sum_{i=1}^{n}(a_i)_L(0)p^i + (a_0)_L(0) =$$

$$-\sum_{i=1}^{n}(a_i)_R(0)p^i - (a_0)_R(0)$$

and

$$\sum_{i=1}^{n} \max\{-(a_i)_L(0), (a_i)_R(0)\}p^i + (a_0)_R(0) = \sum_{i=1}^{n}(a_i)_R(0)p^i + (a_0)_R(0) \leq p$$

Then

$$\sum_{i=1}^{n}(a_i)_R(0)p^i + (a_0)_R(0) \leq p$$

and hence by using $Theorem$ (4.2.2), (4.1) has extremal solutions in the interval

$$[\chi_{\{0\}}, \chi_{[-p,p]}] = \{x \in E^1 \ : \ \chi_{\{0\}} \preceq x \preceq \chi_{[-p,p]}\}.$$

Remark 4.2.2 *To find an appropriate* $p > 0$ *in* $Theorem$ (4.2.5) , *we can solve inequality*

$$(a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + ((a_1)_R(0) - 1)p + (a_0)_R(0) \leq 0.$$

If $(a_1)_R(0) > 1$*, there is no such value of* p.

Remark 4.2.3 *Suppose, in hypotheses of* $Theorem$ (4.2.5)*, we have*

$$0 \leq (a_n)_R(0) + (a_{n-1})_R(0) + \cdots + (a_1)_R(0) \leq 1$$

and

$$\frac{(a_0)_R(0)}{1 - (a_n)_R(0) - (a_{n-1})_R(0) - \cdots - (a_1)_R(0)} \leq 1.$$

Then we can take $0 \le p \le 1$ *such that*

$$p \ge \frac{(a_0)_R(0)}{1 - (a_n)_R(0) - (a_{n-1})_R(0) - \cdots - (a_1)_R(0)}.$$

In this case,

$$(a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + (a_2)_R(0)p^2 \le$$

$$(a_n)_R(0)p + (a_{n-1})_R(0)p + \cdots + (a_2)_R(0)p$$

and

$$(a_0)_R(0) \le p(1 - (a_n)_R(0) - (a_{n-1})_R(0) - \cdots - (a_1)_R(0)),$$

hence

$$(a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + (a_1)_R(0)p^2 + (a_0)_R(0) \le$$

$$p((a_n)_R(0) + (a_{n-1})_R(0) + \cdots + (a_1)_R(0)) + (a_0)_R(0)) \le p.$$

4.3 Numerical applications

In this section we give examples for applications of $Theorems$ (4.2.2) and (4.2.5)

Example 4.3.1 *Consider the fuzzy equation*

$$(\frac{-1}{6}, 0, \frac{1}{16})x^2 + (\frac{-1}{2}, 0, \frac{1}{2})x + (\frac{-1}{6}, 0, \frac{1}{16}) = x.$$

According to $Theorem$ (4.2.5) *this equation has extremal solutions in the interval*

$$[\chi_{\{0\}}, \chi_{[-1.171,1.171]}] = \{x \in E^1 \ \mid \ \chi_{\{0\}} \preceq x \preceq \chi_{[-1.171,1.171]}\}.$$

Symmetric solution of this fuzzy polynomial is drawed in Fig. 4.1.

