SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY

UNIVERSAL APPROXIMATION OF FUZZY FUNCTIONS

Supervisor

Dr. Saeid Abbasbandy

Advisers

Prof. Esmaeil Babolian

By

Majid Amirfakhrian

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY(Ph.D.)
IN APPLIED MATHEMATICS (NUMERICAL ANALYSIS)
SEPTEMBER 2005

SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY
DEPARTMENT OF MATHEMATICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Universal Approximation Of Fuzzy Functions**" by **Majid Amirfakhrian** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

Dated: September 2005

Research Supervisor: ______________________
Dr. Saeid Abbasbandy

External Examiner: ______________________
Dr. Babak Asady

Examing Committee: ______________________
Prof. Kh. Maleknejad

____
Dr. T. Allahviranloo

____
Dr. M. A. Fariborzi Araghi

____
Dr. F. Hosseinzadeh

SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY [1]

Date: **September 2005**

Author: **Majid Amirfakhrian**

Title: **Universal Approximation Of Fuzzy Functions**

Department: **Mathematics**

Degree: **Ph.D.** Convocation: **October** Year: **2005**

Signature of Author

[1]Poonak, Tehran

To my dears:

my honorable mother and father

and my wife

Contents

Acknowledgements

I would like to thank Dr. **Saeid Abbasbandy** my supervisor, for his many suggestions and constant support during this research.

I am grateful to **my parents** for their patience and *love*, without whose support this work would have never come into existence.

Finally, I wish to thank the following:

I am also thankful to Prof. E. Babolian and Prof. Kh. Maleknejad for their useful advices.

My thanks to the referees of this thesis: Dr. M. A. Fariborzi Araghi, Dr. B. Asady and Dr. T. Alahviranloo.

Science and Research Branch, Islamic Azad University, Tehran
Majid Amirfakhrian
September 6, 2005

Abstract

The aim of this study is to use linear programming problem for approximate fuzzy data, fuzzy functions, or fuzzy numbers.

Let $\mathcal{X}$ be a set of m distinct points $x_1, x_2, \ldots, x_m$; of $\mathbb{R}$. Here we approximate a given fuzzy function $\tilde{f}$ defined on $\mathcal{X}$, by fuzzy polynomials of degree at most n, using two types of methods, such that $n \leq m$, first by solving fuzzy linear programming and then by using some independent non-fuzzy linear programming problems. Properties of these approximations are considered in this work. The nearest approximation of a fuzzy number out of a particular subset of all fuzzy numbers is presented. Also the nearest approximation of any power of a fuzzy number and multiplication of two fuzzy numbers are computed.

Originality

In this research we use approximation problem for fuzzy functions from $\mathbb{R}$ to a subset of all fuzzy numbers. In Chapter 1 we represent a method for solving a fuzzy linear programming with fuzzy variables, defined by Mashinchi, Maleki, Peraei and Tata in [23, 37]. They used a particular ranking method for fuzzy numbers to solve this fuzzy linear programming but we use a class of ranking methods to solve it. We use this scheme in next chapters to find approximation of a fuzzy function.

We define a parametric ranking for triangular fuzzy numbers in Section 2.2.1 and use it for approximation (Section 2.2). We define an approximating polynomial based on Fortemps and Roubens defuzzification and introduce two approximating polynomials under the name of Universal and $SAF-$ approximations (Section 2.3).

We define two types of best approximations of a triangular valued fuzzy number in Chapter 3 and we explain about existence and uniqueness of them (Sections 3.2.1, 3.2.2, 3.3.1, 3.3.2).

We define a new distance which is a metric on all trapezoidal fuzzy numbers in Section 4.3, and based on it, we introduce the nearest approximations of a fuzzy number out of a particular subset of all fuzzy numbers (Sections 4.4, 4.5, 4.9). We approximate any power of a trapezoidal fuzzy number and multiplication of two trapezoidal fuzzy numbers by a trapezoidal one in Sections 4.7 and 4.8.

Publications

1. A New Approach to Universal Approximation of Fuzzy Functions on a Discrete Set of Points, *Journal of Applied Mathematical Modelling*, (to appear) (**ISI**).

2. The Nearest Approximation of a Fuzzy Quantity in Parametric Form, *Journal of Applied Mathematics and Computations*, (to appear) (**ISI**).

3. Numerical Approximation of Fuzzy Functions by Fuzzy Polynomials, *Journal of Applied Mathematics and Computations*, (to appear) (**ISI**).

4. Best Approximation of Fuzzy Functions, *Journal of Nonlinear Studies*, (to appear) (**AMS**).

5. Best Approximation of Fuzzy Functions, presented in 33^{rd} Iranian mathematics conference, Ferdowsi University Mashhad, (2002).

6. Best approximation of fuzzy functions by optimization, Proceeding of the 4^{th} seminar on fuzzy sets and its applications, Mazandaran University, Iran, (2003) 8-15.

7. A New Method to Find Best Approximation of Fuzzy Functions by Optimization, presented in 34^{th} Iranian mathematics conference, Shahroud University, (2003).

Introduction

Most of our traditional tools for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. By crisp we mean dichotomous, that is, yes-or-no type rather than more-or-less type. In conventional dual logic, for instance, a statement can be true or false and nothing in between. In set theory, an element can either belong to a set or not; and in optimization, a solution is either feasible or not. Precision assumes that the parameters of a model represent exactly either our perception of the phenomenon modeled or the features of the real system that has been modeled. Generally, precision also implies that the model is unequivocal, that is, it contains no ambiguities, [41].

The question

How can we approximate fuzzy quantities with a good performance?

is our goal to answer.

In Chapter 1 some basic definitions and results on fuzzy numbers, definition of fuzzy linear programming with fuzzy variables and a method for solving it according to a special class of ranking used to find approximating polynomials in next chapters as well as definition of the problem of approximation, that is the main interest of the later chapters, are discussed.

In Chapter 2 the approximation problem on triangular fuzzy numbers leads us to an approximating polynomial name e_ϕ-approximation and on the set of all fuzzy numbers, the approximation problem gives us the $D-$approximation and we present a method to find it, also the universal and SAF approximations which are special cases of $D-$approximation are found in this chapter. In Chapter 3 two best approximations of a triangular valued fuzzy function on a set of points are defined and are computed. Chapter 4 contains an idea for computing the nearest approximation of a fuzzy number out of a particular subset of all fuzzy numbers. Finally is brought Appendix.

Chapter 1

Preliminaries

Fuzziness is not a priori an obvious concept and demands some explanation. ”Fuzziness” is ”vagueness”, i.e. to designate the kind of uncertainty which is both due to fuzziness and ambiguity. Generalizing refers to the application of a symbol to a multiplicity of objects in the field of reference, ambiguity to the association of a finite number of alternative meanings having the same phonetic form. But, the fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol applies.

1.1 Introduction

In this chapter the basic definitions of fuzzy sets and algebraic operations are defined and extension principle are provided, that this is one of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets.

1.2 Fuzzy sets and some basic definitions

Definition 1.2.1. If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \ : \ \ x \in X\},$$

where $\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\tilde{A}$ that maps X to the membership space M (when M contains only the two points 0 and 1, $\tilde{A}$ is nonfuzzy and $\mu_{\tilde{A}}(x)$ is identical to the characteristic function of nonfuzzy set).

The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite. Elements with zero degree of membership are normally not listed. A fuzzy set is obviously a generalization of a classical set and the membership function a generalization of the characteristic function. Since we are generally referring to a universal (crisp) set X, some elements of fuzzy set may have zero degree of membership. Often it is appropriate to consider those elements of the universe that have a nonzero degree of membership in a fuzzy set.

The membership function is not limited to values between 0 and 1. If $\sup_x \mu_{\tilde{A}}(x) = 1$, the fuzzy set $\tilde{A}$ is called *normal*. A nonempty fuzzy set $\tilde{A}$ can always be normalized by dividing $\mu_{\tilde{A}}(x)$ by $\sup_x \mu_{\tilde{A}}(x)$. As a matter of convenience, we will generally assume that fuzzy sets are normalized and $M = [0, 1]$.

Definition 1.2.2. [41] The *support* of a fuzzy set $\tilde{A}$ is the ordinary subset of X:

$$\text{supp}(\tilde{A}) = \{x \in X \ : \ \ \mu_{\tilde{A}}(x) > 0\}.$$

Definition 1.2.3. The *height* of $\tilde{A}$ is the least upper bound of $\mu_{\tilde{A}}(x)$, i.e.

$$\mathrm{hgt}(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x).$$

$\tilde{A}$ is normal if and only if $\exists x \in X, \mu_{\tilde{A}}(x) = 1$; this definition implies $\mathrm{hgt}(\tilde{A}) = 1$. The empty set $\tilde{\emptyset}$ is defined as $\forall x \in X, \mu_{\tilde{\emptyset}}(x) = 0$. A more general and even more useful notion is that of an $\alpha-$cut set.

Definition 1.2.4. The set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree $\alpha \in (0, 1]$ is called the $\alpha-$level set or $\alpha-$cut:

$$[\tilde{A}]^{\alpha} = \{x \in X \ : \ \mu_{\tilde{A}}(x) \geq \alpha\}.$$

If nonequality is hold strictly then $[\tilde{A}]^{\alpha}$ is called "strong $\alpha-$level set", also for $\alpha = 0$ we have

$$[\tilde{A}]^{0} = \overline{\bigcup_{\alpha \in (0,1]} [\tilde{A}]^{\alpha}}.$$

Definition 1.2.5. The m-th power of a fuzzy set $\tilde{A}$ is a fuzzy set with membership function

$$\mu_{\tilde{A}^m}(x) = [\mu_{\tilde{A}}(x)]^m \quad , \quad \forall x \in X \quad , \quad \forall m \in \mathbb{R}^+.$$

Convexity also plays a role in fuzzy set theory. By contrast to classical set theory, however, convexity conditions are defined with reference to the membership function rather than the support of a fuzzy set.

Definition 1.2.6. A fuzzy set $\tilde{A}$ is *convex* if

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \quad , \quad \forall x_1, x_2 \in X, \forall \lambda \in [0, 1].$$

Alternatively, a fuzzy set is convex if all of its $\alpha-$level sets are convex, [41].

Definition 1.2.7. [41] A fuzzy number $\tilde{A}$ is a convex normalized fuzzy set $\tilde{A}$ of the real line $\mathbb{R}$ such that

1. There exists exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{A}}(x_0) = 1$ (x_0 is called mean value of $\tilde{A}$).

2. $\mu_{\tilde{A}}(x)$ is upper semicontinuous and compactly supported.

A fuzzy number $\tilde{A}$ is called *positive* (*negative*) if its membership function is such that $\mu_{\tilde{A}}(x) = 0, \ \forall x < 0 \ (\forall x > 0)$.

We denote by $\mathcal{F}(\mathbb{R})$, the set of all real fuzzy numbers (which are normal, upper semicontinuous, fuzzy convex and compactly supported).

Dubois and Prade [10] suggest a special type of representation for fuzzy numbers of the following type: They call L (and R), which map $\mathbb{R}^+ \longrightarrow [0,1]$, and are decreasing, *shape functions* or *reference functions* if $L(0) = 1$, $L(x) < 1$ for $x > 0$; $L(x) > 0$ for $x < 1$; $L(1) = 0$ or $[L(x) > 0, \forall x$ and $L(+\infty) = 0]$. The most useful type of fuzzy numbers is LR type as follows :

Definition 1.2.8. A fuzzy number $\tilde{A}$ is of LR type if there exist reference functions L (for left) and R (for right), and scalars $\alpha > 0$, $\beta > 0$ with

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & , \quad x \leq m, \\ R(\frac{x-m}{\beta}) & , \quad x \geq m, \end{cases}$$

where m, called the mean value of $\tilde{A}$, is a real number, α and β are called the left and right spreads, respectively. Symbolically $\tilde{A}$ is denoted by $(m, \alpha, \beta)_{LR}$.

Definition 1.2.9. A fuzzy interval $\tilde{A}$ is of LR type if there exist reference functions L and R, and four parameters $\underline{m}, \overline{m} \in \mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$,$\underline{m} \leq \overline{m}$, $\alpha > 0$, $\beta > 0$ with membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{\underline{m}-x}{\alpha}) & , \quad x \leq \underline{m}, \\ 1 & , \quad \underline{m} \leq x \leq \overline{m}, \\ R(\frac{x-\overline{m}}{\beta}) & , \quad x \geq \overline{m}. \end{cases}$$

The fuzzy interval is then denoted by

$$(\underline{m}, \overline{m}, \alpha, \beta)_{LR}.$$

One of the most useful type of fuzzy numbers is triangular :

Definition 1.2.10. A triangular fuzzy number $\tilde{A}$ is defined by triple (m, σ, γ), such that

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{m-x}{\sigma} & , \quad (m-\sigma) \leq x \leq m, \\ 1 + \frac{m-x}{\gamma} & , \quad m \leq x \leq (m+\gamma), \\ 0 & , \quad otherwise, \end{cases}$$

where $\sigma, \gamma > 0$. Denote by $\mathcal{TRF}(\mathbb{R})$, the set of all triangular fuzzy numbers. We will write: (1) $\tilde{A} > 0$ if $m - \sigma > 0$; (2) $\tilde{A} \geq 0$ if $m - \sigma \geq 0$; (3) $\tilde{A} < 0$ if $m + \gamma < 0$; and (4) $\tilde{A} \leq 0$ if $m + \gamma \leq 0$.

Definition 1.2.11. We define a semitriangular fuzzy number, $\tilde{A} = (m, \sigma, \gamma)$, with membership function,

$$\mu_{\tilde{A}}(x) = \begin{cases} l(x) & , \quad (m-\sigma) \leq x \leq m, \\ r(x) & , \quad m \leq x \leq (m+\gamma), \\ 0 & , \quad otherwise, \end{cases}$$

where σ and γ, are non-negative real numbers as the left and right spreads, respectively; $m \in \mathbb{R}$ and

$$l(x) = \begin{cases} 1 - \frac{m-x}{\sigma} & , \quad \sigma \neq 0, \\ 1 & , \quad \sigma = 0, \end{cases} \quad \text{and} \quad r(x) = \begin{cases} 1 + \frac{m-x}{\gamma} & , \quad \gamma \neq 0, \\ 1 & , \quad \gamma = 0. \end{cases}$$

A triangular or semitriangular fuzzy number $\tilde{A} = (m, \sigma, \gamma)$ is *symmetric triangular* if $\sigma = \gamma$. We denote by $\mathcal{TSF}(\mathbb{R})$, the set of all triangular and semitriangular fuzzy numbers. The other most useful type of fuzzy numbers is trapezoidal form as follows:

Definition 1.2.12. A trapezoidal fuzzy number $\tilde{A}$ is defined by $(\underline{m}, \overline{m}, \sigma, \gamma)$, such that

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{\underline{m}-x}{\sigma} & , \quad (\underline{m} - \sigma) \leq x \leq \underline{m}, \\ 1 & , \quad \underline{m} \leq x \leq \overline{m}, \\ 1 + \frac{\overline{m}-x}{\gamma} & , \quad \overline{m} \leq x \leq (\overline{m} + \gamma), \\ 0 & , \quad otherwise, \end{cases}$$

where $\sigma, \gamma > 0$. We will write: (1) $\tilde{A} > 0$ if $\underline{m} - \sigma > 0$; (2) $\tilde{A} \geq 0$ if $\underline{m} - \sigma \geq 0$; (3) $\tilde{A} < 0$ if $\overline{m} + \gamma < 0$; and (4) $\tilde{A} \leq 0$ if $\overline{m} + \gamma \leq 0$.

We define a semitrapezoidal fuzzy number, same as semitriangular fuzzy number defined in Definition 1.2.11. We denote by $\mathcal{TZF}(\mathbb{R})$, the set of all trapezoidal and semitrapezoidal fuzzy numbers, and denote by $\mathcal{TF}(\mathbb{R})$, the union set of $\mathcal{TZF}(\mathbb{R})$ and $\mathcal{TSF}(\mathbb{R})$.

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. In its elementary form, it was already implied in Zadeh's first contribution (1965).

Definition 1.2.13. Let X be a cartesian product of universals $X = X_1 \times X_2 \times \ldots \times X_k$, and $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k$ be k fuzzy sets in $X_1, X_2, \ldots, X_k$, respectively. Suppose f is a mapping from X to a universe Y, $y = f(x_1, \ldots, x_k)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \ : \ y = f(x_1, \ldots, x_k) \ , \ (x_1, \ldots, x_k) \in X\}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x_1,\ldots,x_k)\in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_k}(x_k)\} & , \ f^{-1}(y) \neq \emptyset, \\ 0 & , \ otherwise. \end{cases}$$

A fuzzy function is a generalization of the concept of a classical function. A classical function f is a mapping (correspondence) from the domain D of definition of the function into a space S; $f(D) \subseteq S$ is called the range of f. Different features of the classical concepts of a function can be considered to be fuzzy rather than crisp. Therefore different "degrees" of fuzzification of the classical notion of a function are conceivable :

1. There can be a crisp mapping from a fuzzy set that carries along the fuzziness of the domain and therefore generates a fuzzy set. The image of a crisp argument would again be crisp.

2. The mapping itself can be fuzzy, thus blurring the image of a crisp argument. This we shall call a *fuzzy function.* These are called "fuzzifying function" by Dubois and Prade.

3. Ordinary functions can have fuzzy properties or be constrained by fuzzy constrains, [41].

Definition 1.2.14. A classical function $f : X \to Y$ maps a fuzzy domain $\tilde{A}$ in X into a fuzzy range $\tilde{B}$ in Y if and only if

$$\forall x \in X \ , \ \mu_{\tilde{B}}(f(x)) \geq \mu_{\tilde{A}}(x).$$

Given a classical function $f : X \to Y$ and a fuzzy domain $\tilde{A}$ in X, the extension principle yields the fuzzy range $\tilde{B}$ with the membership function

$$\mu_{\tilde{B}}(y) = \sup_{x \in f^{-1}(y)} \mu_{\tilde{A}}(x),$$

hence f is a function according to the above definition, [41].

For the purpose of ordering fuzzy numbers we use a function $E : \mathcal{F}(\mathbb{R}) \longrightarrow \mathbb{R}$, named ranking function. In this case we say $\tilde{a} \leq_E \tilde{b}$ if and only if $E(\tilde{a}) \leq E(\tilde{b})$.

Let E be an ordering method, $\tilde{S}$ the set of fuzzy quantities for which the method E can be applied and $\mathcal{A}$ an arbitrary finite subset of $\tilde{S}$. In [33, 34], Wang and Kerre put worth the following seven axioms as the reasonable properties of ordering fuzzy quantities for E:

$\mathbf{A_1}$. For $\tilde{A} \in \mathcal{A}$, $\tilde{A} \leq_E \tilde{A}$ on $\mathcal{A}$.

$\mathbf{A_2}$. For $(\tilde{A}, \tilde{B}) \in \mathcal{A}^2$; $\tilde{A} \leq_E \tilde{B}$ and $\tilde{B} \leq_E \tilde{A}$ on $\mathcal{A}$, we should have $\tilde{A} =_E \tilde{B}$ on $\mathcal{A}$.

$\mathbf{A_3}$. For $(\tilde{A}, \tilde{B}, \tilde{C}) \in \mathcal{A}^3$; $\tilde{A} \leq_E \tilde{B}$ and $\tilde{B} \leq_E \tilde{C}$ on $\mathcal{A}$, we should have $\tilde{A} \leq_E \tilde{C}$ on $\mathcal{A}$.

$\mathbf{A_4}$. For $(\tilde{A}, \tilde{B}) \in \mathcal{A}^2$; $\sup \mathrm{supp}(\tilde{B}) < \inf \mathrm{supp}(\tilde{A})$, we should have $\tilde{B} \leq_E \tilde{A}$ on $\mathcal{A}$.

One stronger version of this axiom is as follows:

$\mathbf{A'_4}$. For $(\tilde{A}, \tilde{B}) \in \mathcal{A}^2$; $\sup \mathrm{supp}(\tilde{B}) < \inf \mathrm{supp}(\tilde{A})$, we should have $\tilde{B} <_E \tilde{A}$ on $\mathcal{A}$.

$\mathbf{A_5}$. Let $\tilde{S}$ and $\tilde{S}'$ be two arbitrary finite sets of fuzzy quantities in which E can be applied and $\tilde{A}$ and $\tilde{B}$ are in $\tilde{S} \cap \tilde{S}'$. We obtain $\tilde{B} <_E \tilde{A}$ on $\tilde{S}'$ if and only if $\tilde{B} <_E \tilde{A}$ on $\tilde{S}$.

$\mathbf{A_6}$. Let $\tilde{A}, \tilde{B}, \tilde{A} + \tilde{C}$ and $\tilde{B} + \tilde{C}$ be elements of $\tilde{S}$. If $\tilde{B} \leq_E \tilde{A}$ on $\{\tilde{A}, \tilde{B}\}$, then $\tilde{B} + \tilde{C} \leq_E \tilde{A} + \tilde{C}$ on $\{\tilde{A} + \tilde{C}, \tilde{B} + \tilde{C}\}$. Concerning $<_E$ a similar axiom is as follows:

$\mathbf{A'_6}$. Let $\tilde{A}, \tilde{B}, \tilde{A} + \tilde{C}$ and $\tilde{B} + \tilde{C}$ be elements of $\tilde{S}$. If $\tilde{B} <_E \tilde{A}$ on $\{\tilde{A}, \tilde{B}\}$, then $\tilde{B} + \tilde{C} <_E \tilde{A} + \tilde{C}$ on $\{\tilde{A} + \tilde{C}, \tilde{B} + \tilde{C}\}$ when $\tilde{C} \neq \tilde{\emptyset}$.

$\mathbf{A_7}$. Let $\tilde{A}, \tilde{B}, \tilde{A}\tilde{C}$ and $\tilde{B}\tilde{C}$ be elements of $\tilde{S}$. If $\tilde{B} \leq_E \tilde{A}$ and $\tilde{C} \geq_E \tilde{0}$ on $\{\tilde{A}, \tilde{B}\}$, then $\tilde{B}\tilde{C} \leq_E \tilde{A}\tilde{C}$ on $\{\tilde{A}\tilde{C}, \tilde{B}\tilde{C}\}$.

Definition 1.2.15. Let E be an arbitrary ranking function on $\mathcal{F}(\mathbb{R})$, and let $\tilde{S}$ be a subset of $\mathcal{F}(\mathbb{R})$. We say $\tilde{S}$ is bounded from below if and only if there exist a $\tilde{u} \in \mathcal{F}(\mathbb{R})$, such that $\tilde{u} \leq_E \tilde{s}$ for all $\tilde{s} \in \tilde{S}$.

Definition 1.2.16. A continuous function $s : [0, 1] \longrightarrow [0, 1]$ with the following properties is a *regular reducing* function[32]:

1. $s(0) = 0$,

2. $s(1) = 1$,

3. $s(r)$ is increasing.

4. $\int_0^1 s(r)dr = \frac{1}{2}$.

Definition 1.2.17. The *parametric form* of a fuzzy number is shown by $\tilde{u} = (\underline{u}(r), \overline{u}(r))$, where the functions $\underline{u}(r)$ and $\overline{u}(r)$; $0 \leq r \leq 1$ satisfy the following requirements:

1. $\underline{u}(r)$ is monotonically increasing left continuous function.

2. $\overline{u}(r)$ is monotonically decreasing left continuous function.

3. $\underline{u}(r) \leq \overline{u}(r) \quad , \quad 0 \leq r \leq 1.$

4. $\overline{u}(r) = \underline{u}(r) = 0$ for $r < 0$ or $r > 1$.

If a is a crisp number then $\overline{u}(r) = \underline{u}(r) = a$, for all $r \in [0, 1]$.

Definition 1.2.18. The *Value* of a fuzzy number $\tilde{u}$ is defined as follows, [32],

$$Val(\tilde{u}) = \int_0^1 s(r)[\overline{u}(r) + \underline{u}(r)]dr,$$

where $s(r)$ is a regular reducing function.

The quantity Val, is a ranking function for fuzzy numbers.

Definition 1.2.19. The *Ambiguity* of a fuzzy number $\tilde{u}$ is defined as follows, [32],

$$Amb(\tilde{u}) = \int_0^1 s(r)[\overline{u}(r) - \underline{u}(r)]dr,$$

where $s(r)$ is a regular reducing function.

Definition 1.2.20. The fuzzy distance function on $\mathcal{F}(\mathbb{R})$, $\delta : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$, is defined by

$$\delta(\tilde{\mu}, \tilde{v})(z) = \sup_{|x-y|=z} \min(\tilde{\mu}(x), \tilde{v}(y)).$$

1.3 Hausdorff metric

Denote by κ^n the set of all nonempty compact subsets of $\mathbb{R}^n$ and by κ_c^n the subset of κ^n consisting of nonempty convex compact sets.

In this section we define a metric space by Hausdorff separation. Recall that

$$\rho(x, A) = \min_{a \in A} \|x - a\|$$

is the distance of a point $x \in \mathbb{R}^n$ from $A \in \kappa^n$ and that the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \kappa^n$ is defined as

$$\rho(A, B) = \max_{a \in A} \rho(a, B).$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$.

An open ϵ-neighborhood of $\mathrm{A} \in \kappa^n$ is the set

$$\mathrm{N}(\mathrm{A}, \epsilon) = \{x \in \mathbb{R}^n \ : \ \rho(x, \mathrm{A}) < \epsilon\} = \mathrm{A} + \epsilon \mathrm{B}^n,$$

where B^n is the open unit ball in $\mathbb{R}^n$, [8].

Definition 1.3.1. A mapping $F : \mathbb{R}^n \to \kappa^n$ is *upper semicontinuous* (usc) at x_0 if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0)$ such that

$$F(x) \subset N(F(x_0), \epsilon) = F(x_0) + \epsilon B^n,$$

for all $x \in N(x_0, \delta)$.

Definition 1.3.2. The *Hausdorff metric* d_H on κ^n is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The space (κ^n, d_H) is a complete metric space. Let D^n denote the set of usc normal fuzzy sets on $\mathbb{R}^n$ with compact support. That is, $\tilde{u} \in D^n$, then $\tilde{u} : \mathbb{R}^n \to [0,1]$ is usc, $\text{supp}(\tilde{u})$ is compact and there exists at least one $\xi \in \text{supp}(\tilde{u})$ for which $u(\xi) = 1$. The β-level set of $\tilde{u}$, $0 < \beta \le 1$ is

$$[\tilde{u}]^\beta = \{x \in \mathbb{R}^n \ : \ \mu_{\tilde{u}}(x) \ge \beta\}.$$

Clearly, for $\alpha \le \beta, [\tilde{u}]^\alpha \supseteq [\tilde{u}]^\beta$. The level sets are nonempty from normality and compact by usc and compact support. The metric d_∞ is defined on D^n as

$$d_\infty(\tilde{u}, \tilde{v}) = \sup\{d_H([\tilde{u}]^\alpha, [\tilde{v}]^\alpha) \ : \ 0 \le \alpha \le 1\}, \quad \tilde{u}, \tilde{v} \in D^n$$

and (D^n, d_∞) is a complete metric space. $\mathcal{F}^n(\mathbb{R})$ is the subset of fuzzy convex elements of D^n. The metric space $(\mathcal{F}^n(\mathbb{R}), d_\infty)$ is also complete, [9].

If $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, then, according to Zadeh extension principle, we can extend f to $\mathcal{F}^n(\mathbb{R}) \times \mathcal{F}^n(\mathbb{R}) \to \mathcal{F}^n(\mathbb{R})$ by the equation

$$f(u,v)(z) = \sup_{z=f(x,y)} \min\{\mu_u(x), \mu_v(y)\}.$$

It is well known that

$$[f(u,v)]^\alpha = f([u]^\alpha, [v]^\alpha)$$

for all $u, v \in \mathcal{F}^n(\mathbb{R}), \alpha \in [0,1]$ and f continuous. Especially for addition and scalar multiplication, we define algebraic operations with $\alpha-$level sets.

Definition 1.3.3. Let $\tilde{u}, \tilde{v} \in \mathcal{F}(\mathbb{R})$ and k scalar, then for $\alpha \in [0,1]$,

1. $[\tilde{u} + \tilde{v}]^\alpha = [u_1(\alpha) + v_1(\alpha), u_2(\alpha) + v_2(\alpha)]$,

2. $[\tilde{u} - \tilde{v}]^\alpha = [u_1(\alpha) - v_2(\alpha), u_2(\alpha) - v_1(\alpha)]$,

3. $[\tilde{u}.\tilde{v}]^{\alpha} = [\min\{u_1(\alpha).v_1(\alpha), u_1(\alpha).v_2(\alpha), u_2(\alpha).v_1(\alpha), u_2(\alpha).v_2(\alpha)\},$
$\max\{u_1(\alpha).v_1(\alpha), u_1(\alpha).v_2(\alpha), u_2(\alpha).v_1(\alpha), u_2(\alpha).v_2(\alpha)\}],$

4. $[k\tilde{u}]^{\alpha} = k[\tilde{u}]^{\alpha}$,

where $[\tilde{u}]^{\alpha} = [u_1(\alpha), u_2(\alpha)]$ and $[\tilde{v}]^{\alpha} = [v_1(\alpha), v_2(\alpha)]$, [30].

1.4 Interpolation and Approximation of Fuzzy Functions

Approximation of fuzzy functions on a finite set of distinct points, has been studied by several authors. In [1, 4, 17, 20, 19], the problem of fuzzy interpolation is considered. Also In [14], a method to construct an approximating fuzzy valued polynomial of a fuzzy function, on a set of distinct points, is given.

We will introduce two type of fuzzy valued polynomials of degree at most n in next chapter and we will use them for approximation of a fuzzy functions.

Interpolation problem is the following: Given $n+1$ different points in $\mathbb{R}$ with the corresponding fuzzy values in $\mathbb{R}$, find a fuzzy polynomial of degree at most n which coincides on these points, with the given fuzzy values. In other words, let the values of a fuzzy function $\tilde{f}$ on the set $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ are given, i.e. the points $(x_i, \tilde{f}_i)$ for $i = 1, 2, \ldots, m$ are given. In interpolation problem we find a polynomial $\tilde{P}(x)$ of degree at most $n = m - 1$ where m is the number of points such that

$$\tilde{P}(x_i) = \tilde{f}_i \quad , \quad i = 1, 2, \ldots, m.$$

But, when we have lots of points (m is very large) it is not good or even possible to find such polynomials. In this case, we will find a polynomial from arbitrary degree which is an approximation to original function. In this case, we have m points but we want to find a polynomial with degree at most $n < m$ but not $n = m - 1$ necessarily.

Definition 1.4.1. For each $\alpha \in [0, 1]$, the lower and upper spreads of a fuzzy function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{F}(\mathbb{R})$, on its α-cut, are $[\tilde{f}]_-^\alpha$ and $[\tilde{f}]_+^\alpha$, respectively [19], such that, for all $x \in \mathbb{R}$,

$$\begin{aligned} [\tilde{f}]_-^\alpha(x) &= \inf\{t \in \mathbb{R} \mid t \in [\tilde{f}(x)]^\alpha\}, \\ [\tilde{f}]_+^\alpha(x) &= \sup\{t \in \mathbb{R} \mid t \in [\tilde{f}(x)]^\alpha\}. \end{aligned} \tag{1.4.1}$$

Lodwick and Santos [19], represented two properties for interpolating polynomials $\tilde{P}(x)$ as follows:

Property 1 : Polynomials with larger α-cut values are contained within polynomials with lower α-cut values. That is, for $\alpha_1 \leq \alpha_2$ and for all x,

$$[\tilde{P}]_-^{\alpha_1}(x) \leq [\tilde{P}]_-^{\alpha_2}(x) \leq [\tilde{P}]_+^{\alpha_2}(x) \leq [\tilde{P}]_+^{\alpha_1}(x).$$

Property 2 : All generated polynomials $[\tilde{P}]_-^\alpha$ and $[\tilde{P}]_+^\alpha$, posses the underlying smoothness and continuity conditions associated with the interpolation method (in this work approximation method) being used.

We call these properties, Lodwick and Santos properties.

1.5 Fuzzy Linear Programming with Fuzzy Variables

In this section, we represent a method for solving a fuzzy linear programming with fuzzy variables by using a class of ranking methods.

1.5.1 Fuzzy Linear Programming with Fuzzy Variables

Let $A_{m\times n}$ be a matrix with real number arrays, $\tilde{X} \in \mathcal{F}^n(\mathbb{R})$ and $\tilde{C} \in \mathcal{F}^m(\mathbb{R})$ be two fuzzy vectors and $b \in \mathbb{R}^n$ be a real vector. A fuzzy linear programming with fuzzy variables can be put in the following form:

$$(FLP)\begin{cases} \min & \tilde{\beta} = b^T\tilde{X}, \\ s.t. & \\ & A\tilde{X} \geq_E \tilde{C}, \end{cases} \tag{1.5.1}$$

where, E is a ranking function on fuzzy numbers which for three fuzzy numbers $\tilde{a}, \tilde{b}$ and $\tilde{c}$ has the following properties:

1. $E(-\tilde{a}) = -E(\tilde{a})$,
2. $E(\tilde{b} + \tilde{c}) = E(\tilde{b}) + E(\tilde{c})$,
3. $E(\alpha\tilde{a}) = \alpha E(\tilde{a}) \quad , \quad \text{for all} \quad \alpha \in \mathbb{R}$,
4. $\tilde{a} \leq_E \tilde{b} + \tilde{c} \quad \Longleftrightarrow \quad \tilde{a} - \tilde{c} \leq_E \tilde{b}$.

1.5.2 Solving a Fuzzy Linear Programming

A method for solving a linear programming with fuzzy variables such as (FLP) is introduced in [23, 37]. According to this method we use the following *auxiliary* linear programming problem:

$$(AFLP)\begin{cases} \max & \tilde{w} = \tilde{C}^TY, \\ s.t. & \\ & A^TY = b, \\ & Y \geq 0. \end{cases} \tag{1.5.2}$$

The relations between problem (FLP) and problem $(AFLP)$ are as follows:

Lemma 1.5.1. *If $\tilde{X}$ is any feasible solution for (FLP) and Y is any feasible solution for $(AFLP)$, then $\tilde{C}^T Y \leq_E b^T \tilde{X}$.*

Lemma 1.5.2. *If $\tilde{X}_0$ is a feasible solution for (FLP) and Y_0 is a feasible solution for $(AFLP)$, such that $\tilde{C}^T Y_0 = b^T \tilde{X}_0$, then $\tilde{X}_0$ is an optimal solution of (FLP) and Y_0 is an optimal solution of $(AFLP)$.*

Theorem 1.5.3. *If the auxiliary problem $(AFLP)$ has an optimal solution, then problem (FLP) has a fuzzy optimal solution.*

To solve $(AFLP)$, we must solve the following (LP):

$$(LP)\begin{cases} \max & w = C^T Y, \\ s.t. & \\ & A^T Y = b, \\ & Y \geq 0, \end{cases} \tag{1.5.3}$$

where $C^T = (c_1, c_2, \ldots, c_m)$ and $c_j = E(\tilde{c}_j)\quad,\quad j = 1, 2, \ldots, m$.

By solving (LP), we get an optimal solution with basis B from matrix A^T. Variables and objective coefficients are named according to basis, Y_B^* and C_B, respectively. Thus $Y_B^* = B^{-1}b$ and optimal value of (LP) is $w^* = C_B^T Y_B^*$. Therefore the optimal solution of $(AFLP)$ is Y_B^* too, but the optimal value of $(AFLP)$ is $\tilde{w}^* = \tilde{C}_B^T Y_B^*$. It is a fuzzy number because components of $\tilde{C}_B$ are fuzzy numbers. Consequently optimal solution of (FLP) is $\tilde{X}^{*T} = \tilde{C}_B^T B^{-1}$, and optimal value of (FLP), is $\tilde{\beta}^* = \tilde{C}_B^T B^{-1} b$.

Chapter 2

Approximation of Fuzzy Functions

Approximation of fuzzy functions on a finite set of distinct points, has been studied by several authors. In [20], the problem of fuzzy interpolation is found. Kaleva [17], presented some properties of fuzzy Lagrange and fuzzy spline interpolating functions, and properties of natural and complete splines of odd degree, are introduced in [1, 4].

2.1 Introduction

Let $\mathcal{X}$ be a set of m distinct points $x_1, x_2, \ldots, x_m$; of $\mathbb{R}$. In Section 2.2 we introduce a fuzzy valued polynomial on $\mathcal{TSF}(\mathbb{R})$, the set of all triangular and semitriangular fuzzy numbers, and we consider the approximation of a given fuzzy function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{TSF}(\mathbb{R})$, on a discrete point set $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, by a fuzzy valued polynomial $\tilde{P}_n$ of degree at most n, where the integers m and n, are given. (Throughout this chapter we consider $n < m$.)

In this section, we introduce a ranking method and an approximation of a fuzzy function, called e_ϕ-approximation on $\mathcal{X}$. We show the existence of this approximation and some examples are given.

Also, in Section 2.3 we approximate a given fuzzy function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{F}(\mathbb{R})$ defined on $\mathcal{X}$ in this chapter. We introduce a fuzzy valued polynomial on $\mathcal{F}(\mathbb{R})$ and we consider the problem of approximating a given fuzzy function $\tilde{f}$, on $\mathcal{X}$, by a fuzzy valued polynomial $\tilde{P}_n$ of degree at most n.

In this section, we use the ranking method of Fortemps and Roubens on the set of all fuzzy numbers, also D-approximation of a fuzzy function on $\mathcal{X}$. We explain existence and uniqueness of this approximation and we present two special types of this approximation, also some examples are given.

Throughout this chapter we use standard difference for fuzzy numbers.

2.2 e_ϕ-Approximation of Fuzzy Functions

We introduce e_ϕ-approximation of a fuzzy function on a finite set of distinct points $\mathcal{X}$ and as we demonstrate in this section, an approach to the problem of finding an e_ϕ-approximation of a given fuzzy function $\tilde{f}$ on $\mathcal{X}$, is to express the problem as a fuzzy linear programming problem.

2.2.1 Polynomials and Ranking

For two arbitrary fuzzy numbers $\tilde{a} = (a, \sigma, \gamma), \tilde{b} = (b, \sigma', \gamma') \in \mathcal{TSF}(\mathbb{R})$, and a real number k, some results of applying fuzzy arithmetic on fuzzy numbers $\tilde{a}$ and $\tilde{b}$, are as follows,

- if $k > 0$, then $k\tilde{a} = (ak, \sigma k, \gamma k)$,

- if $k < 0$, then $k\tilde{a} = (ak, -\gamma k, -\sigma k)$,
- $\tilde{a} + \tilde{b} = (a + b, \sigma + \sigma', \gamma + \gamma')$,
- $\tilde{a} - \tilde{b} = (a - b, \sigma + \gamma', \gamma + \sigma')$.

Denote by $\prod_n$, the set of all real valued polynomials of degree at most n, and by $\prod_n^+$, the set of all nonnegative real valued piecewise polynomials of degree at most n.

We employ a class of fuzzy valued polynomials on $\mathcal{TSF}(\mathbb{R})$, and we approximate a fuzzy function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{TSF}(\mathbb{R})$, by such fuzzy valued polynomials.

Definition 2.2.1. A triangular fuzzy valued polynomial of degree at most n on $\mathcal{TSF}(\mathbb{R})$, is a function $\tilde{P} : \mathbb{R} \longrightarrow \mathcal{TSF}(\mathbb{R})$, such that, for all $x \in \mathbb{R}$,

$$\tilde{P}(x) = (P(x), \underline{P}(x), \overline{P}(x)) \in \mathcal{TSF}(\mathbb{R}),$$

where, $P \in \prod_n$ and $\underline{P}, \overline{P} \in \prod_n^+$. We denote the set of all fuzzy valued polynomials by $\tilde{\prod}_n$. Such fuzzy valued polynomials have already been used for interpolation problem before [1, 4, 17, 14, 19].

Definition 2.2.2. Let $\tilde{a}, \tilde{b} \in \mathcal{TSF}(\mathbb{R})$, and

$$\begin{cases} e_\phi : \mathcal{TSF}(\mathbb{R}) \longrightarrow \mathbb{R}, \\ e_\phi(a, \sigma, \gamma) = a - \phi\sigma + \phi\gamma, \end{cases} \tag{2.2.1}$$

where $\phi \in [0, 1]$. We define $\tilde{a} \leq_e \tilde{b}$, iff $e_\phi(\tilde{a}) \leq e_\phi(\tilde{b})$. Also, we use $\tilde{a} \succeq 0 (\succ 0)$, iff $e_\phi(\tilde{a}) \geq 0 (> 0)$, and $\tilde{a} \preceq 0 (\prec 0)$, iff $e_\phi(\tilde{a}) \leq 0 (< 0)$.

For $\phi = \frac{1}{4}$, this ranking method is the same as Yager's ranking method [35] and Fortemps-Roubens' defuzzification [15, 29] and Miguel-Campos-Munoz's method [25] with unite probability measure and 0.5 as optimism-pessimism indicator for triangular fuzzy numbers. Also, $Val(\tilde{a}) = e_{\frac{1}{6}}(\tilde{a})$. Indeed, all symmetric triangular fuzzy numbers of equal modes have same ranking. This ranking will be used to construct an approximating fuzzy valued polynomial, in this section. The relation between e_ϕ and nonnegativity of a fuzzy number is explained in the next lemma.