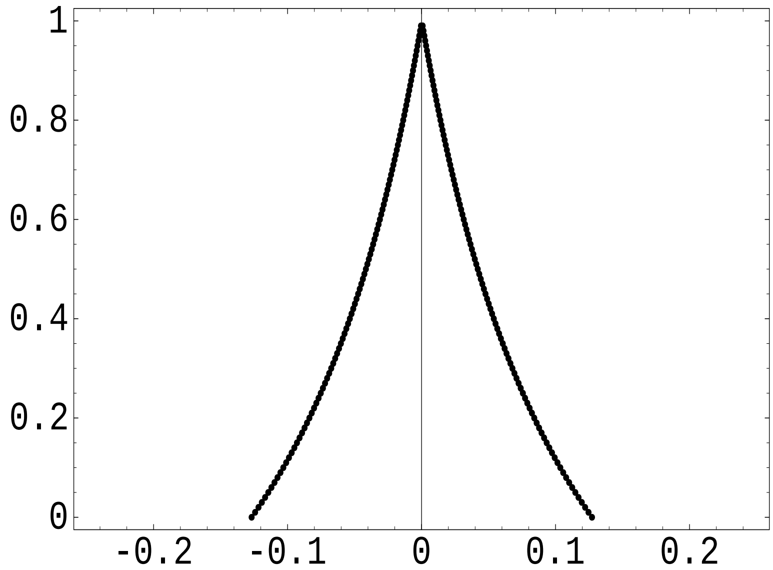

Figure 4.1. Symmetric solution of Example 1

Example 4.3.2 *Consider fuzzy equation*

$$(-3,0,3)x^7 + (-5,0,5)x^6 + (-3,0,3)x^3 + (-1,0,1)x^2 + (-.1,0,.1)x + (-.1,0,.1) = x.$$

It is clear that for all $p \in [0.145, 0.29]$

$$3p^7 + 5p^6 + 3p^3 + p^2 + .1p + .1 \leq p.$$

By using Theorem $(4.2.5)$*, this fuzzy polynomial has extremal solutions in the interval*

$$[\chi_{\{0\}}, \chi_{[-0.145,0.145]}] = \{x \in E^1 \mid \chi_{\{0\}} \preceq x \preceq \chi_{[-0.145,0.145]}\}.$$

For symmetric solution of this equation see Fig. 4.2.

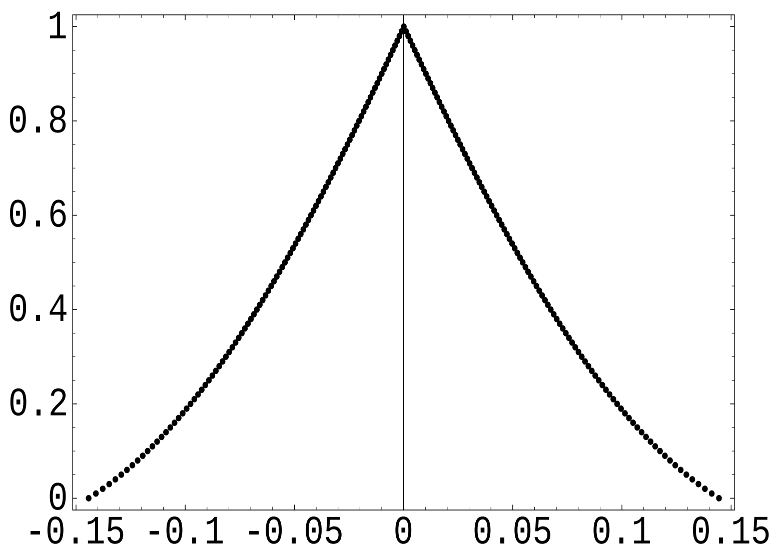

Figure 4.2. Symmetric solution of Example 2

Example 4.3.3 *Consider fuzzy equation*

$$(-1, 0, 1)x^3 + (-3, 0, 2)x^2 + (-.5, 0, .1)x = x.$$

Since for all $p \in [0, .16]$ *we have*

$$-p \leq -p^3 - 3p^2 - .5p$$

and

$$p^3 + 3p^2 + .5p \leq p,$$

hence, according to Theorem (4.2.2), *this equation has extremal solutions in the interval*

$$[\chi_{\{0\}}, \chi_{[-p,p]}] = \{x \in E^1 \ \mid \ \chi_{\{0\}} \preceq x \preceq \chi_{[-p,p]}\}.$$

Open problems

1. In chapter 2 we applied LU Decomposition method for solving $n \times n$ system of fuzzy linear equation as $Ax = b$ whose coefficients matrix, A, is crisp and the right hand side column, b, is an arbitrary fuzzy number vector. Solving $n \times n$ system of fuzzy linear equation as $Ax = b$ when the elements in A and b are fuzzy numbers, by numerical methods, can be a topic for future research.

2. In chapter 3 we found a numerical solution of fuzzy nonlinear equations by Homotopy method and in chapter 4 we considered fuzzy polynomials such as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = x,$$

where x and a_i, $i = 0, 1, \ldots n$, are positive fuzzy numbers and gave interval which contains extremal solutions. We suspect numerical methods as Homotopy and Newton be true for finding numerical solutions of these fuzzy polynomials and will be a topic for future research.

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