Lemma 2.2.1. *If $\tilde{a}$ is a nonnegative fuzzy number, then $\tilde{a} \succeq 0$, and if it is a nonpositive fuzzy number, then $\tilde{a} \preceq 0$.*

Proof. Since $\phi \in [0,1]$, we have $\phi\gamma \leq \gamma$, and hence $a + \phi\gamma - \phi\sigma \leq a + \gamma$, therefore if $\tilde{a}$ is nonpositive fuzzy number, then $\tilde{a} \preceq 0$. In a same way, we can show that, if $\tilde{a}$ is a nonnegative fuzzy number, then $\tilde{a} \succeq 0$. $\square$

Lemma 2.2.2. *If $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{TSF}(\mathbb{R})$ then,*

1. $e_\phi(-\tilde{a}) = -e_\phi(\tilde{a})$,
2. $e_\phi(\tilde{b} + \tilde{c}) = e_\phi(\tilde{b}) + e_\phi(\tilde{c})$,
3. $e_\phi(\alpha\tilde{a}) = \alpha e_\phi(\tilde{a}) \quad , \quad$ *for all* $\alpha \in \mathbb{R}$,
4. $\tilde{a} \leq_e \tilde{b} + \tilde{c} \qquad \Longleftrightarrow \qquad \tilde{a} - \tilde{c} \leq_e \tilde{b}$,
5. $\tilde{a} \leq_e \tilde{b} \qquad \Longleftrightarrow \qquad \tilde{a} + \tilde{c} \leq_e \tilde{b} + \tilde{c}$.
6. $\tilde{a} \leq_e \tilde{b} \qquad \Longleftrightarrow \qquad -\tilde{b} \leq_e -\tilde{a}$.

Proof. Let $\tilde{a} = (a, \sigma, \gamma)$, $\tilde{b} = (b, \sigma', \gamma')$ and $\tilde{c} = (c, \sigma'', \gamma'')$.

1. $e_\phi(-\tilde{a}) = e_\phi(-a, \gamma, \sigma) = -a - \phi\gamma + \phi\sigma = -(a - \phi\sigma + \phi\gamma) = -e_\phi(\tilde{a})$.

2. $e_\phi(\tilde{a}+\tilde{c}) = e_\phi(a+c, \sigma+\sigma'', \gamma+\gamma'') = (a+c) - \phi(\sigma+\sigma'') + \phi(\gamma+\gamma'') = (a-\phi\sigma+\phi\gamma) + (c-\phi\sigma''+\phi\gamma'') = e_\phi(\tilde{a}) + e_\phi(\tilde{c})$.

3. If $\alpha \geq 0$,then $e_\phi(\alpha\tilde{a}) = e_\phi(\alpha a, \alpha\sigma, \alpha\gamma) = \alpha a - \phi\alpha\sigma + \phi\alpha\gamma = \alpha(a-\phi\sigma+\phi\gamma) = \alpha e_\phi(\tilde{a})$, and if $\alpha < 0$, then $e_\phi(\alpha\tilde{a}) = e_\phi(\alpha a, -\alpha\gamma, -\alpha\sigma) = \alpha a + \phi\alpha\gamma - \phi\alpha\sigma = \alpha(a-\phi\sigma+\phi\gamma) = \alpha e_\phi(\tilde{a})$.

4. $e_\phi(\tilde{a}) \leq e_\phi(\tilde{b}+\tilde{c}) \iff a-\phi\sigma+\phi\gamma \leq (b+c)-\phi(\sigma'+\sigma'')+\phi(\gamma'+\gamma'') \iff (a-c)-\phi(\sigma+\gamma'')+\phi(\gamma+\sigma'') \leq b-\phi\sigma'+\phi\gamma' \iff e_\phi(\tilde{a}-\tilde{c}) \leq e_\phi(\tilde{b}) \iff \tilde{a}-\tilde{c} \leq_e \tilde{b}$.

5. $\tilde{a} \leq_e \tilde{b} \iff e_\phi(\tilde{a}) \leq e_\phi(\tilde{b}) \iff e_\phi(\tilde{a}) + e_\phi(\tilde{c}) \leq e_\phi(\tilde{b}) + e_\phi(\tilde{c}) \iff e_\phi(\tilde{a}+\tilde{c}) \leq e_\phi(\tilde{b}+\tilde{c}) \iff \tilde{a}+\tilde{c} \leq_e \tilde{b}+\tilde{c}$.

$\square$

Definition 2.2.3. If $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m \in \mathcal{TSF}(\mathbb{R})$, then

$$\tilde{\theta} = \max_{i=1,2,\ldots,m} \tilde{a}_i \iff e_\phi(\tilde{\theta}) = \max_{i=1,2,\ldots,m} e_\phi(\tilde{a}_i).$$

Taking into consideration, the Definition 2.2.2, for all $\tilde{a} \in \mathcal{TSF}(\mathbb{R})$; either $\tilde{a} \succeq 0$ is true, or $\tilde{a} \prec 0$.

Definition 2.2.4. The weak fuzzy absolute value of a fuzzy number $\tilde{a} \in \mathcal{TSF}(\mathbb{R})$, is,

$$|\tilde{a}|_F = \begin{cases} \tilde{a} & , \quad \tilde{a} \succeq 0, \\ -\tilde{a} & , \quad otherwise(\tilde{a} \prec 0), \end{cases} \tag{2.2.2}$$

according to Definition 2.2.2.

Lemma 2.2.3. *If $\tilde{a}, \tilde{b} \in \mathcal{TSF}(\mathbb{R})$ then,*

1. $e_\phi(|\tilde{a}|_F) = |e_\phi(\tilde{a})|_F$.

2. $|\tilde{a}|_F \succeq 0$.

3. $\tilde{a} \leq_e |\tilde{a}|_F$.

4. $|\tilde{a}|_F \leq_e \tilde{b}\ (\tilde{b} \succeq 0) \quad \Longleftrightarrow \quad -\tilde{b} \leq_e \tilde{a} \leq_e \tilde{b}$.

5. $|\tilde{a}|_F \geq_e \tilde{b}\ (\tilde{b} \succeq 0) \quad \Longleftrightarrow \quad \tilde{a} \leq_e -\tilde{b}\ \ or\ \ \tilde{a} \geq_e \tilde{b}$.

Proof. Let $\tilde{a}, \tilde{b} \in \mathcal{TSF}(\mathbb{R})$,

1. If $\tilde{a} \succeq 0$, then $|\tilde{a}|_F = \tilde{a}$, hence $e_\phi(|\tilde{a}|_F) = e_\phi(\tilde{a})$ and since $e_\phi(\tilde{a}) \geq 0$, we observe that $|e_\phi(\tilde{a})| = e_\phi(\tilde{a})$. Now let $\tilde{a} \prec 0$, therefore $|\tilde{a}|_F = -\tilde{a}$, and $e_\phi(|\tilde{a}|_F) = e_\phi(-\tilde{a}) = -e_\phi(\tilde{a})$. In this case $e_\phi(\tilde{a}) < 0$ thus $|e_\phi(\tilde{a})| = -e_\phi(\tilde{a})$.

2. $e_\phi(|\tilde{a}|_F) = |e_\phi(\tilde{a})| \geq 0 \quad \Longrightarrow \quad |\tilde{a}|_F \succeq 0$.

3. $e_\phi(\tilde{a}) \leq |e_\phi(\tilde{a})| = e_\phi(|\tilde{a}|_F) \quad \Longrightarrow \quad \tilde{a} \leq_e |\tilde{a}|_F$.

4. $e_\phi(|\tilde{a}|_F) = |e_\phi(\tilde{a})| \leq e_\phi(\tilde{b}) \quad \Longleftrightarrow \quad -e_\phi(\tilde{b}) \leq e_\phi(\tilde{a}) \leq e_\phi(\tilde{b}) \quad \Longleftrightarrow \quad e_\phi(-\tilde{b}) \leq e_\phi(\tilde{a}) \leq e_\phi(\tilde{b}) \quad \Longleftrightarrow \quad -\tilde{b} \leq_e \tilde{a} \leq_e \tilde{b}$.

5. $e_\phi(|\tilde{a}|_F) = |e_\phi(\tilde{a})| \geq e_\phi(\tilde{b}) \quad \Longleftrightarrow \quad e_\phi(\tilde{a}) \geq e_\phi(\tilde{b})$ or $e_\phi(\tilde{a}) \leq -e_\phi(\tilde{b}) \quad \Longleftrightarrow \quad \tilde{a} \geq_e \tilde{b}$ or $\tilde{a} \leq_e -\tilde{b}$.

□

2.2.2 e_ϕ-Approximation of a Fuzzy Function

In this section, we introduce a method to approximate a fuzzy function on a finite set of distinct points using e_ϕ ranking, and we call it e_ϕ-approximation.

Let $\tilde{P}_n(x) = \sum_{j=0}^{n} x^j \tilde{a}_j$ be a fuzzy valued polynomial, where $\tilde{a}_j \in \mathcal{TSF}(\mathbb{R})$. This is a particular case of fuzzy valued polynomials, defined in Definition 2.2.1, because here P, $\underline{P}$ and $\overline{P}$, belong to $\prod_n$.

Let $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ be a set of m distinct points of $\mathbb{R}$, and $\tilde{f}_i$, be the value of a fuzzy function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{TSF}(\mathbb{R})$, at the point x_i, $i = 1, 2, \ldots m$.

For every fuzzy valued polynomial $\tilde{P}_n \in \tilde{\prod}_n$, let $\tilde{\beta}_{\tilde{P}_n} = (\eta, \sigma, \gamma) \succeq 0$, be a fuzzy number, such that:

$$\tilde{\beta}_{\tilde{P}_n} = \max_{i=1,2,\ldots,m} |\tilde{P}_n(x_i) - \tilde{f}_i|_F, \tag{2.2.3}$$

based on Definition 2.2.3.

Definition 2.2.5. The fuzzy valued polynomial $\tilde{P}_n^e \in \tilde{\prod}_n$, is an e_ϕ-approximation to $\tilde{f}$ on $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, if for every $\tilde{P}_n \in \tilde{\prod}_n$, we have $\tilde{\beta}_{\tilde{P}_n^e} \leq_e \tilde{\beta}_{\tilde{P}_n}$, i.e.

$$\tilde{\beta}_{\tilde{P}_n^e} = \min_{\tilde{P}_n \in \tilde{\prod}_n} \tilde{\beta}_{\tilde{P}_n}. \tag{2.2.4}$$

Applying Lemma 2.2.2 and Definition 2.2.3, the relation (2.2.3) can be written as

$$-\tilde{\beta}_{\tilde{P}_n} \leq_e \tilde{P}_n(x_i) - \tilde{f}_i \leq_e \tilde{\beta}_{\tilde{P}_n} \quad , \quad i = 1, 2, \ldots, m. \tag{2.2.5}$$

Our problem then is to minimize the linear fuzzy function of $\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_n, \tilde{\beta}_{\tilde{P}_n}$;

$$\tilde{\beta}_{\tilde{P}_n},$$

subject to the $2m$ linear constraints,

$$\begin{cases} \tilde{\beta}_{\tilde{P}_n} + \sum_{j=0}^{n} x_i^j \tilde{a}_j \geq_e \tilde{f}_i & , \quad i = 1, 2, \ldots, m, \\ \tilde{\beta}_{\tilde{P}_n} - \sum_{j=0}^{n} x_i^j \tilde{a}_j \geq_e -\tilde{f}_i & , \quad i = 1, 2, \ldots, m. \end{cases}$$

Let's write $\tilde{\beta} = \tilde{\beta}_{\tilde{P}_n}$. The problem of finding e_ϕ-approximation out of $\tilde{\prod}_n$ on $\mathcal{X}$, is equivalent to the following fuzzy linear programming problem:

$$\begin{cases} \min \ \tilde{\beta} \\ s.t. \\ \tilde{\beta} + \sum_{j=0}^{n} x_i^j \tilde{a}_j \geq_e \tilde{f}_i \quad , \quad i = 1, 2, \dots, m, \\ \tilde{\beta} - \sum_{j=0}^{n} x_i^j \tilde{a}_j \geq_e -\tilde{f}_i \quad , \quad i = 1, 2, \dots, m. \end{cases} \tag{2.2.6}$$

Defining

$$A = \begin{pmatrix} 1 & 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & x_m & \dots & x_m^n \\ 1 & -1 & -x_1 & \dots & -x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & -1 & -x_m & \dots & -x_m^n \end{pmatrix}_{2m \times (n+2)} , \tag{2.2.7}$$

and taking

$$\begin{cases} \tilde{X}^T = (\tilde{\beta}, \tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_n), \\ \tilde{C}^T = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_m, -\tilde{f}_1, -\tilde{f}_2, \dots, -\tilde{f}_m), \\ b^T = (1, 0, 0, \dots, 0), \end{cases} \tag{2.2.8}$$

our problem changes to the following form:

$$(FLP) \begin{cases} \min \quad \tilde{z} = b^T \tilde{X}, \\ s.t. \\ \qquad A\tilde{X} \geq_e \tilde{C}. \end{cases} \tag{2.2.9}$$

Using the scheme defined in Section 1.5, we find the optimal solution. Define linear programming problems $(AFLP)$ and (LP) the same as (1.5.2) and (1.5.3), with respect to ranking function e_ϕ.

Let $C^T = (c_1, c_2, \dots, c_{2m})$ where, $c_j = e_\phi(\tilde{c}_j)$, $j = 1, 2, \dots, 2m$. Solving (LP), we obtain an optimal solution with basis B. Call the variables and objective coefficients corresponding to this basis, Y_B^* and C_B, respectively. $Y_B^* = B^{-1}b$ and optimal value of

(LP) is $w^* = C_B^T Y_B^*$. The optimal solution of $(AFLP)$ is Y_B^*, and the optimal value of $(DFLP)$ is $\tilde{w}^* = \tilde{C}_B^T Y_B^*$. Consequently, optimal solution of (FLP) is $\tilde{X}^{*^T} = \tilde{C}_B^T B^{-1}$, and optimal value is $\tilde{\beta}^* = \tilde{z}^* = \tilde{C}_B^T B^{-1} b$.

Since the points are distinct, if $m > n$, then $rank(A) = n + 2$, and B will be a $(n+2)$ by $(n+2)$ matrix. Therefore all coefficients of fuzzy valued polynomial are computable, because we have $n+2$ unknown coefficients. Thus, we consider $m > n$.

2.2.3 Existence of e_ϕ-Approximation

Theorem 2.2.4. *e_ϕ-approximation of a fuzzy function exists.*

Proof. We want to show that e_ϕ-approximation of a fuzzy function exists and for this purpose, we show that the problem (FLP), has a solution. Let $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{TSF}(\mathbb{R})$ be an arbitrary fuzzy function whose values on the points of $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ are $\{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m\}$, respectively. Defining $\tilde{X}_0^T = (\tilde{\beta}_0, 0, 0, \ldots, 0)$, where

$$\tilde{\beta}_0 = \sum_{\tilde{f}_i \succeq 0} \tilde{f}_i - \sum_{\tilde{f}_i \prec 0} \tilde{f}_i,$$

we have,

$$A\tilde{X}_0 = \tilde{\beta}_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \geq_e \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_m \end{pmatrix} = \tilde{C}.$$

Therefore, $\tilde{X}_0^T = (\tilde{\beta}_0, 0, 0, \ldots, 0)$ is a feasible solution for (FLP). It is clear that $\tilde{z} \succeq 0$, and for the feasible point $\tilde{X}_0$, we have

$$\tilde{z}_0 = b^T \tilde{X}_0 = \sum_{\tilde{f}_i \succeq 0} \tilde{f}_i - \sum_{\tilde{f}_i \prec 0} \tilde{f}_i.$$

Thus,

$$0 \preceq \min \quad \tilde{z} \leq_e \sum_{\tilde{f}_i \succeq 0} \tilde{f}_i - \sum_{\tilde{f}_i \prec 0} \tilde{f}_i.$$

It follows that, the set of all feasible points, is bounded from below with respect to e_ϕ, and (FLP) has a solution. $\square$

Theorem 2.2.4 shows the existence of the e_ϕ-approximation of a fuzzy function on a set of distinct points, however by an example we show that the e_ϕ-approximation of a fuzzy function, is not unique. Let $\mathcal{X} = \{1, 2, 3\}$ and the values of a fuzzy function $\tilde{f}$ on $\mathcal{X}$, are

$$\{(1, 1, 1), (2, 1, 1), (3, 1, 1)\},$$

respectively. An e_ϕ-approximating fuzzy valued polynomial of degree 2, to $\tilde{f}$ on $\mathcal{X}$, depending on $\tilde{C}_B^T = ((1, 1, 1), (3, 1, 1), (-2, 1, 1), (-3, 1, 1))$, is $\tilde{P}_2(x) = x^2(0, 3, 3) + x(1, 13, 13) + (0, 12, 12)$ and $\tilde{\beta} = (0, 1, 1)$. But, it can be shown that $\tilde{Q}_2(x) = x^2(0, 2, 2) + x(1, 8, 8) + (0, 7, 7)$ is also an e_ϕ-approximation to $\tilde{f}$ on $\mathcal{X}$, depending on $\tilde{C}_B'^T = ((3, 1, 1), (-1, 1, 1), (-2, 1, 1), (-3, 1, 1))$, and $\tilde{\beta} = (0, 1, 1)$.

It is easy to show that for all e_ϕ-approximating fuzzy valued polynomials of degree at most n, to a fuzzy function $\tilde{f}$, on a set of m distinct points; the fuzzy quantity $\tilde{\beta}$, is equal, because of this fact we can say all of them are acceptable.

Definition 2.2.6. A fuzzy valued polynomial $\tilde{P}_n \in \tilde{\prod}_n$, with symmetric triangular coefficients, is a **STC** polynomial, i.e.

$$\tilde{P}_n(x) = \sum_{j=0}^{n} x^j \tilde{a}_j = \sum_{j=0}^{n} x^j (a_j, \sigma_j, \gamma_j),$$

where $\sigma_j = \gamma_j$, for $j = 0, 1, \ldots, n$.

Theorem 2.2.5. *Let $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ and $\tilde{f} \in \tilde{\prod}_n$, be a* ***STC*** *polynomial, then the e_ϕ-approximation to $\tilde{f}$ on $\mathcal{X}$, is a* ***STC*** *polynomial $\tilde{P}_n^e$, such that, for all $x \in \mathbb{R}$,*

$$[\tilde{P}_n^e]^1(x) = [\tilde{f}]^1(x).$$

Proof. Taking

$$\tilde{f}(x) = \sum_{j=0}^{n} x^j (a_j, \sigma_j, \sigma_j),$$

we have,

$$(x_i, \tilde{f}_i) = (x_i, (\sum_{j=0}^{n} x_i^j a_j, \sum_{j=0}^{n} |x_i^j| \sigma_j, \sum_{j=0}^{n} |x_i^j| \sigma_j)) \quad , \quad i = 1, 2, \ldots, m.$$

We consider two cases:

Case 1: Suppose $\sigma_j = 0$, $j = 0, 1, \ldots, n$, then $\tilde{f} = P_n \in \prod_n$is a crisp polynomial and our problem is a crisp best approximation problem and it is proved that $P_n^e = P_n$, where P_n^e is the e_ϕ-approximation to P_n[6, 24, 28]. On the other hand, $(x_i, \tilde{f}_i) = (x_i, (\sum_{j=0}^{n} x_i^j a_j, 0, 0))$, $i = 1, 2, \ldots, m$; therefore $e_\phi(\tilde{f}_i) = \sum_{j=0}^{n} x_i^j a_j$, and it is a common crisp problem. Let B be an optimal basis (the corresponding basis to an optimal solution), thus solution of this problem is $(\tilde{C}_B^T B^{-1})^T = ((C_B^T B^{-1})^T, 0, 0)$, where $(\tilde{C}_B^T B^{-1})_j = ((C_B^T B^{-1})_j, 0, 0)$ is a crisp number, for $j = 0, 1, \ldots, n$. It follows that $\tilde{P}_n^e = \tilde{f}$. (In crisp form the e_ϕ approximation is the best uniform approximation. [28])

Case 2: Let σ_j be a nonnegative number for $j = 0, 1, \ldots, n$. Then $(x_i, \tilde{f}_i) = (x_i, (\sum_{j=0}^{n} x_i^j a_j, \theta_i, \theta_i))$, where $\theta_i = \sum_{j=0}^{n} |x_i^j| \sigma_j$. Therefore $e_\phi(\tilde{f}_i) = \sum_{j=0}^{n} x_i^j a_j$ and the problem is similar to the first case and $(\tilde{C}_B^T B^{-1})_j = ((C_B^T B^{-1})_j, \lambda_j, \lambda_j)$, for $j = 0, 1, \ldots, n$. That is

$$(\tilde{C}_B^T B^{-1})^T = ((C_B^T B^{-1})^T, \Lambda, \Lambda),$$

where, $\Lambda^T = (\lambda_0, \lambda_1, \ldots, \lambda_{n+1})$. It follows that,

$$\tilde{P}_n^e(x) = \sum_{j=0}^{n} x^j (a_j, \sigma'_j, \sigma'_j),$$

and the proof is completed. $\square$

Lemma 2.2.6. *The e_ϕ-approximating polynomial satisfies both properties 1 and 2 of Lodwick and Santos properties.*

Proof. An e_ϕ-approximating polynomial obtained from the first method, is as follows

$$\widetilde{P}_n(x) = \sum_{j=0}^{n} x^j (a_j, \sigma_j, \gamma_j) = (P(x), T(x), S(x))$$

where $P(x) = \sum_{j=0}^{n} a_j x^j$,

$$T(x) = \begin{cases} \sum_{j=0}^{n} \sigma_j x^j & x \geq 0, \\ \sum_{j=0}^{[\frac{n}{2}]} \sigma_{2j} x^{2j} - \sum_{j=0}^{[\frac{n}{2}]} \gamma_{2j+1} x^{2j+1} & x < 0, \end{cases}$$

and

$$S(x) = \begin{cases} \sum_{j=0}^{n} \gamma_j x^j & x \geq 0, \\ \sum_{j=0}^{[\frac{n}{2}]} \gamma_{2j} x^{2j} - \sum_{j=0}^{[\frac{n}{2}]} \sigma_{2j+1} x^{2j+1} & x < 0. \end{cases}$$

$T(x)$ and $S(x)$, are nonnegative polynomials, therefore, for all $x \in \mathbb{R}$, $P(x) - T(x) \leq P(x)$ and $P(x) + S(x) \geq P(x)$. It yields property 1. For all $x \in \mathbb{R}$ and $\alpha \in [0, 1]$, we observe that, P, $[\widetilde{P}_n]_-^\alpha$ and $[\widetilde{P}_n]_+^\alpha$, are polynomials of degree at most n, and

$$[\widetilde{P}_n]_-^\alpha(x) \leq P(x) \leq [\widetilde{P}_n]_+^\alpha(x).$$

It yields property 2, for e_ϕ-approximating polynomial. $\square$

2.2.4 Numerical Examples of e_ϕ-approximation

Example 2.2.1. *$m = 6$ and $n = 5$,*

x	-2	-1	0	1	2	3
$\tilde{f}(x)$	(-33,1,2)	(-2,2,1)	(-1,1,2)	(0,1,1)	(31,2,1)	(242,2,1)

$\tilde{P}_5(x) = x^5(1, \frac{4}{15}, \frac{4}{15}) + x^4(0, \frac{2}{3}, \frac{2}{3}) + x^3(0, \frac{17}{12}, \frac{17}{12}) + x^2(0, \frac{8}{3}, \frac{8}{3}) + x(0, \frac{13}{6}, \frac{13}{6}) + (-1, 2, 2)$, $\tilde{\beta} = (0, 1, 1)$.

Example 2.2.2. *$m = 6$ and $n = 1$,*

x	1	1.1	1.2	3	3.5	4
$\tilde{f}(x)$	(1,2,1)	(1.1,1,2)	(1.2,0.1,1)	(3,1,1)	(3.5,0.5,0.5)	(4,1.2,0.5)

$\tilde{P}_1(x) = x(1, 0.7333, .8333) + (0, 2.375, 2.27)$, $\tilde{\beta} = (0, 1.9867, 0.9917)$.

Example 2.2.3. *$m = 6$ and $n = 2$,*

x	-1	-0.9	-0.8	0.8	0.9	1
$\tilde{f}(x)$	(16,1,1)	(12.96,1,1)	(10.24,1,1)	(10.24,1,1)	(12.96,1,1)	(16,1,1)

$\tilde{P}_2(x) = x^2(16, \frac{25}{9}, \frac{25}{9}) + x(0, 1, 1) + (0, \frac{5}{2}, \frac{5}{2})$, $\tilde{\beta} = (0, 1, 1)$.

The e_ϕ-approximating function in the last example is compared with linear spline and Lagrange interpolating functions, on the given points. See Figures 2.1-2.3.

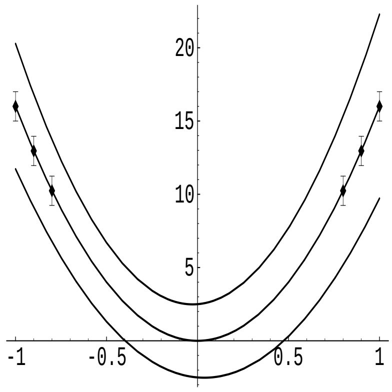
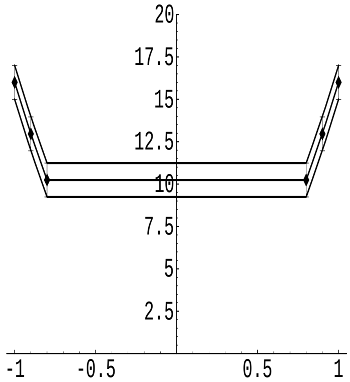


Figure 2.1 : e_ϕ-approximating function



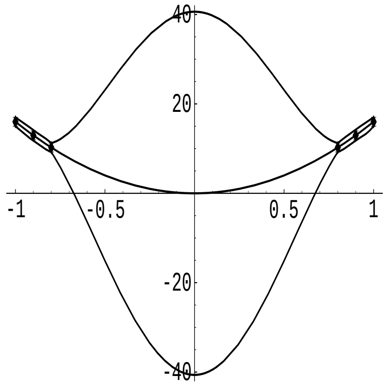


Figure 2.2 : Linear spline interpolation Figure 2.3 : Lagrange interpolation

The maximum error of e_ϕ-approximating function is $er_{e_\phi}(1) = (0, 7.2778, 7.7778)$. Also, maximum errors of linear spline interpolating function and Lagrange interpolating function, are $er_S(0) = (10.24, 2, 2)$ and $er_L(0) = (0, 41.6285, 41.6285)$, respectively, where, $er_{e_\phi}(x)$, $er_S(x)$ and $er_L(x)$ are error functions of e_ϕ-approximation, spline interpolation and Lagrange interpolation, respectively.

2.3 Approximation of Fuzzy Functions by Distance Method

We introduce D-approximation of a fuzzy function on a finite set of distinct points $\mathcal{X}$ and as we demonstrate in this section, an approach to the problem of finding a D-approximation of a given fuzzy function $\tilde{f}$ on $\mathcal{X}$, is to express the problem as a fuzzy linear programming problem. Also, we introduce universal and SAF approximations in this section.

2.3.1 Polynomials, Fuzziness and Ambiguity

Let $\tilde{v} = (\underline{v}(r), \overline{v}(r)), \tilde{u} = (\underline{u}(r), \overline{u}(r)) \in \mathcal{F}(\mathbb{R})$ in parametric form. Some results of applying fuzzy arithmetic on fuzzy numbers $\tilde{v}$ and $\tilde{u}$ are as follows:

- $x > 0 : x\tilde{v} = (x\underline{v}(r), x\overline{v}(r))$;
- $x < 0 : x\tilde{v} = (x\overline{v}(r), x\underline{v}(r))$;
- $\tilde{v} + \tilde{u} = (\underline{v}(r) + \underline{u}(r), \overline{v}(r) + \overline{u}(r))$;
- $\tilde{v} - \tilde{u} = (\underline{v}(r) - \overline{u}(r), \overline{v}(r) - \underline{u}(r))$.

Parametric form of a triangular fuzzy number $\tilde{v} = (v_s, v_l, v_r)$ is

$$\tilde{v} = (v_l(r-1) + v_s, v_r(1-r) + v_s).$$

A fuzzy polynomial of degree at most n is a function $\tilde{P}_n$ from $\mathbb{R}$ to $\mathcal{F}(\mathbb{R})$ such that $\tilde{P}_n(x) = \sum_{j=0}^{n} x^j \tilde{a}_j$. Denote by $\tilde{\prod}_n^{\mathcal{F}}$ the set of all fuzzy polynomials $\tilde{P}_n(x) = \sum_{j=0}^{n} x^j \tilde{a}_j$

of degree at most n on $\mathcal{F}(\mathbb{R})$. A fuzzy polynomial of degree at most n on $\mathcal{F}(\mathbb{R})$ can be put in the following parametric form:

$$\tilde{P}_n(x) = (\underline{v}_x(r), \overline{v}_x(r)) \tag{2.3.1}$$

where

$$\underline{v}_x(r) = \sum_{x^j \geq 0} x^j \underline{a_j}(r) + \sum_{x^j < 0} x^j \overline{a_j}(r),$$
$$\overline{v}_x(r) = \sum_{x^j \geq 0} x^j \overline{a_j}(r) + \sum_{x^j < 0} x^j \underline{a_j}(r),$$

for $0 \leq r \leq 1$.

Definition 2.3.1. For an arbitrary fuzzy number $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, the quantity

$$D(\tilde{v}) = \frac{1}{2}\{\int_0^1 \overline{v}(r)dr + \int_0^1 \underline{v}(r)dr\}, \tag{2.3.2}$$

defines a defuzzification [15, 29].

It means that $\tilde{v} \leq_D \tilde{u}$ if and only if $D(\tilde{v}) \leq D(\tilde{u})$, and we use $\tilde{v} \succeq 0$ if and only if $D(\tilde{v}) \geq 0$, in this case, we say $\tilde{v}$ is nonnegative. It should be mentioned that $\geq_D$ compares two fuzzy numbers and $\succeq$ compares a fuzzy number with singleton zero.

Lemma 2.3.1. *For a trapezoidal fuzzy number $\tilde{v}$, Definition 2.3.1 implies that*

$$D(\tilde{v}) = \frac{1}{4}\{\underline{v}(0) + \overline{v}(0) + \overline{v}(1) + \underline{v}(1)\}. \tag{2.3.3}$$

Lemma 2.3.2. *If $\tilde{v}, \tilde{u}, \tilde{w} \in \mathcal{F}(\mathbb{R})$, then*

1. $D(-\tilde{v}) = -D(\tilde{v})$,

2. $D(\tilde{u} + \tilde{w}) = D(\tilde{u}) + D(\tilde{w})$,

3. $D(\alpha\tilde{v}) = \alpha D(\tilde{v})$,

4. $\tilde{v} \leq_D \tilde{u} + \tilde{w} \quad \Longleftrightarrow \quad \tilde{v} - \tilde{w} \leq_D \tilde{u},$

5. $\tilde{v} \leq_D \tilde{u} \quad \Longleftrightarrow \quad \tilde{v} + \tilde{w} \leq_D \tilde{u} + \tilde{w},$

6. $\tilde{v} \leq_D \tilde{u} \quad \Longleftrightarrow \quad -\tilde{u} \leq_D -\tilde{v}.$

Proof. Let $\tilde{v} = (\underline{v}, \overline{v})$, $\tilde{u} = (\underline{u}, \overline{u})$ and $\tilde{w} = (\underline{w}, \overline{w})$:

1. $D(-\tilde{v}) = D(-(\underline{v}, \overline{v})) = D(-\overline{v}, -\underline{v}) = -\frac{1}{2}\{\int_0^1 [\overline{v}(r) + \underline{v}(r)]dr\} = -D(\tilde{v}).$

2. $D(\tilde{u} + \tilde{w}) = D((\underline{v} + \underline{u}, \overline{v} + \overline{u})) = \frac{1}{2}\{\int_0^1 [\overline{v}(r) + \overline{u}(r)]dr + \int_0^1 [\underline{v}(r) + \underline{u}(r)]dr\} = \frac{1}{2}\{\int_0^1 [\overline{v}(r) + \underline{v}(r)]dr\} + \frac{1}{2}\{\int_0^1 [\overline{u}(r) + \underline{u}(r)]dr\} = D(\tilde{u}) + D(\tilde{w}).$

3. If $\alpha \geq 0$ then $D(\alpha\tilde{v}) = D((\alpha\underline{v}, \alpha\overline{v})) = \frac{1}{2}\{\int_0^1 [\alpha\overline{v}(r) + \alpha\underline{v}(r)]dr\} = \alpha D(\tilde{v})$, and if $\alpha < 0$ then $D(\alpha\tilde{v}) = D((\alpha\overline{v}, \alpha\underline{v})) = \frac{1}{2}\{\int_0^1 [\alpha\overline{v}(r) + \alpha\underline{v}(r)]dr\} = \alpha D(\tilde{v}).$

4. $\tilde{v} \leq_D \tilde{u} + \tilde{w} \quad \Longleftrightarrow \quad D(\tilde{v}) \leq D(\tilde{u} + \tilde{w}) \quad \Longleftrightarrow \quad \frac{1}{2}\{\int_0^1 [\overline{v}(r) + \underline{v}(r)]dr\} \leq \frac{1}{2}\{\int_0^1 [\overline{w}(r) + \overline{u}(r) + \underline{w}(r) + \underline{u}(r)]dr\} \quad \Longleftrightarrow \quad \int_0^1 [\overline{v}(r) - \underline{w}(r) + \underline{v}(r) - \overline{w}(r)]dr\} \leq \frac{1}{2}\{\int_0^1 [\overline{u}(r) + \underline{u}(r)]dr\} \quad \Longleftrightarrow \quad \tilde{v} - \tilde{w} \leq_D \tilde{u},$

5. $\tilde{v} \leq_D \tilde{u} \quad \Longleftrightarrow \quad D(\tilde{v}) \leq D(\tilde{u}) \quad \Longleftrightarrow \quad D(\tilde{v}) + D(\tilde{w}) \leq D(\tilde{u}) + D(\tilde{w}) \quad \Longleftrightarrow \quad D(\tilde{v} + \tilde{w}) \leq D(\tilde{u} + \tilde{w}) \quad \Longleftrightarrow \quad \tilde{v} + \tilde{w} \leq_D \tilde{u} + \tilde{w}.$

6. $\tilde{v} \leq_D \tilde{u} \quad \Longleftrightarrow \quad D(\tilde{v}) \leq D(\tilde{u}) \quad \Longleftrightarrow \quad -D(\tilde{u}) \leq -D(\tilde{v}) \quad \Longleftrightarrow \quad D(-\tilde{u}) \leq D(-\tilde{v}) \quad \Longleftrightarrow \quad -\tilde{u} \leq_D -\tilde{v}.$

$\square$

Remark 2.3.1. Let $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_m \in \mathcal{F}(\mathbb{R})$, then $\tilde{a}_k = \max_{i=1,2,\ldots,m} \tilde{a}_i$ if and only if $D(\tilde{a}_k) = \max_{i=1,2,\ldots,m} D(\tilde{a}_i)$.

Definition 2.3.2. If for a fuzzy number $\tilde{a}$, there exists a monotonically decreasing left continuous function $\delta(r)$ on $[0,1]$ and three real numbers $\alpha_1, \alpha_2 \geq 0$ and b where $\tilde{a} = (-\alpha_1\delta(r) + b, \alpha_2\delta(r) + b)$, we call $\delta(r)$, a **source** function.

For all triangular or trapezoidal fuzzy numbers we have, $\delta(r) = (1 - r)$.

Two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ have the same type if source functions of both fuzzy numbers are the same.

Definition 2.3.3. The weak absolute value of a fuzzy number $\tilde{v}$, is

$$|\tilde{v}| = \begin{cases} \tilde{v}, & \tilde{v} \succeq 0, \\ -\tilde{v}, & \tilde{v} \prec 0. \end{cases} \tag{2.3.4}$$

Lemma 2.3.3. *If $\tilde{v}, \tilde{u} \in \mathcal{F}(\mathbb{R})$, then*

1. $\tilde{v}$ and $|\tilde{v}|$ have the same type,
2. $D(|\tilde{v}|) = |D(\tilde{v})|$,
3. $\tilde{v} \leq_D |\tilde{v}|$,
4. $|\tilde{v}| \leq_D \tilde{u}$ $(\tilde{u} \succeq 0) \iff -\tilde{u} \leq_D \tilde{v} \leq_D \tilde{u}$,
5. $|\tilde{v}| \geq_D \tilde{u}$ $(\tilde{u} \succeq 0) \iff \tilde{v} \leq_D -\tilde{u}$ or $\tilde{v} \geq_D \tilde{u}$.

Proof. By Definition 2.3.3 and Lemma 2.3.2, it is clear that, if $\tilde{v} = (\underline{v}, \overline{v}), \tilde{u} = (\underline{u}, \overline{u}) \in \mathcal{F}(\mathbb{R})$, then:

1. it is trivial.
2. If $\tilde{v} \succeq 0$, then $|\tilde{v}| = \tilde{v}$, hence $D(|\tilde{v}|) = D(\tilde{v})$ and since $D(\tilde{v}) \geq 0$, we observe that $|D(\tilde{v})| = D(\tilde{v})$. Now let $\tilde{v} \prec 0$, therefore $|\tilde{v}| = -\tilde{v}$, and $D(|\tilde{v}|) = D(-\tilde{v}) = -D(\tilde{v})$. In this case $D(\tilde{v}) < 0$, thus $|D(\tilde{v})| = -D(\tilde{v})$.

3. $D(|\tilde{v}|) = |D(\tilde{v})| \geq 0 \implies |\tilde{v}| \succeq 0;$ and
$D(\tilde{v}) \leq |D(\tilde{v})| = D(|\tilde{v}|) \implies \tilde{v} \leq_D |\tilde{v}|.$

4. $D(|\tilde{v}|) = |D(\tilde{v})| \leq D(\tilde{u}) \iff -D(\tilde{u}) \leq D(\tilde{v}) \leq D(\tilde{u}) \iff D(-\tilde{u}) \leq D(\tilde{v}) \leq D(\tilde{u}) \iff -\tilde{u} \leq_D \tilde{v} \leq_D \tilde{u}.$

5. $D(|\tilde{v}|) = |D(\tilde{v})| \geq D(\tilde{u}) \iff D(\tilde{v}) \geq D(\tilde{u})$ or $D(\tilde{v}) \leq -D(\tilde{u}) \iff \tilde{v} \geq_D \tilde{u}$ or $\tilde{v} \leq_D -\tilde{u}.$

□

Remark 2.3.2. It is important to know that the definition of the absolute value of a fuzzy number based on extension principle does not satisfy all properties of Lemma 2.3.3, for example it does not satisfy property 2.

Remark 2.3.3. If $\tilde{u}, \tilde{v} \in \mathcal{F}(\mathbb{R})$ and $\alpha \in \mathbb{R}$, then

1. $Amb(\tilde{u} + \tilde{v}) = Amb(\tilde{u}) + Amb(\tilde{v}),$

2. $Amb(\tilde{u} - \tilde{v}) = Amb(\tilde{u}) + Amb(\tilde{v}),$

3. $Amb(\alpha\tilde{v}) = |\alpha| Amb(\tilde{v}).$

Definition 2.3.4. We define *Ambiguity* of a fuzzy function $\tilde{f}(x)$ with respect to a set of points $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ by

$$\mathbf{A}_{\mathcal{X}}(\tilde{f}) = \max_{1 \leq i \leq m} Amb(\tilde{f}_i).$$

By Definition 2.3.4 and Remark 2.3.3, it can be shown that if $\tilde{f}(x) = \sum_{j=0}^{n} x^j \tilde{a}_j$ be a fuzzy polynomial of degree at most n then

$$\mathbf{A}_{\mathcal{X}}(\tilde{f}) = \max_{1 \leq i \leq m} \sum_{j=0}^{n} |x_i^j| Amb(\tilde{a}_j). \tag{2.3.5}$$

Definition 2.3.5. We define *Fuzziness* of a fuzzy polynomial $\tilde{P}_n(x) = \sum_{j=0}^{n} x^j \tilde{a}_j$ with respect to a set of points $X = \{x_1, x_2, \ldots, x_m\}$ by

$$\mathbf{F}_X(\tilde{P}_n) = \max_{1 \le i \le m} Fuzz(\tilde{P}_n(x_i)),$$

where, $Fuzz(\tilde{a}) = \overline{a}(0) - \underline{a}(0)$.

Remark *2.3.4.* If $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$ and $\alpha \in \mathbb{R}$, then

1. $Fuzz(\tilde{a} + \tilde{b}) = Fuzz(\tilde{a}) + Fuzz(\tilde{b})$,
2. $Fuzz(\tilde{a} - \tilde{b}) = Fuzz(\tilde{a}) + Fuzz(\tilde{b})$,
3. $Fuzz(\alpha\tilde{a}) = |\alpha| Fuzz(\tilde{a})$.

By Definition 2.3.5 and Remark 2.3.4, it can be shown that

$$\mathbf{F}_{\mathcal{X}}(\tilde{P}_n) = \max_{1 \le i \le m} \sum_{j=0}^{n} |x_i^j| Fuzz(\tilde{a}_j). \tag{2.3.6}$$

2.3.2 $D-$Approximation of a Fuzzy Function

Let $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ be a set of m distinct points of $\mathbb{R}$, and the values of a fuzzy function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{F}(\mathbb{R})$ are $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m$ at these points.

Definition 2.3.6. For every polynomial $\tilde{P}_n(x) \in \tilde{\prod}_n^{\mathcal{F}}$, let $\tilde{\beta}(\tilde{P}_n) = (\underline{\beta}(r), \overline{\beta}(r))$ be a nonnegative fuzzy number which has the same type as $\tilde{f}_i$'s and

$$\tilde{\beta}(\tilde{P}_n) = \max_{i=1,2,\ldots,m} |\tilde{P}_n(x_i) - \tilde{f}_i|. \tag{2.3.7}$$

Definition 2.3.7. The polynomial $\tilde{P}_n^D(x) \in \tilde{\prod}_n^{\mathcal{F}}$ is a $D-$approximation of $\tilde{f}$ at $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, if for every $\tilde{P}_n(x) \in \tilde{\prod}_n^{\mathcal{F}}$ we have $\tilde{\beta}(\tilde{P}_n^D) \le_D \tilde{\beta}(\tilde{P}_n)$, i.e.

$$\tilde{\beta}(\tilde{P}_n^D) = \min_{\tilde{P}_n \in \tilde{\prod}_n^{\mathcal{F}}} \tilde{\beta}(\tilde{P}_n). \tag{2.3.8}$$

It is obvious that $\tilde{P}_n^D(x)$ exists when

$$\max_{i=1,2,\ldots,m} |\tilde{P}_n^D(x_i) - \tilde{f}_i| = \min_{\tilde{P}_n \in \tilde{\prod}_n^{\mathcal{F}}} \{ \max_{i=1,2,\ldots,m} |\tilde{P}_n(x_i) - \tilde{f}_i| \}. \tag{2.3.9}$$

Let's write $\tilde{\beta} = \tilde{\beta}(\tilde{P}_n)$. Hence, one must minimize $\tilde{\beta}$.

By a simple computation, it can be shown that we should solve the following fuzzy linear programming problem

$$\begin{cases} \min \tilde{\beta} \\ s.t. \\ \tilde{\beta} + \sum_{j=0}^{n} x_i^j \tilde{a}_j \geq_D \tilde{f}_i, \qquad i = 1, 2, \ldots, m, \\ \tilde{\beta} - \sum_{j=0}^{n} x_i^j \tilde{a}_j \geq_D -\tilde{f}_i, \qquad i = 1, 2, \ldots, m. \end{cases} \tag{2.3.10}$$

Defining A, $\tilde{X}$, $\tilde{C}$ and b the same as relations (2.2.7) and (2.2.8), our problem changes to the following form:

$$(FLP) \begin{cases} \min \quad \tilde{z} = b^T \tilde{X}, \\ s.t. \\ \qquad A\tilde{X} \geq_D \tilde{C}. \end{cases} \tag{2.3.11}$$

Using the scheme defined in Section 1.5 we find the optimal solution. Define linear programming problems $(AFLP)$ and (LP) the same as (1.5.2) and (1.5.3), with respect to ranking function D.

Let $C^T = (c_1, c_2, \ldots, c_{2m})$ where $c_j = D(\tilde{c}_j)$, $j = 1, 2, \ldots, 2m$.

Solving (LP), we obtain an optimal solution with basis B from matrix A^T. Call the variables and objective coefficients corresponding to this basis, Y_B^* and C_B, respectively. $Y_B^* = B^{-1}b$ and optimal value of (LP) is $w^* = C_B^T Y_B^*$. The optimal solution of $(AFLP)$ is Y_B^*, and the optimal value of $(DFLP)$ is $\tilde{w}^* = \tilde{C}_B^T Y_B^*$. Consequently optimal solution of (FLP) is $\tilde{X}^{*^T} = \tilde{C}_B^T B^{-1}$, and optimal value is $\tilde{\beta}^* = \tilde{z}^* = \tilde{C}_B^T B^{-1} b$.

2.3.3 Existence of $D-$Approximation of a Fuzzy Function

Since we want to compute $n+2$ variables ($\tilde{a}_j$'s for $j = 0, \dots, n$, and $\tilde{\beta}$), rank of matrix A should be $n+2$. If $n < m$, then $rank(A) = n+2$. Thus, we consider $n < m$.

Theorem 2.3.4. *$D-$approximation of a fuzzy function exists.*

Proof. Let $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{F}(\mathbb{R})$ be an arbitrary fuzzy function whose values on the points of $\mathcal{X} = \{x_1, x_2, \dots, x_m\}$ are $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_m$. It is clear that $\tilde{X}_0^T = (\tilde{\beta}_0, 0, 0, \dots, 0)$ is a feasible solution of (FLP) such that

$$\tilde{\beta}_0 = \sum_{\tilde{f}_i \succeq 0} \tilde{f}_i - \sum_{\tilde{f}_i \prec 0} \tilde{f}_i,$$

because

$$A\tilde{X} = \tilde{\beta}_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \geq_D \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_m \end{pmatrix} = \tilde{C}.$$

Now, $\tilde{z} \succeq 0$ (by definition of $\tilde{\beta}$) and the set of all $\tilde{z}$ for feasible points, is bounded from below, therefore (FLP) has a solution and it means that $D-$approximation of a fuzzy function exists. □

However by an example we show that $D-$approximation of a fuzzy function, is not unique. Let $\mathcal{X} = \{1, 2, 3\}$ and the values of a fuzzy function $\tilde{f}$, on $\mathcal{X}$ are

$\{(r, -r+2), (r+1, -r+3), (r+2, -r+4)\}$ respectively. Then

$$\tilde{P}_2(x) = x^2(3r-3, -3r+3) + x(11r-10, -11r+12) + (8r-8, -8r+8),$$

is a $D-$approximation of $\tilde{f}$ on $\mathcal{X}$, and $\tilde{\beta} = (r-1, -r+1)$. But, it can be shown that $\tilde{Q}_2(x) = x^2(2r-2, -2r+2) + x(8r-7, -8r+9) + (7r-7, -7r+7)$ is also a $D-$approximation of $\tilde{f}$ on $\mathcal{X}$, and $\tilde{\beta}' = (r-1, -r+1)$. However, if we can find all of the optimal solutions of (2.3.11), after solving equations (2.3.13) or (2.3.15), which will present in next sections, we can choose a polynomial with small fuzziness or small ambiguity.

Definition 2.3.8. $\tilde{a}$ is a symmetric L-R fuzzy number(***SLR***) if there exists a source function $\delta(r)$ and two real numbers α and b where $\tilde{a} = (-\alpha\delta(r) + b, \alpha\delta(r) + b)$.

Definition 2.3.9. A fuzzy polynomial $\tilde{P}(x) = \sum_{j=0}^{n} x^j \tilde{a}_j$ with symmetric coefficients is ***SC*** polynomial, if all coefficients $\tilde{a}_j$'s are ***SLR*** with the same source function $\delta(r)$.

Theorem 2.3.5. *Let $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ and $\tilde{f} \in \tilde{\prod}_n^{\mathcal{F}}$, be a **SC** polynomial with source function $\delta(r)$, then the D-approximation to $\tilde{f}$ on $\mathcal{X}$, is a **SC** polynomial $\tilde{P}_n^D$ with the same source function $\delta(r)$, such that, for all $x \in \mathbb{R}$,*

$$[\tilde{P}_n^D]^1(x) = [\tilde{f}]^1(x).$$

Proof. Taking

$$\tilde{f}(x) = \sum_{j=0}^{n} x^j \tilde{a}_j = \sum_{j=0}^{n} x^j(-a_j\delta(r) + a_{j_s}, a_j\delta(r) + a_{j_s}).$$

we have

$$(x_i, \tilde{f}(x_i)) = (x_i, (\underline{v}_i(r), \overline{v}_i(r))),$$

such that

$$\underline{v}_i(r) = -(\sum_{j=0}^{n} |x_i^j| a_j)\delta(r) + (\sum_{j=0}^{n} x_i^j a_{j_s}),$$

$$\overline{v}_i(r) = (\sum_{j=0}^{n} |x_i^j| a_j)\delta(r) + (\sum_{j=0}^{n} x_i^j a_{j_s}),$$

where $0 \leq r \leq 1$. Now we prove this theorem in two steps. For the first step, suppose $a_j = 0$, $j = 0, 1, \ldots, n$, then $\tilde{f}(x) = P_n(x) \in \prod_n$is a crisp polynomial and it has been proved that $P_n^D(x) = P_n(x)$. On the other hand

$$(x_i, \tilde{f}(x_i)) = (x_i, (\sum_{j=0}^{n} x_i^j a_{j_s}, \sum_{j=0}^{n} x_i^j a_{j_s})), \quad i = 1, 2, \ldots, m,$$

and $D(\tilde{f}_i) = \sum_{j=0}^{n} x_i^j a_{j_s}$, for $i = 1, 2, \ldots, m$, and it is a common crisp problem. Suppose B as an optimal basis (The basis corresponding to an optimal solution) for problem, then solution of this problem is $(\tilde{C}_B^T B^{-1})^T = ((C_B^T B^{-1})^T, (C_B^T B^{-1})^T)$, where for $j = 0, 1, \ldots, n+1$,

$$(\tilde{C}_B^T B^{-1})_j^T = ((C_B^T B^{-1})_j^T, (C_B^T B^{-1})_j^T),$$

is a ***SLR***. Therefore, $\tilde{P}_n^D(x) = \tilde{f}(x)$.

For the second step, suppose a_j as a nonnegative number for $j = 0, 1, \ldots, n$. Then

$$(x_i, \tilde{f}(x_i)) = (x_i, (-\theta_i\delta(r) + \sum_{j=0}^{n} x_i^j a_{j_s}, \theta_i\delta(r) + \sum_{j=0}^{n} x_i^j a_{j_s})),$$

which $\theta_i = \sum_{j=0}^{n} |x_i^j| a_j$. Thus $D(\tilde{f}_i) = \sum_{j=0}^{n} x_i^j a_{j_s}$ and the problem is similar to the first case and for $j = 0, 1, \ldots, n+1$,

$$(\tilde{C}_B^T B^{-1})_j^T = (-\lambda_{j1}\delta(r) + (C_B^T B^{-1})_j^T, \lambda_{j2}\delta(r) + (C_B^T B^{-1})_j^T).$$

That is,

$$(\tilde{C}_B^T B^{-1})^T = (-\Lambda_1\delta(r) + (C_B^T B^{-1})^T, \Lambda_2\delta(r) + (C_B^T B^{-1})^T),$$

where, $\Lambda_k^T = (\lambda_{0k}, \lambda_{1k}, \ldots, \lambda_{n+1,k})$, $k = 1, 2$. Also, it can be shown that $\Lambda_1 = \Lambda_2$. Thus,

$$\tilde{P}_n^D(x) = \sum_{j=0}^{n} x^j \tilde{a}'_j = \sum_{j=0}^{n} x^j(-b_j\delta(r) + a_{j_s}, b_j\delta(r) + a_{j_s}),$$

and $[\tilde{P}_n^D]^1(x) = [\tilde{f}]^1(x)$. □

Lemma 2.3.6. *The D-approximating polynomial satisfies both properties 1 and 2 of Lodwick and Santos properties.*

Proof. From definition of fuzzy polynomial (2.3.1), for $\alpha \in [0, 1]$ we have

$$[\tilde{P}_n^D]_-^\alpha(x) = \underline{v}_x(\alpha) = \sum_{x^j \geq 0} x^j \underline{a_j}(\alpha) + \sum_{x^j < 0} x^j \overline{a_j}(\alpha)$$

and

$$[\tilde{P}_n^D]_+^\alpha(x) = \overline{v}_x(\alpha) = \sum_{x^j \geq 0} x^j \overline{a_j}(\alpha) + \sum_{x^j < 0} x^j \underline{a_j}(\alpha)$$

therefore $[\tilde{P}_n^D]_-^\alpha(x) \leq [\tilde{P}_n^D]_+^\alpha(x)$, and it yields property 1. Also property 2 is clearly instates. □

2.3.4 Special Cases of $D-$Approximation

We can observe that:

> $D-$approximation of a fuzzy function is not unique, but all of these approximations have the same $\tilde{\beta}$-distance, like points on circumference of a circle which are equidistant from center.

Let $\mathcal{G}_{\mathcal{X}}(\tilde{f})$ be the set of all $D-$approximations of $\tilde{f}$ on the set of points $\mathcal{X}$.

Since we have used standard difference, fuzziness and ambiguity of our approximation $\tilde{P}_n(x)$ are not less than fuzziness and ambiguity of function $\tilde{f}(x)$ respectively. Thus we define universal and SAF approximations which are two special cases of $D-$approximation.

2.3.5 Universal Approximation of a Fuzzy Function

As we said before ambiguity of our approximation $\tilde{P}_n(x)$ is not less than ambiguity of function $\tilde{f}(x)$, thus we define universal approximation out of $\mathcal{G}_{\mathcal{X}}(\tilde{f})$ to minimize difference between ambiguity of $\tilde{P}_n(x)$ and ambiguity of $\tilde{f}(x)$.

Definition 2.3.10. A fuzzy polynomial $\tilde{P}_n^U$ is *universal approximation* of $\tilde{f}$, if $\tilde{P}_n^U \in \mathcal{G}_{\mathcal{X}}(\tilde{f})$ and it satisfies the following equation:

$$\max_{1\le i\le m} Amb(\tilde{f}_i - \tilde{P}_n^U(x_i)) = \min_{\tilde{P}_n\in\mathcal{G}_{\mathcal{X}}(\tilde{f})} \max_{1\le i\le m} Amb(\tilde{f}_i - \tilde{P}_n(x_i)). \tag{2.3.12}$$

If $\tilde{P}_n(x)$ is a $D-$ approximation of $\tilde{f}$ on $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, then

$$\begin{aligned}\max_{1\le i\le m} Amb(\tilde{f}_i - \tilde{P}_n(x_i)) &= Amb(\tilde{f}_i) + \max_{1\le i\le m} Amb(\tilde{P}_n(x_i))\\ &= Amb(\tilde{f}_i) + \mathbf{A}_{\mathcal{X}}(\tilde{P}_n).\end{aligned}$$

Thus $\tilde{P}_n^S$ is a universal approximation of $\tilde{f}$, if it satisfies the following equation:

$$\mathbf{A}_{\mathcal{X}}(\tilde{P}_n^U) = \min_{\tilde{P}_n\in\mathcal{G}_{\mathcal{X}}(\tilde{f})} \mathbf{A}_{\mathcal{X}}(\tilde{P}_n). \tag{2.3.13}$$

Corollary 2.3.7. *The universal approximating polynomial satisfies both properties 1 and 2 of Lodwick and Santos properties.*

Proof. Proof is straightforward, because universal approximation is a special case of D-approximation. □

2.3.6 Numerical Examples

Example 2.3.1. *Let $m = 3$ and $n = 1$,*

x	1	2	3
$\tilde{f}(x)$	$(r, 2-r)$	$(r+1, 3-r)$	$(r+2, 4-r)$

$\mathcal{G}_{\mathcal{X}}(\tilde{f}) = \{\tilde{P}_1(x), \tilde{Q}_1(x), \tilde{R}_1(x), \tilde{S}_1(x)\}$, where

$$\begin{aligned}
\tilde{P}_1(x) &= x(r, 2-r) + (3r-3, 3-3r) &&, \quad \tilde{\beta} = (r-1, -r+1),\\
\tilde{Q}_1(x) &= x(2r-1, 3-2r) + (6r-6, 6-6r) &&, \quad \tilde{\beta} = (r-1, -r+1),\\
\tilde{R}_1(x) &= x(\tfrac{2}{5}r + \tfrac{3}{5}, \tfrac{7}{5} - \tfrac{2}{5}r) + (\tfrac{6}{5}r - \tfrac{6}{5}, \tfrac{6}{5} - \tfrac{6}{5}r) &&, \quad \tilde{\beta} = (r-1, -r+1),\\
\tilde{S}_1(x) &= x(\tfrac{1}{2}r + \tfrac{1}{2}, \tfrac{3}{2} - \tfrac{1}{2}r) + (\tfrac{3}{2}r - \tfrac{3}{2}, \tfrac{3}{2} - \tfrac{3}{2}r) &&, \quad \tilde{\beta} = (r-1, -r+1),
\end{aligned}$$

$\mathbf{A}_{\mathcal{X}}(\tilde{P}_1) = 2$, $\mathbf{A}_{\mathcal{X}}(\tilde{Q}_1) = 4$, $\mathbf{A}_{\mathcal{X}}(\tilde{R}_1) = 0.8$ and $\mathbf{A}_{\mathcal{X}}(\tilde{S}_1) = 1$. According to (2.3.13), $\tilde{R}_1(x)$ is universal approximation of $\tilde{f}$ on $\mathcal{X}$.

Example 2.3.2. *Let $m = 6$ and $n = 2$,*

x	$\tilde{f}(x)$
-1	(15+r,17-r)
-0.9	(11.96+r,13.96-r)
-0.8	(9.24+r,11.24-r)
0.8	(9.24+r,11.24-r)
0.9	(11.96+r,13.96-r)
1	(15+r,17-r)

$\mathcal{G}_{\mathcal{X}}(\tilde{f}) = \{\tilde{P}_2(x)\}$, where

$$\tilde{P}_2(x) = x^2(16 - \frac{25}{9}(1-r), 16 + \frac{25}{9}(1-r)) + x(r-1, -r+1) + (-\frac{5}{2}(1-r), \frac{5}{2}(1-r)),$$

$$\tilde{\beta} = (r-1, -r+1).$$

Hence $P_2(x)$ is universal approximation of $\tilde{f}$ on $\mathcal{X}$.

2.3.7 $SAF-$Approximation of a Fuzzy Function

Fuzziness of our $D-$approximating polynomials $\tilde{P}_n(x)$ are not less than fuzziness of function $\tilde{f}(x)$, thus we define SAF-approximation out of $\mathcal{G}_{\mathcal{X}}(\tilde{f})$ to minimize difference between fuzziness of $\tilde{P}_n(x)$ and fuzziness of $\tilde{f}(x)$.

Definition 2.3.11. A fuzzy polynomial $\tilde{P}_n^S$ is $SAF-$*approximation* of $\tilde{f}$, if $\tilde{P}_n^S \in \mathcal{G}_{\mathcal{X}}(\tilde{f})$ and it satisfies the following equation:

$$\max_{1 \le i \le m} Fuzz(\tilde{f}_i - \tilde{P}_n^S(x_i)) = \min_{\tilde{P}_n \in \mathcal{G}_{\mathcal{X}}(\tilde{f})} \max_{1 \le i \le m} Fuzz(\tilde{f}_i - \tilde{P}_n(x_i)). \tag{2.3.14}$$

If $\tilde{P}_n(x)$ is a $D-$ approximation of $\tilde{f}$ on $X = \{x_1, x_2, \ldots, x_m\}$, then

$$\max_{1 \le i \le m} Fuzz(\tilde{f}_i - \tilde{P}_n(x_i)) = Fuzz(\tilde{f}_i) + \max_{1 \le i \le m} Fuzz(\tilde{P}_n(x_i))$$
$$= Fuzz(\tilde{f}_i) + \mathbf{F}_{\mathcal{X}}(\tilde{P}_n).$$

Thus $\tilde{P}_n^S$ is a SAF-approximation of $\tilde{f}$, if it satisfies the following equation:

$$\mathbf{F}_{\mathcal{X}}(\tilde{P}_n^S) = \min_{\tilde{P}_n \in \mathcal{G}_{\mathcal{X}}(\tilde{f})} \mathbf{F}_{\mathcal{X}}(\tilde{P}_n). \tag{2.3.15}$$

Corollary 2.3.8. *The SAF-approximating polynomial satisfies both properties 1 and 2 of Lodwick and Santos properties.*

Proof. Proof is straightforward, because SAF-approximation is a special case of D-approximation. $\square$

2.3.8 Numerical Examples

Example 2.3.3. *Let $m = 2$ and $n = 1$,*

x_i	*1*	*2*
$\tilde{f}_i$	$(r, 2-r)$	$(r+1, 3-r)$

$$\tilde{P}_1(x) = x(2r-1, 3-2r) + (4r-4, 4-4r),$$

$$\tilde{\beta} = (r-1, -r+1).$$

$\mathcal{G}_{\mathcal{X}}(\tilde{f}) = \{\tilde{P}_1(x)\}$, Thus

$\tilde{P}_1(x)$ is $SAF-$approximation of $\tilde{f}$ on $\mathcal{X}$.

Example 2.3.4. *Let $\tilde{f}(x) = x^2(10r-9, -r+2) - x(r-1, -5r+5) + (r, -r+2)$ and $m = 3$, $n = 2$,*

x_i	*0*	*1*	*2*
$\tilde{f}_i$	$(r, -r+2)$	$(16r-14, -3r+5)$	$(51r-46, -7r+12)$

$\mathcal{G}_{\mathcal{X}}(\tilde{f}) = \{\tilde{P}_2(x), \tilde{Q}_2(x), \tilde{R}_2(x)\}$, *where*

$$\tilde{P}_2(x) = x^2(43r-42, -8r+9) + x(12r-12, -60r+60) + (r, -r+2),$$
$$\tilde{Q}_2(x) = x^2(29r-28, -20r+21) + x(37r-37, -33r+33) + (r, -r+2),$$
$$\tilde{R}_2(x) = x^2(30r-29, -21r+22) + x(38r-38, -34r+34) + (r, -r+2),$$

Since $\boldsymbol{F}_{\mathcal{X}}(\tilde{P}_2) = 276$, $\boldsymbol{F}_{\mathcal{X}}(\tilde{Q}_2) = 266$ and $\boldsymbol{F}_{\mathcal{X}}(\tilde{R}_2) = 276$; $\tilde{Q}_2(x)$ is SAF-approximation of $\tilde{f}$ on $\mathcal{X}$ with $\tilde{\beta} = (r-1, -r+1)$.

Example 2.3.5. *Let $m = 4$, $n = 1$,*

x_i	*1*	*2*	*3*	*4*
$\tilde{f}_i$	$(r, -r+2)$	$(r+1, -2r+4)$	$(2r+1, -r+4)$	$(2r+2, -2r+6)$

$\mathcal{G}_{\mathcal{X}}(\tilde{f}) = \{\tilde{P}_1(x)\}$, *where* $\tilde{P}_1(x) = x(2r-1, -\frac{3}{2}r + \frac{5}{2}) + (\frac{21}{4}r - \frac{21}{4}, -\frac{13}{2}r + \frac{13}{2})$,

$\tilde{\beta} = (2r-2, -\frac{5}{4}r + \frac{5}{4})$.

Hence $\tilde{P}_1(x)$ *is* $SAF-$*approximation of* $\tilde{f}$ *on* $\mathcal{X}$.

2.4 Conclusion

In this chapter, we proposed a method to find e_ϕ-approximation of a fuzzy function on a finite set of distinct points in $\mathcal{TSF}(\mathbb{R})$. Unfortunately, the e_ϕ-approximation of a fuzzy function is not unique but all of the e_ϕ-approximating polynomials, have the same $\tilde{\beta}$-distance, like points on circumference of a circle which are equidistant from its center. Also, it is clear that if all fuzzy numbers are symmetric triangular fuzzy numbers then the method gives us the same e_ϕ-approximating polynomials, however ϕ varies. As it seems, the spreads of an e_ϕ-approximating polynomial become large when x is far from origin, and this is an expected property of fuzzy numbers.

Also, in this chapter we proposed a method to find $D-$approximation of a fuzzy function on a set of points in $\mathcal{F}(\mathbb{R})$. Unfortunately, the $D-$approximation of a fuzzy function is not unique too, and again all of these approximations have the same $\tilde{\beta}$-distance, like points on circumference of a circle which are equidistant from center, and since we use standard difference for fuzzy numbers, fuzziness of our approximations increases when we are far from origin.

In this chapter we introduced two approximating fuzzy polynomials as universal and SAF approximations out of all $D-$approximations to minimize ambiguity or amount of fuzziness.

Chapter 3

Best Approximation of Fuzzy Functions

In this chapter, we approximate a triangular valued fuzzy function $\tilde{f}$ on a set of m distinct real points $\mathcal{X}$ by a fuzzy valued polynomial $\tilde{P}_n$ of degree at most n defined in previous chapter by two methods, where the integers m and n, are given.

3.1 Introduction

Let $\mathcal{X}$ be a set of m distinct points $x_1, x_2, \ldots, x_m$; of $\mathbb{R}$. In this chapter, we approximate a given fuzzy function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{TSF}(\mathbb{R})$ defined on $\mathcal{X}$.

In Section 3.2 we introduce a metric on $\mathcal{TSF}(\mathbb{R})$. We introduce distance best approximation of a fuzzy function on the finite set of distinct points $\mathcal{X}$ by solving three linear programming problems. Also, we present some theorems for its existence and uniqueness. Some examples are given.

In Section 3.3, we introduce the standard best polynomial approximation of a fuzzy function and a method for computing the standard best approximation of a fuzzy function by linear programming. But does a standard best approximation of a fuzzy function always exist? We will answer this question. Finally, some examples are given in this section.

3.2 Distance Best Approximation of Fuzzy Functions

In this section, we introduce a metric on $\mathcal{TSF}(\mathbb{R})$. We introduce distance best approximation of a fuzzy function on a finite set of distinct points $\mathcal{X}$ and as we demonstrate in this section, an approach to the problem of finding the distance best approximation of a given fuzzy function $\tilde{f}$ on $\mathcal{X}$, is to express the problem as three linear programming problems.

3.2.1 Distance Best Approximation of a Fuzzy Function

In this chapter, we introduce the distance best approximation of a fuzzy function on a finite set of distinct points and we present a method to find it.

Definition 3.2.1. For $\tilde{a} = (a, \sigma, \gamma)$ and $\tilde{b} = (b, \sigma', \gamma')$, belonging to $\mathcal{TSF}(\mathbb{R})$, we define a distance by

$$D(\tilde{a}, \tilde{b}) = |a - b| + |\sigma - \sigma'| + |\gamma - \gamma'|. \tag{3.2.1}$$

Lemma 3.2.1. *D is a metric on $\mathcal{TSF}(\mathbb{R})$.*

Proof. For $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{TSF}(\mathbb{R})$, we can prove the following properties easily:

1. $D(\tilde{a}, \tilde{b}) \geq 0$,

2. if $\tilde{a} \neq \tilde{b}$, then $D(\tilde{a}, \tilde{b}) > 0$,

3. $D(\tilde{a}, \tilde{b}) = D(\tilde{b}, \tilde{a})$,

4. $D(\tilde{a}, \tilde{b}) \leq D(\tilde{a}, \tilde{b}) + D(\tilde{b}, \tilde{c})$.

$\square$

We use this new metric to compare two triangular and semitriangular fuzzy numbers. We call the distance between an arbitrary fuzzy number $\tilde{a} \in \mathcal{TSF}(\mathbb{R})$ and $\tilde{0} = (0, 0, 0)$, the *norm* of $\tilde{a}$, thus the norm of a fuzzy number $\tilde{a} = (a, \sigma, \gamma) \in \mathcal{TSF}(\mathbb{R})$, is

$$\mathbf{N}(\tilde{a}) = |a| + \sigma + \gamma. \tag{3.2.2}$$

Let $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ be a set of m distinct points of $\mathbb{R}$, and $\tilde{f}_i = (f_i, \sigma_i, \gamma_i) \in \mathcal{TSF}(\mathbb{R})$ be the value of a fuzzy function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{TSF}(\mathbb{R})$, at the point x_i, $i = 1, 2, \ldots, m$.

Definition 3.2.2. A fuzzy valued polynomial $\tilde{P}^* \in \tilde{\prod}_n$, is the distance best approximation to $\tilde{f}$ on $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, if

$$\max_{i=1,2,\ldots,m} D(\tilde{P}^*(x_i), \tilde{f}_i) = \min_{\tilde{P} \in \tilde{\prod}_n} \left\{ \max_{i=1,2,\ldots,m} D(\tilde{P}(x_i), \tilde{f}_i) \right\}. \tag{3.2.3}$$

Suppose $\tilde{\beta} = (\eta, \sigma, \gamma)$, is a fuzzy number, where $\eta \geq 0$, and

$$\mathbf{N}(\tilde{\beta}) = \max_{i=1,2,\ldots,m} D(\tilde{P}(x_i), \tilde{f}_i), \tag{3.2.4}$$

that is

$$\eta + \sigma + \gamma = \max_{i=1,2,\ldots,m} \left\{ |P(x_i) - f_i| + |\underline{P}(x_i) - \sigma_i| + |\overline{P}(x_i) - \gamma_i| \right\}. \tag{3.2.5}$$

Taking

$$\eta = \max_{i=1,2,\ldots,m} |P(x_i) - f_i|,$$
$$\sigma = \max_{i=1,2,\ldots,m} |\underline{P}(x_i) - \sigma_i|,$$
$$\gamma = \max_{i=1,2,\ldots,m} |\overline{P}(x_i) - \gamma_i|,$$

we have

$$\eta \geq |P(x_i) - f_i|,$$
$$\sigma \geq |\underline{P}(x_i) - \sigma_i|,$$
$$\gamma \geq |\overline{P}(x_i) - \gamma_i|,$$

for $i = 1, 2, \ldots, m$, and we have three linear programming problems to be solved:

$$(I) \quad \begin{cases} \min \eta \\ s.t. \\ \eta + \sum_{j=0}^{n} a_j x_i^j \geq f_i \quad , \quad i = 1, 2, \ldots, m, \\ \eta - \sum_{j=0}^{n} a_j x_i^j \geq -f_i \quad , \quad i = 1, 2, \ldots, m, \end{cases} \tag{3.2.6}$$

$$(II) \quad \begin{cases} \min \sigma \\ s.t. \\ \sigma + \sum_{j=0}^{n} \underline{a}_j x_i^j \geq \sigma_i \quad , \quad i = 1, 2, \ldots, m, \\ \sigma - \sum_{j=0}^{n} \underline{a}_j x_i^j \geq -\sigma_i \quad , \quad i = 1, 2, \ldots, m, \end{cases} \tag{3.2.7}$$

$$(III) \quad \begin{cases} \min \gamma \\ s.t. \\ \gamma + \sum_{j=0}^{n} \overline{a}_j x_i^j \geq \gamma_i \quad , \quad i = 1, 2, \ldots, m, \\ \gamma - \sum_{j=0}^{n} \overline{a}_j x_i^j \geq -\gamma_i \quad , \quad i = 1, 2, \ldots, m. \end{cases} \tag{3.2.8}$$

Using (I),(II) and (III), we find three crisp best approximations on data (x_i, f_i) , (x_i, σ_i) and (x_i, γ_i) for $i = 1, 2, \ldots, m$, call them P, $\underline{P}$ and $\overline{P}$, respectively. Thus the distance best approximation to $\tilde{f}$ out of $\tilde{\prod}_n$ on $\mathcal{X}$, at point x, is $\tilde{P}(x) = (P(x), \underline{P}(x), \overline{P}(x))$. Also the error of this approximation, is $\tilde{\beta} = (\eta, \sigma, \gamma)$. In general, we can use the following definition for the distance best approximation to $\tilde{f}$ out of $\tilde{\prod}_n$ on $\mathcal{X}$, at point x,

$$(P(x), \max\{0, \underline{P}(x), -\overline{P}(x)\}, \max\{0, -\underline{P}(x), \overline{P}(x)\}). \tag{3.2.9}$$

Solving (I), (II) and (III), we have three independent polynomials P, $\underline{P}$ and $\overline{P}$ of degree at most n. But, applying (3.2.9), right and left spreads of fuzzy valued polynomial may be piecewise polynomials of degree at most n.

3.2.2 Existence and Uniqueness of Distance Best Approximation

Theorem 3.2.2. *Distance best approximation of a fuzzy function exists and is unique.*

Proof. Best approximation on data (x_i, f_i), for $i = 1, 2, \ldots, m$; exists and is unique, because it is a crisp problem and existence and uniqueness are known [6, 24, 28], thus (I) has a unique solution. With a similar discussion it can be shown that (II) and (III) have unique solutions on data (x_i, σ_i) and (x_i, γ_i) for $i = 1, 2, \ldots, m$, respectively. Therefore, distance best approximation of a fuzzy function exists and is unique. □

Corollary 3.2.3. *Distance best approximation of a crisp function is the usual polynomial approximation.*

Proof. For $\tilde{f}(x) = (f(x), 0, 0)$, we have $\underline{P}(x) = \overline{P}(x) = 0$, because best approximation of crisp zero polynomial, is itself, thus $\tilde{P}(x) = (P(x), 0, 0)$. □

Theorem 3.2.4. *Distance best approximation of a fuzzy valued polynomial $\tilde{P}$, where $P, \underline{P}$ and $\overline{P} \in \prod_n$, on the set of distinct points, $\mathcal{X} = \{x_1, x_2, \ldots, x_{n+1}\}$, is itself.*

Proof. Solving problem (I), we obtain the best approximation to P, since $P \in \prod_n$, and it is a crisp polynomial, then best approximation to P, is itself. By the same arguments it can be shown that best approximations to $\underline{P}, \overline{P}$, are themselves. This argument implies the result. $\square$

Lemma 3.2.5. *The best approximating polynomial satisfies only the property 1 of Lodwick and Santos properties.*

Proof. Applying (3.2.9), we instate property 1, for distance best approximation, but here, there are two piecewise polynomials as the spreads of this fuzzy polynomial, for all $\alpha \in [0, 1]$, and it follows that, property 2, does not yield necessarily. $\square$

3.2.3 Numerical Examples of Distance Best Approximation

Example 3.2.1. $m = 3$ *and* $n = 2$.

x	*0*	*1*	*2*
$\tilde{f}(x)$	*(1,1,1)*	*(1,7,1)*	*(2,1,5)*

$$\tilde{P}(x) = (\tfrac{1}{2}x^2 - \tfrac{1}{2}x + 1, -6x^2 + 12x + 1, 2x^2 - 2x + 1) \text{ and } \tilde{\beta} = (0, 0, 0).$$

Example 3.2.2. $m = 5$ *and* $n = 2$.

x	*0*	*1*	*2*	*3*	*4*
$\tilde{f}(x)$	*(1,1,1)*	*(2,1,2)*	*(4,2,3)*	*(7,3,1)*	*(9,1,4)*

$$\tilde{P}(x) = (\tfrac{1}{6}x^2 + \tfrac{3}{2}x + \tfrac{2}{3}, -\tfrac{1}{3}x^2 + \tfrac{5}{3}x + \tfrac{1}{3}, \tfrac{1}{6}x^2 - \tfrac{1}{2}x + \tfrac{13}{6}) \text{ and } \tilde{\beta} = (\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{7}{6}).$$

Example 3.2.3. *$m = 3$ and $n = 2$.*

x	*0*	*2*	$\frac{5}{2}$
$\tilde{f}(x)$	$(0, 1, \frac{5}{2})$	$(2, 0, \frac{1}{2})$	$(\frac{5}{2}, 1, 0)$

$\tilde{P}(x) = (x, x^2 - \frac{5}{2}x + 1, -x + \frac{5}{2})$ *and* $\tilde{\beta} = (0, 0, 0)$.

Using the introduced method in this chapter, we obtain $P(x) = x$, $\underline{P}(x) = x^2 - \frac{5}{2}x + 1$, $\overline{P}(x) = -x + \frac{5}{2}$. Hence we must apply (3.2.9) to obtain the best approximating function $\tilde{P}(x)$. See Figures 3.1-3.2.

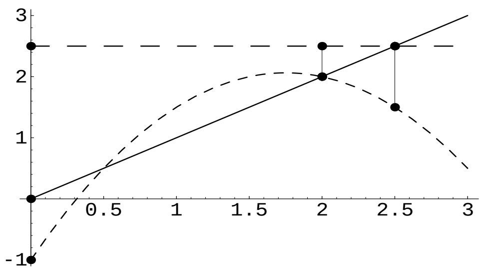


Figure 3.1 : $P(x), \underline{P}(x)$ and $\overline{P}(x)$

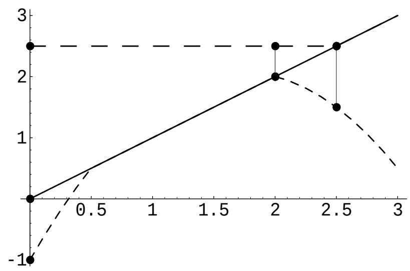


Figure 3.2 : $\tilde{P}(x)$ using (3.2.9).

Example 3.2.4. *Let us find the best approximating polynomials of degrees 1-4 and 10, related to the following data [14].*

x_i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
f_i	6	5	8	10	13	19	20	23	25	27	26	25	5	8	9	10	12	13	20	28
σ_i	2	3	5	9	9	5	8	9	5	6	7	4	3	5	7	4	6	3	6	7
γ_i	8	6	8	9	8	7	5	5	7	8	6	6	3	3	8	3	8	9	7	9

21	22	23	24	25	26	27	28	29	30
29	30	35	44	45	43	47	48	50	49
8	4	2	3	6	5	3	4	6	5
9	7	3	5	6	4	9	5	4	4

Graphs of the best approximating polynomials of degree $1-4$, are shown in Figure 3.3, and the best approximating polynomial of degree 10, on these data, is shown in Figure 3.4.

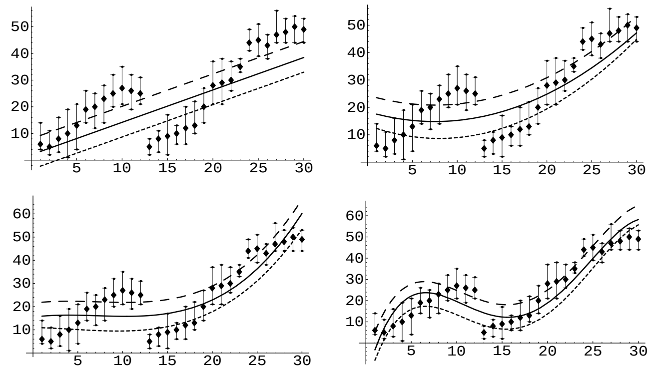


Figure 3.3 : Graphs of fuzzy polynomials of degrees $1, 2, 3, 4$.

The errors of approximating polynomials of degrees 1 to 4, are $\tilde{\beta}_1 = (12.8214, 3.5, 3)$, $\tilde{\beta}_2 = (11.797, 3.2305, 3)$, $\tilde{\beta}_3 = (11.1666, 2.9982, 3)$ and $\tilde{\beta}_4 = (9.1934, 2.8918, 2.9558)$ respectively, and error of approximating polynomial of degree 10, is
$\tilde{\beta}_{10} = (6.0562, 2.3405, 2.5644)$.

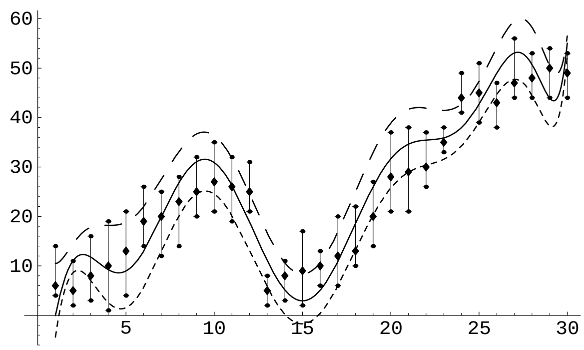


Figure 3.4 : Graph of fuzzy polynomial of degree 10.

3.3 Standard Best Approximation of a Fuzzy Polynomial

In this section, we introduce the standard best polynomial approximation of a fuzzy function and we demonstrate in this that an approach to the problem of finding the standard best approximation of a given fuzzy function $\tilde{f}$ on $\mathcal{X}$, is to express the problem as three linear programming problems. Also, we show the existence of standard best approximation of a fuzzy function.

3.3.1 Standard Best Approximation of a Fuzzy Function

Let $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$ be a set of m points in $\mathbb{R}$, and the values of a function $\tilde{f} : \mathbb{R} \longrightarrow \mathcal{TSF}(\mathbb{R})$ at these points are $(x_i, \tilde{f}_i = (f_{i_s}, f_{i_l}, f_{i_r})) \in \mathcal{TSF}(\mathbb{R})$. That is $(x_i, \tilde{f}_i) \in \mathbb{R} \times \mathcal{TSF}(\mathbb{R})$ for $i = 1, 2, \ldots, m$ are given.

Definition 3.3.1. A polynomial $\tilde{P}^*(x)$ of degree at most n is a standard best approximation of $\tilde{f}$ at $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, if we have

$$\max_{i=1,2,\ldots,m} \mathbf{N}(\tilde{P}^*(x_i) - \tilde{f}_i) = \min_{\tilde{P}\in\tilde{\prod}_n} \left\{ \max_{i=1,2,\ldots,m} \mathbf{N}(\tilde{P}(x_i) - \tilde{f}_i) \right\}. \tag{3.3.1}$$

Now suppose that $\tilde{\beta} = (\beta, \beta_l, \beta_r)$ is a triangular fuzzy number such that $\beta \geq 0$, and

$$\mathbf{N}(\tilde{\beta}) = \max_{i=1,2,\ldots,m} \mathbf{N}(\tilde{P}(x_i) - \tilde{f}_i), \tag{3.3.2}$$

that is

$$\beta + \beta_l + \beta_r = \max_{i=1,2,\ldots,m} \left\{ |P(x_i) - f_{i_s}| + (\underline{P}(x_i) + f_{i_r}) + (\overline{P}(x_i) + f_{i_l}) \right\}. \tag{3.3.3}$$

Set

$$\beta = \max_{i=1,2,\ldots,m} |P(x_i) - f_{i_s}|,$$
$$\beta_l = \max_{i=1,2,\ldots,m}\{\underline{P}(x_i) + f_{i_r}\},$$
$$\beta_r = \max_{i=1,2,\ldots,m}\{\overline{P}(x_i) + f_{i_l}\};$$

to express the problem as some linear programming problems. Therefore

$$\beta \geq |P(x_i) - f_{i_s}|,$$
$$\beta_l \geq \underline{P}(x_i) + f_{i_r},$$
$$\beta_r \geq \overline{P}(x_i) + f_{i_l},$$

for every $i = 1, 2, \ldots, m$, and we have three problems to solve.

$$(I) \quad \begin{cases} \min \beta \\ s.t. \\ \beta + \sum_{j=0}^{n} a_j x_i^j \geq f_{i_s} \quad , \quad i = 1, 2, \ldots, m, \\ \beta - \sum_{j=0}^{n} a_j x_i^j \geq -f_{i_s} \quad , \quad i = 1, 2, \ldots, m, \end{cases} \tag{3.3.4}$$

$$(II) \quad \begin{cases} \min \beta_l \\ s.t. \\ \beta_l - \sum_{j=0}^{n} \underline{a}_j x_i^j \geq f_{i_r} \quad , \quad i = 1, 2, \ldots, m, \\ \beta_l \geq 0. \end{cases} \tag{3.3.5}$$

$$(III) \quad \begin{cases} \min \beta_r \\ s.t. \\ \beta_r - \sum_{j=0}^{n} \overline{a}_j x_i^j \geq f_{i_l} \quad , \quad i = 1, 2, \ldots, m, \\ \beta_r \geq 0. \end{cases} \tag{3.3.6}$$

Therefore, using (I),(II) and (III), we find three crisp approximations on data (x_i, f_{i_s}) , (x_i, f_{i_r}) and (x_i, f_{i_l}) for $i = 1, 2, \ldots, m$; separately by linear programming and call them $P(x)$, $\underline{P}(x)$ and $\overline{P}(x)$ respectively. Thus, standard best approximation of $\tilde{f}$ at point x, is $\tilde{P}(x) = (P(x), \underline{P}(x), \overline{P}(x))$. Also, the error of this approximation on $\mathcal{X}$, is $\tilde{\beta} = (\beta, \beta_l, \beta_r)$.

But, the fuzziness of this function may be negative, for the purpose of nonnegativity of fuzziness, we can consider the following form as standard best approximation

$$(P(x), |\underline{P}(x)|, |\overline{P}(x)|). \tag{3.3.7}$$

3.3.2 Existence of Standard Best Approximation of a Fuzzy Function

Theorem 3.3.1. *Suppose $m \geq n+1$. In solution of problems (II) and (III) we have $\beta_l = \beta_r = 0$.*

Proof. We will show that in the solution of problem (II), $\beta_l = 0$. (The same argument is confirmed for β_r.) For this purpose, we find a feasible point of (II) such that for this point $\beta_l = 0$ and because of the fact $\beta_l \geq 0$, we conclude the desired result.

We find a polynomial $\underline{P}(x) = \sum_{j=0}^{n} \underline{a}_j x_i^j$, such that

$$\sum_{j=0}^{n} \underline{a}_j x_i^j \leq -f_{i_r} \quad , \quad i = 1, 2, \ldots, m. \tag{3.3.8}$$

We choose $n+1$ points through the whole points $\{(x_i, f_{i_r}) | i = 1, \ldots, m\}$ and find the interpolating polynomial $\underline{P}(x) = \sum_{j=0}^{n} \underline{a}_j x_i^j$ on these data. Let

$$h = \max_{i=1,\ldots,m} (\underline{P}(x_i) + f_{i_r}),$$

if $h > 0$ then we can change this polynomial to new polynomial $\underline{P}(x) - h$. This new polynomial satisfies (3.3.8). Thus, the point $X = (0, \underline{a}_0, \underline{a}_1, \ldots, \underline{a}_n)$ is a feasible solution of (II) where $\underline{a}_j$'s are coefficients of obtained polynomial. $\square$

Theorem 3.3.2. *Standard best approximation of a fuzzy function exists.*

Proof. Standard best approximation of degree at most n on data (x_i, f_{i_s}), for $i = 1, 2, \ldots, m$; exists and is unique, because it is a crisp problem and existence and uniqueness are known [6, 24, 28]. Thus, (I) has a unique solution and according to Theorem 3.3.1, it is clear that $\underline{P}(x)$ and $\overline{P}(x)$ exist. Therefore, standard best approximation of a fuzzy function exists. $\square$

Lemma 3.3.3. *Value of standard best approximation of $\tilde{f}$ at point x_i is*

$$(P(x_i), -\underline{P}(x_i), -\overline{P}(x_i)). \tag{3.3.9}$$

Proof. From Theorem 3.3.1 it is clear that in solution of (3.3.5), $\underline{P}(x_i) \leq -f_{i_r} \leq 0$, thus $|\underline{P}(x_i)| = -\underline{P}(x_i)$ and similarly $|\overline{P}(x_i)| = -\overline{P}(x_i)$. $\square$

Lemma 3.3.4. *The standard best approximating polynomial satisfies both properties 1 and 2 of Lodwick and Santos properties.*

Proof. $P, \underline{P}$ and $\overline{P}$ are three independent polynomials and it yields property 2 for standard best approximation also for $\alpha \in [0, 1]$ we have

$$[\tilde{P}_n^*]_-^\alpha(x) = [P - |\underline{P}|]_-^\alpha(x) \leq [P + |\overline{P}|]_+^\alpha(x) = [\tilde{P}_n^*]_+^\alpha(x),$$

and it proves the property 1. $\square$

3.3.3 Numerical Examples of Standard Best Approximation

Example 3.3.1. $m = 3$ *and* $n = 2$.

x	*0*	*1*	*2*
$\tilde{f}(x)$	*(1,1,1)*	*(2,1,1)*	*(5,1,1)*

$P(x) = x^2 + 1, \underline{P}(x) = -1, \overline{P}(x) = -1$

$$\tilde{P}(x) = (x^2 + 1, 1, 1) \quad , \quad \tilde{\beta} = (0, 0, 0).$$

Example 3.3.2. *$m = 5$ and $n = 2$.*

x	*0*	*0.5*	*1*	*1.5*	*2*
$\tilde{f}(x)$	*(1,1,1)*	*(1,2,1)*	*(2,1,2)*	*(3,2,2)*	*(1,1,2)*

$P(x) = -\frac{4}{3}x^2 + \frac{10}{3}x + \frac{1}{3}, \underline{P}(x) = -2, \overline{P}(x) = -2$

$$\tilde{P}(x) = (-\frac{4}{3}x^2 + \frac{10}{3}x + \frac{1}{3}, 2, 2) \quad , \quad \tilde{\beta} = (\frac{2}{3}, 0, 0).$$

3.4 Conclusion

In this chapter, we proposed two methods to find the best approximation of a fuzzy function, one method gives us the distance best approximation and the other one gives us the standard best approximation. Also, we showed that distance best approximation of a fuzzy function on a discrete set of points exists and is unique. Also, the distance best approximation to a fuzzy valued polynomial of degree at most n with left and right polynomial spreads, on a set consisting of $n + 1$ distinct points, is itself. In this chapter, we showed existence of the standard best approximation of a fuzzy function on a set of points.

Chapter 4

The Nearest Approximation of a LR Fuzzy Number

In some applications of fuzzy logic such as control theory, it may be better to use the same fuzzy numbers. Obviously, if we use a defuzzification rule which replaces a fuzzy set by a single number, we generally lose too many important information. Also, an interval approximation for fuzzy numbers is considered in [16], where a fuzzy computation problem is converted into interval arithmetic problem. But, in this case, we lose the fuzzy central concept. Even in some works such as [2, 21, 22], we solve an optimization problem to obtain the nearest trapezoidal fuzzy number which is related to an arbitrary fuzzy number however in these cases it is not guaranteed to have the same modal value (or interval).

4.1 Introduction

In this chapter, we use value and ambiguity of fuzzy numbers the same as [7, 32]. For almost all distances we can find two different triangular or trapezoidal fuzzy numbers $\tilde{u}, \tilde{v}$ which $\tilde{u} \approx \tilde{v}$ but actually $\tilde{u} \neq \tilde{v}$.

The structure of this chapter is as follows. In Section 4.2, the basic concepts of our work are introduced. In Section 4.3, we introduce a metric D on $\mathcal{TF}(\mathbb{R})$, which is a pseudo-metric on $\mathcal{F}(\mathbb{R})$. In Sections 4.4 and 4.5, near and the nearest trapezoidal fuzzy numbers to an arbitrary generalized left-right fuzzy number are introduced and simple methods for computing them, are presented. In Section 4.6, the nearest trapezoidal approximation of a fuzzy number is introduced. In Section 4.7, any power of a trapezoidal fuzzy number is approximated with a trapezoidal one, and in Section 4.8, we approximate multiplication of two trapezoidal fuzzy numbers with a trapezoidal one. Finally, in Section 4.9, some numerical examples are given.

4.2 Basic Concepts

Definition 4.2.1. A fuzzy set $\tilde{u}$ is called a trapezoidal shape fuzzy number, or generalized left-right fuzzy number (LR fuzzy number) if its membership function satisfy the following [13]

$$\mu_{\tilde{u}}(x) = \begin{cases} l_{\tilde{u}}(x) & , \quad l \leq x \leq \underline{m}, \\ 1 & , \quad \underline{m} \leq x \leq \overline{m}, \\ r_{\tilde{u}}(x) & , \quad \overline{m} \leq x \leq r, \\ 0 & , \quad otherwise, \end{cases}$$

where $l_{\tilde{u}}(x)$ is the left spread membership function that is an increasing continuous function on $[l, \underline{m}]$ and $r_{\tilde{u}}(x)$ is the right spread membership function that is a decreasing continuous function on $[\overline{m}, r]$ such that

$$l_{\tilde{u}}(l) = \begin{cases} 1, & l = \underline{m}; \\ 0, & l \neq \underline{m}; \end{cases} \quad , \quad r_{\tilde{u}}(r) = \begin{cases} 1, & \overline{m} = r; \\ 0, & \overline{m} \neq r, \end{cases}$$

and $l_{\tilde{u}}(\underline{m}) = r_{\tilde{u}}(\overline{m}) = 1$.

Definition 4.2.2. Let s be a regular reducing function. For a nonnegative integer $j \geq 1$, we define j^{th}-*source number* with respect to s, by

$$I_j = \int_0^1 r^j s(r) dr. \tag{4.2.1}$$

Specially we call I_1, the source number with respect to s.

Definition 4.2.3. Let for positive integer $m > 0$, the m-th derivation of $\overline{u}$ and $\underline{u}$ (i.e. $\overline{u}^{(m)}$ and $\underline{u}^{(m)}$) exist. We define *m-validity* and *m-unvalublity* of a fuzzy number $\tilde{u}$, by the following relations,

$$V(\tilde{u}^{(m)}) = \int_0^1 s(r)[\overline{u}^{(m)}(r) + \underline{u}^{(m)}(r)]dr,$$

$$A(\tilde{u}^{(m)}) = \int_0^1 s(r)[\overline{u}^{(m)}(r) - \underline{u}^{(m)}(r)]dr.$$

4.3 Source Metric

Definition 4.3.1. For $\tilde{u}, \tilde{v} \in \mathcal{F}(\mathbb{R})$, we define ***source distance*** of $\tilde{u}$ and $\tilde{v}$ by

$$D_s(\tilde{u}, \tilde{v}) = \frac{1}{2}\big\{ |Val(\tilde{u}) - Val(\tilde{v})| + |Amb(\tilde{u}) - Amb(\tilde{v})| + d_H([\tilde{u}]^1, [\tilde{v}]^1) \big\},$$

where d_H is the Hausdorff metric, and $[.]^1$ is the 1-cut representation of a fuzzy number.

Theorem 4.3.1. *For $\tilde{u}, \tilde{v}, \tilde{w} \in \mathcal{F}(\mathbb{R})$ the source distance, D, satisfies the following properties:*

1. $D_s(\tilde{u}, \tilde{u}) = 0,$
2. $D_s(\tilde{u}, \tilde{v}) = D_s(\tilde{v}, \tilde{u}),$
3. $D_s(\tilde{u}, \tilde{w}) \leq D_s(\tilde{u}, \tilde{v}) + D_s(\tilde{v}, \tilde{w}).$

Proof. Let $\tilde{u}, \tilde{v}, \tilde{w} \in \mathcal{F}(\mathbb{R})$,

1. $d_H([\tilde{u}]^1, [\tilde{u}]^1) = 0$, thus $D_s(\tilde{u}, \tilde{u}) = 0$,
2. It is straightforward.
3. $$\begin{aligned} D_s(\tilde{u}, \tilde{v}) + D_s(\tilde{v}, \tilde{w}) &= |Val(\tilde{u}) - Val(\tilde{v})| + |Amb(\tilde{u}) - Amb(\tilde{v})| + d_H([\tilde{u}]^1, [\tilde{v}]^1) \\ &+ |Val(\tilde{v}) - Val(\tilde{w})| + |Amb(\tilde{v}) - Amb(\tilde{w})| + d_H([\tilde{v}]^1, [\tilde{w}]^1) \\ &\geq |Val(\tilde{u}) - Val(\tilde{w})| + |Amb(\tilde{u}) - Amb(\tilde{w})| + d_H([\tilde{u}]^1, [\tilde{w}]^1) = D_s(\tilde{u}, \tilde{w}). \end{aligned}$$

□

Example 4.3.1. *Let $\mu_{\tilde{u}}(x) = \chi_{\{a\}}(x)$ and $\mu_{\tilde{v}}(x) = \chi_{\{b\}}(x)$, then*

$$D_s(\tilde{u}, \tilde{v}) = \frac{1}{2}(|a - b| + |0 - 0| + |a - b|) = |a - b|.$$

Theorem 4.3.2. *For $\tilde{u}, \tilde{v}, \tilde{u}', \tilde{v}' \in \mathcal{TF}(\mathbb{R})$ and nonnegative real number k, the source distance D satisfies the following properties:*

1. $D_s(k\tilde{u}, k\tilde{v}) = kD_s(\tilde{u}, \tilde{v}),$

2. $D_s(\tilde{u}+\tilde{v},\tilde{u}'+\tilde{v}') \le D_s(\tilde{u},\tilde{u}') + D_s(\tilde{v},\tilde{v}')$.

Proof. Let $\tilde{u} = (\underline{m}_u, \overline{m}_u, \sigma_u, \gamma_u)$ thus

$$\begin{cases} \underline{u}(r) = \underline{m}_u - (1-r)\sigma_u, \\ \overline{u}(r) = \overline{m}_u + (1-r)\gamma_u, \end{cases}$$

and if $k \ge 0$ then

$$\begin{cases} \underline{k\tilde{u}}(r) = k\underline{m}_u - k(1-r)\sigma_u = k\underline{u}(r), \\ \overline{k\tilde{u}}(r) = k\overline{m}_u + k(1-r)\gamma_u = k\overline{u}(r), \end{cases}$$

thus

$$\begin{cases} Val(k\tilde{u}) = kVal(\tilde{u}), \\ Amb(k\tilde{u}) = kAmb(\tilde{u}), \end{cases}$$

and also we have, $d_H(k[\tilde{u}]^1, k[\tilde{v}]^1) = kd_H([\tilde{u}]^1, [\tilde{v}]^1)$, consequently

1. If $\tilde{u}, \tilde{v} \in \mathcal{TF}(\mathbb{R})$ and $k \ge 0$, then $D_s(k\tilde{u}, k\tilde{v}) = kD_s(\tilde{u},\tilde{v})$.

2. If $\tilde{u}, \tilde{v}, \tilde{u}', \tilde{v}' \in \mathcal{TF}(\mathbb{R})$ then

$$D_s(\tilde{u}+\tilde{v},\tilde{u}'+\tilde{v}') \le D_s(\tilde{u},\tilde{u}') + D_s(\tilde{v},\tilde{v}'),$$

   because,

$$d_H([\tilde{u}+\tilde{v}]^1, [\tilde{u}'+\tilde{v}']^1) = d_H([\tilde{u}]^1+[\tilde{v}]^1, [\tilde{u}']^1+[\tilde{v}']^1) \le d_H([\tilde{u}]^1, [\tilde{u}']^1) + d_H([\tilde{v}]^1, [\tilde{v}']^1),$$

   also we have

$$\begin{cases} \underline{\tilde{u}+\tilde{v}}(r) = \underline{u}(r) + \underline{v}(r), \\ \overline{\tilde{u}+\tilde{v}}(r) = \overline{u}(r) + \overline{v}(r), \end{cases}$$

and we have,

$$
\begin{aligned}
&|Val(\tilde{u}+\tilde{v}) - Val(\tilde{u}'+\tilde{v}')| \\
&= \left|\int_0^1 [\overline{\tilde{u}+\tilde{v}}(r) + \underline{\tilde{u}+\tilde{v}}(r) - \overline{\tilde{u}'+\tilde{v}'}(r) - \underline{\tilde{u}'+\tilde{v}'}(r)]s(r)d(r)\right| \\
&= \left|\int_0^1 [\overline{u}(r) + \underline{u}(r) - \overline{u'}(r) - \underline{u'}(r) + \overline{v}(r) + \underline{v}(r) - \overline{v'}(r) - \underline{v'}(r)]s(r)d(r)\right| \\
&\leq |Val(\tilde{u}) - Val(\tilde{u}')| + |Val(\tilde{v}) - Val(\tilde{v}')|.
\end{aligned}
$$

We have similar relation for Amb, thus the result is obtained.

□

Theorem 4.3.3. *Let $\tilde{u}, \tilde{v} \in \mathcal{TF}(\mathbb{R})$, then $D_s(\tilde{u}, \tilde{v}) = 0$ if and only if $\tilde{u} = \tilde{v}$.*

Proof. If $\tilde{u} = \tilde{v}$, from Theorem (4.3.1) we have $D_s(\tilde{u}, \tilde{v}) = 0$. Let $\tilde{u} = (\underline{m}_u, \overline{m}_u, \sigma_u, \gamma_u)$ and $\tilde{v} = (\underline{m}_v, \overline{m}_v, \sigma_v, \gamma_v)$ are two trapezoidal fuzzy numbers. If $D_s(\tilde{u}, \tilde{v}) = 0$ then

$$
\begin{cases}
a) \quad d_H([\tilde{u}]^1, [\tilde{v}]^1) = 0, \\
b) \quad Val(\tilde{u}) = Val(\tilde{v}), \\
c) \quad Amb(\tilde{u}) = Amb(\tilde{v}).
\end{cases}
$$

From (a), we have $\max\{|\overline{m}_v - \overline{m}_u|, |\underline{m}_v - \underline{m}_u|\} = 0$ and hence, $\underline{m}_u = \underline{m}_v$ and $\overline{m}_u = \overline{m}_v$. From (b) and (c),

$$
\begin{cases}
Val(\tilde{u}) + Amb(\tilde{u}) = 2\int_0^1 s(r)\overline{u}(r)dr = 2\int_0^1 s(r)\overline{v}(r)dr = Val(\tilde{v}) + Amb(\tilde{v}), \\
Val(\tilde{u}) - Amb(\tilde{u}) = 2\int_0^1 s(r)\underline{u}(r)dr = 2\int_0^1 s(r)\underline{v}(r)dr = Val(\tilde{v}) - Amb(\tilde{v}),
\end{cases}
$$

and hence,

$$
\begin{cases}
\overline{m}_u + (1-2I_1)\gamma_u = \overline{m}_v + (1-2I_1)\gamma_v, \\
\underline{m}_u - (1-2I_1)\sigma_u = \underline{m}_v - (1-2I_1)\sigma_v,
\end{cases}
$$

which implies $\gamma_u = \gamma_v$, $\sigma_u = \sigma_v$ and $\tilde{u} = \tilde{v}$. □

Corollary 4.3.4. *The source distance, D, is a metric on $\mathcal{TF}(\mathbb{R})$.*

Proof. By Theorems (4.3.3) and (4.3.1) the proof is clear. $\square$

By an example, we show that the source distance D, is a pseudo-metric on $\mathcal{F}(\mathbb{R})$. Let $\tilde{u} = (3, 6 - \frac{3\pi}{2}, 6 - \frac{3\pi}{2})$ and

$$\mu_{\tilde{v}}(x) = \begin{cases} \frac{2}{(x-3)^2+1} - 1, & 2 \leq x \leq 4, \\ 0, & otherwise, \end{cases}$$

thus $D_s(\tilde{u}, \tilde{v}) = 0$, but $\tilde{u} \neq \tilde{v}$.

Thus we can say that:

Lemma 4.3.5. *The source distance, D, is a metric on $\mathcal{TF}(\mathbb{R})$ and a pseudo-metric on $\mathcal{F}(\mathbb{R})$.*

4.4 Near Approximation of a Fuzzy Number

In this section, we introduce a method for approximating fuzzy numbers.

Definition 4.4.1. Let $\mathcal{A}$ be a subset of $\mathcal{F}(\mathbb{R})$. $\tilde{v}$, is a ***near approximation*** of an arbitrary fuzzy number $\tilde{u} \in \mathcal{F}(\mathbb{R})$ out of $\mathcal{A}$, if and only if

$$D_s(\tilde{v}, \tilde{u}) = \min_{\tilde{w} \in \mathcal{A}} D_s(\tilde{w}, \tilde{u}).$$

Let PF_n be the set of all fuzzy numbers $\tilde{v}$, where $\underline{v}(r)$ and $\overline{v}(r)$ are polynomials of degree at most n.

Let $N_{\mathcal{A}}(\tilde{u})$ be the set of all near approximations $\tilde{v}$ of a fuzzy number $\tilde{u}$ out of $\mathcal{A} \subset \mathcal{F}(\mathbb{R})$. Thus for finding $N_{\mathcal{A}}(\tilde{u})$, one should solve the following problems concurrently

$$I) \quad \min \left| \int_0^1 s(r)[\overline{u}(r) + \underline{u}(r) - \overline{v}(r) - \underline{v}(r)]dr \right|,$$

$$II) \quad \min\left|\int_0^1 s(r)[\overline{u}(r)-\underline{u}(r)-\overline{v}(r)+\underline{v}(r)]dr\right|,$$

$$III) \qquad \min\max \ \{|\overline{u}(1)-\overline{v}(1)|\,,|\underline{u}(1)-\underline{v}(1)|\}.$$

These problems are equivalent to the following linear programming problems (see Appendix):

$$\begin{cases} \min z_1 = \alpha^+ + \alpha^- \\ s.t. \\ \alpha^+ - \alpha^- + Val(\tilde{v}) = Val(\tilde{u}), \\ \alpha^+, \alpha^- \geq 0, \end{cases} \tag{4.4.1}$$

$$\begin{cases} \min z_2 = \beta^+ + \beta^- \\ s.t. \\ \beta^+ - \beta^- + Amb(\tilde{v}) = Amb(\tilde{u}), \\ \beta^+, \beta^- \geq 0, \end{cases} \tag{4.4.2}$$

$$\begin{cases} \min z_3 = \theta \\ s.t. \\ \qquad \overline{v}(1) + \theta \geq \overline{u}(1), \\ \qquad \underline{v}(1) + \theta \geq \underline{u}(1), \\ \qquad \overline{v}(1) - \theta \leq \overline{u}(1), \\ \qquad \underline{v}(1) - \theta \leq \underline{u}(1), \\ \qquad \theta \geq 0. \end{cases} \tag{4.4.3}$$

Theorem 4.4.1. *$\tilde{v}$ is a near approximation of fuzzy number $\tilde{u}$, if and only if*

$$\begin{cases} a) \quad \overline{v}(1) = \overline{u}(1), \\ b) \quad \underline{v}(1) = \underline{u}(1), \\ c) \quad Val(\tilde{v}) = Val(\tilde{u}), \\ d) \quad Amb(\tilde{v}) = Amb(\tilde{u}). \end{cases}$$

Proof. $(\theta, \overline{v}(1), \underline{v}(1)) = (0, \overline{u}(1), \underline{u}(1))$ is optimal solution of (4.4.3). $(\alpha^+, \alpha^-, Val(\tilde{v})) = (0, 0, Val(\tilde{u}))$ and $(\beta^+, \beta^-, Amb(\tilde{v})) = (0, 0, Amb(\tilde{u}))$ are optimal solutions of (4.4.1) and (4.4.2), respectively. □

Let $\tilde{v} = (\sum_{j=0}^{n} a_j r^j, \sum_{j=0}^{n} b_j r^j) \in PF_n$ is a near approximation of an arbitrary fuzzy number $\tilde{u}$, it can be obtained from the following systems:

$$\begin{cases} \sum_{j=0}^{n} a_j I_j = \frac{1}{2}(Val(\tilde{u}) - Amb(\tilde{u})), \\ \sum_{j=0}^{n} a_j = \underline{u}(1), \end{cases} \tag{4.4.4}$$

and

$$\begin{cases} \sum_{j=0}^{n} b_j I_j = \frac{1}{2}(Val(\tilde{u}) + Amb(\tilde{u})), \\ \sum_{j=0}^{n} b_j = \overline{u}(1). \end{cases} \tag{4.4.5}$$

Lemma 4.4.2. *Let I_j be the j^{th}-source number, then $0 \leq \ldots < I_2 < I_1 < I_0 = \frac{1}{2}$,*

Proof. See Appendix. □

Lemma 4.4.3. *The near trapezoidal approximation of an arbitrary fuzzy number $\tilde{u}$, is unique, i.e. $|N_{\mathcal{TF}(\mathbb{R})}(\tilde{u})| = 1$.*

Proof. Let $n = 1$, therefore $\underline{v} = (\underline{v}(1) - \underline{v}(0))r + \underline{v}(0)$ also $\overline{v} = (\overline{v}(1) - \overline{v}(0))r + \overline{v}(0)$. By solving (4.4.4) and (4.4.5) we have

$$\overline{v}(0) = \frac{Val(\tilde{u}) + Amb(\tilde{u}) - 2\overline{u}(1)I_1}{1 - 2I_1} \quad , \quad \underline{v}(0) = \frac{Val(\tilde{u}) - Amb(\tilde{u}) - 2\underline{u}(1)I_1}{1 - 2I_1}.$$

which defines a unique trapezoidal fuzzy number. □

It can be proved that the near approximation of a trapezoidal fuzzy number is itself, the near approximation of a crisp number a is itself, and the near approximation of an interval $[c, d]$ is also itself.

4.5 The Nearest Approximation of a Fuzzy Quantity

In this section, we introduce the nearest approximation of a fuzzy number out of PF_n.

Definition 4.5.1. $\tilde{v}^* \in PF_n$, is the nearest approximation of an arbitrary fuzzy number $\tilde{u} \in \mathcal{F}(\mathbb{R})$ if and only if,

1. $\tilde{v}^* \in N_{PF_n}(\tilde{u})$. i.e. $\tilde{v}^*$ is a near approximation of $\tilde{u}$.

2. If $n \geq 2$ then,

$$D_n^*(\tilde{v}^*, \tilde{u}) = \min_{\tilde{v} \in N_{PF_n}(\tilde{u})} D_n^*(\tilde{v}, \tilde{u}),$$

   where,

$$D_n^*(\tilde{v}, \tilde{u}) = \sum_{m=1}^{n-1} \{|V(\tilde{v}^{(m)}) - V(\tilde{u}^{(m)})| + |A(\tilde{v}^{(m)}) - A(\tilde{u}^{(m)})|\}.$$

We denote the set of all the nearest approximations of a fuzzy number $\tilde{u}$, out of PF_n, by $N^*_{PF_n}(\tilde{u})$.

Lemma 4.5.1. *Let $\tilde{u} \in \mathcal{F}(\mathbb{R})$, then a near trapezoidal approximation of $\tilde{u}$ is the nearest trapezoidal fuzzy number of $\tilde{u}$, i.e.*

$$\tilde{v}^* \in N_{PF_1(\mathbb{R})}(\tilde{u}) \Longleftrightarrow \tilde{v}^* \in N^*_{PF_1(\mathbb{R})}(\tilde{u}).$$

Proof. By Definition 4.5.1 and Lemma 4.4.3 proof is clear. □

Now let $n \geq 2$. For the purpose of obtaining the nearest approximation, one should solve the following linear programming problems for $m = 1, \ldots, n-1$, concurrently

$$\begin{cases} \min w_1 = \alpha_m^+ + \alpha_m^- \\ s.t. \\ \alpha_m^+ - \alpha_m^- + V(\tilde{v}^{(m)}) = V(\tilde{u}^{(m)}), \\ \alpha_m^+, \alpha_m^- \geq 0, \end{cases} \tag{4.5.1}$$

and

$$\begin{cases} \min w_2 = \beta_m^+ + \beta_m^- \\ s.t. \\ \beta_m^+ - \beta_m^- + A(\tilde{v}^{(m)}) = A(\tilde{u}^{(m)}), \\ \beta_m^+, \beta_m^- \geq 0. \end{cases} \tag{4.5.2}$$

Theorem 4.5.2. *Let $n \geq 2$ and for an arbitrary fuzzy number $\tilde{u}$; $\tilde{v} \in N_{PF_n}(\tilde{u})$, then $\tilde{v}$ is the nearest approximation of $\tilde{u}$, if and only if for $m = 1, \ldots, n-1$, we have*

$$\begin{cases} a') \qquad V(\tilde{v}^{(m)}) = V(\tilde{u}^{(m)}), \\ b') \qquad A(\tilde{v}^{(m)}) = A(\tilde{u}^{(m)}). \end{cases}$$

Proof. Straightforward . ☐

Therefore, we should solve the two following system of equations:

$$\begin{cases} \overline{v}(1) & = & \overline{u}(1), \\ \int_0^1 s(r)\overline{v}(r)dr & = & \frac{1}{2}(Val(\tilde{u}) + Amb(\tilde{u})), \\ \int_0^1 s(r)\overline{v}^{(m)}(r)dr & = & \frac{1}{2}(V(\tilde{u}^{(m)}) + A(\tilde{u}^{(m)})), \qquad m = 1, \ldots, n-1, \end{cases} \tag{4.5.3}$$

and

$$\begin{cases} \underline{v}(1) & = & \underline{u}(1), \\ \int_0^1 s(r)\underline{v}(r)dr & = & \frac{1}{2}(Val(\tilde{u}) - Amb(\tilde{u})), \\ \int_0^1 s(r)\underline{v}^{(m)}(r)dr & = & \frac{1}{2}(V(\tilde{u}^{(m)}) - A(\tilde{u}^{(m)})), \qquad m = 1, \ldots, n-1. \end{cases} \tag{4.5.4}$$

The functions $\overline{v}(r)$ and $\underline{v}(r)$, obtained from solutions of (4.5.3) and (4.5.4), should be monotonically decreasing and monotonically increasing, respectively.

4.6 The Nearest Trapezoidal Fuzzy Number

We observed that the near approximation of a fuzzy number out of $\mathcal{TF}(\mathbb{R})$ is also the nearest approximation of that fuzzy number. In this section, we specially introduce the nearest trapezoidal approximation of a fuzzy number.

Definition 4.6.1. The $\tilde{u}^* \in \mathcal{TF}(\mathbb{R})$, is the nearest trapezoidal fuzzy number to an arbitrary LR fuzzy number $\tilde{v}$ if and only if

$$D_s(\tilde{u}^*, \tilde{v}) = \min_{\tilde{u} \in \mathcal{TF}(\mathbb{R})} D_s(\tilde{u}, \tilde{v}).$$

Theorem 4.6.1. *The $\tilde{u}^* \in \mathcal{TF}(\mathbb{R})$, is the nearest trapezoidal fuzzy number to an arbitrary LR fuzzy number $\tilde{w}$ if and only if*

$$D_s(\tilde{u}^*, \tilde{w}) = 0.$$

Proof. If $D_s(\tilde{u}^*, \tilde{w}) = 0$ then for all $\tilde{u} \in \mathcal{TF}(\mathbb{R})$ we have $D_s(\tilde{u}, \tilde{w}) \geq D_s(\tilde{u}^*, \tilde{w})$, i.e. $\tilde{u}^*$ is the nearest trapezoidal fuzzy number to $\tilde{w}$. Conversely, If $D_s(\tilde{u}^*, \tilde{w}) = \beta$ then we show that $\beta = 0$. Suppose $\beta > 0$. We will show that there is a $\tilde{v} \in \mathcal{TF}(\mathbb{R})$ such that $D_s(\tilde{v}, \tilde{w}) < D_s(\tilde{u}^*, \tilde{w})$. Let for fuzzy number $\tilde{w}$, the quantities $Val(\tilde{w})$ and $Amb(\tilde{w})$ with respect to source function s, are specified. We want to find a trapezoidal fuzzy number which has the same value and ambiguity as $\tilde{w}$, also modal values of both fuzzy numbers are the same, with these properties we will have $D_s(\tilde{v}, \tilde{w}) = 0$. Let $\tilde{v}$

be a trapezoidal fuzzy number $(\underline{m}_v, \overline{m}_v, \sigma_v, \gamma_v)$. We want to find $\underline{m}_v$, $\overline{m}_v$, σ_v and γ_v. It is clear that

$$\begin{cases} \underline{v}(r) = \underline{m}_v - (1-r)\sigma_v, \\ \overline{v}(r) = \overline{m}_v + (1-r)\gamma_v. \end{cases}$$

Thus, we should have

$$\begin{cases} \int_0^1 s(r)[\overline{v}(r) + \underline{v}(r)]dr = Val(\tilde{w}), \\ \int_0^1 s(r)[\overline{v}(r) - \underline{v}(r)]dr = Amb(\tilde{w}), \end{cases}$$

and

$$\begin{cases} \frac{1}{2}(Val(\tilde{w}) + Amb(\tilde{w})) = \int_0^1 s(r)\overline{v}(r)dr = \overline{m}_v \int_0^1 s(r)dr - \gamma_v \int_0^1 s(r)(1-r)dr, \\ \frac{1}{2}(Val(\tilde{w}) - Amb(\tilde{w})) = \int_0^1 s(r)\underline{v}(r)dr = \underline{m}_v \int_0^1 s(r)dr + \sigma_v \int_0^1 s(r)(1-r)dr, \end{cases}$$

and hence,

$$\begin{cases} Val(\tilde{w}) + Amb(\tilde{w}) = \overline{m}_v + (1-2I_1)\gamma_v, \\ Val(\tilde{w}) - Amb(\tilde{w}) = \underline{m}_v - (1-2I_1)\sigma_v. \end{cases}$$

Thus,

$$\begin{cases} \underline{m}_v & = & \underline{m}_w, \\ \overline{m}_v & = & \overline{m}_w, \\ \sigma_v & = & \frac{\underline{m}_w - (Val(\tilde{w}) - Amb(\tilde{w}))}{1-2I_1}, \\ \gamma_v & = & \frac{Val(\tilde{w}) + Amb(\tilde{w}) - \overline{m}_w}{1-2I_1}. \end{cases} \tag{4.6.1}$$

It remains to show that $\tilde{v} = (\underline{m}_v, \overline{m}_v, \sigma_v, \gamma_v)$ defined by (4.6.1), is well defined, i.e. $\sigma_v, \gamma_v \geq 0$. It is clear that

$$1 - 2I_1 = 1 - 2\psi \int_0^1 s(r)dr = 1 - \psi,$$

where $\psi \in (0, 1)$, and

$$Val(\tilde{w}) + Amb(\tilde{w}) = 2\int_0^1 s(r)\overline{v}(r)dr = 2\overline{v}(\phi)\int_0^1 s(r)dr = \overline{v}(\phi) \geq \overline{m}_v,$$

$$Val(\tilde{w}) - Amb(\tilde{w}) = 2\int_0^1 s(r)\underline{v}(r)dr = 2\underline{v}(\tau)\int_0^1 s(r)dr = \underline{v}(\tau) \leq \underline{m}_v,$$

where $\tau, \phi \in (0,1)$, thus

$$\sigma_v = \frac{\underline{m}_w - (Val(\tilde{w}) - Amb(\tilde{w}))}{1-2I_1} \geq \frac{\underline{m}_v - \underline{m}_v}{1-2I_1} \geq 0,$$

$$\gamma_v = \frac{Val(\tilde{w}) + Amb(\tilde{w}) - \overline{m}_w}{1-2I_1} \geq \frac{\overline{m}_v - \overline{m}_v}{1-2I_1} \geq 0,$$

and it completed the proof. □

Theorem 4.6.2. *The nearest trapezoidal fuzzy number related to an arbitrary LR fuzzy number is unique.*

Proof. Let $\tilde{u}_1^*, \tilde{u}_2^*$ be two nearest trapezoidal fuzzy numbers to an arbitrary LR fuzzy number $\tilde{w}$, i.e. $D_s(\tilde{u}_1^*, \tilde{w}) = 0$ and $D_s(\tilde{u}_2^*, \tilde{w}) = 0$. By Theorem (4.3.1)

$$0 \leq D_s(\tilde{u}_1^*, \tilde{u}_2^*) \leq D_s(\tilde{u}_1^*, \tilde{w}) + D_s(\tilde{w}, \tilde{u}_2^*) = 0$$

thus $D_s(\tilde{u}_1^*, \tilde{u}_2^*) = 0$ and from Theorem (4.3.3) it is clear that $\tilde{u}_1^* = \tilde{u}_2^*$. □

Corollary 4.6.3. *$\tilde{u}$, is the nearest trapezoidal fuzzy number to an arbitrary LR fuzzy number $\tilde{v}$ if and only if*

$$\begin{cases} d_H([\tilde{u}]^1, [\tilde{v}]^1) = 0, \\ Val(\tilde{u}) = Val(\tilde{v}), \\ Amb(\tilde{u}) = Amb(\tilde{v}). \end{cases}$$

Example 4.6.1. *Let $s(r) = r$ and*

$$\mu_{\tilde{u}}(x) = \begin{cases} \log_3(x) & , \quad 1 \leq x \leq 3, \\ 1 & , \quad 3 \leq x \leq 4, \\ \log_3(7-x) & , \quad 4 \leq x \leq 6, \\ 0 & , \quad otherwise. \end{cases}$$

Therefore $\tilde{v} = (3, 4, 2.55812, 2.55812)$ is the nearest trapezoidal approximation of $\tilde{u}$, see Figure 4.1.

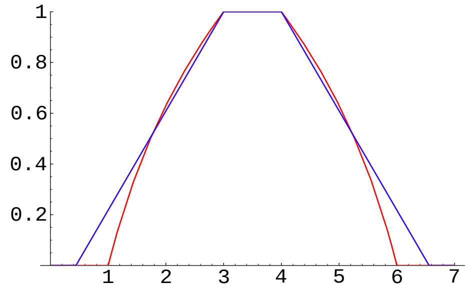


Figure 4.1 : $\tilde{v}$ is the nearest trapezoidal fuzzy number to $\tilde{u}$

The nearest trapezoidal fuzzy number to $\tilde{u}$ in this example using the introduced method in [2] is $(2.80092, 3.80092, 1.96089, 1.96089)$, see Figure 4.2.

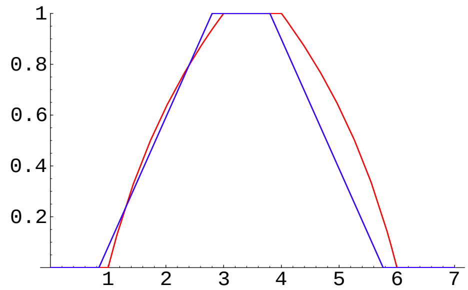


Figure 4.2 : The nearest trapezoidal fuzzy number to $\tilde{u}$ using method of [2]

We compare four methods (1) used in this chapter - (2) used in [2] - (3) used in [21, 22] and (4) used in [7], by some examples, Table 4.1. In this table,

$$\begin{cases} a = 3\sqrt{\frac{\pi}{8}}, \\ b = (8 - 3\sqrt{2})\frac{\sqrt{\pi}}{4}, \\ c = (3\sqrt{2} - 4)\frac{\sqrt{\pi}}{4}, \\ d = (4 - \sqrt{2})\frac{3\sqrt{\pi}}{4}, \\ p = \frac{1}{4}(3\sqrt{2\pi} - 8\sqrt{\ln 2}), \\ q = -\frac{3}{2}\sqrt{\frac{\pi}{2}} + 4\sqrt{\ln 2}. \end{cases}$$

$\underline{u}(r)$ / $\overline{u}(r)$	(1)	(2)	(3)	(4)
$1-0.3\sqrt{-\ln r}$ $2+0.7\sqrt{-\ln r}$	$\underline{m}=1$ $\overline{m}=2$ $\sigma=0.56399$ $\gamma=1.31598$	$\underline{m}=0.96775$ $\overline{m}=2.07526$ $\sigma=0.46723$ $\gamma=1.09020$	$\underline{m}=1.67725$ $\overline{m}=1.67725$ $\sigma=1.60935$ $\gamma=1.60935$	$\underline{m}=0.89256$ $\overline{m}=2.10743$ $\sigma=0.37061$ $\gamma=0.86477$
3^r $7-3^r$	$\underline{m}=3$ $\overline{m}=4$ $\sigma=2.55812$ $\gamma=2.55812$	$\underline{m}=2.80092$ $\overline{m}=4.19908$ $\sigma=1.96089$ $\gamma=1.96089$	$\underline{m}=3.5$ $\overline{m}=3.5$ $\sigma=3.0095$ $\gamma=3.0095$	$\underline{m}=2.97779$ $\overline{m}=4.02221$ $\sigma=2.49146$ $\gamma=2.49146$
1 2	$\underline{m}=1$ $\overline{m}=2$ $\sigma=0$ $\gamma=0$	$\underline{m}=1$ $\overline{m}=2$ $\sigma=0$ $\gamma=0$	$\underline{m}=1.5$ $\overline{m}=1.5$ $\sigma=0.75$ $\gamma=0.75$	$\underline{m}=1$ $\overline{m}=1$ $\sigma=0$ $\gamma=1$
$r+1$ $5-3r$	$\underline{m}=2$ $\overline{m}=2$ $\sigma=1$ $\gamma=3$	$\underline{m}=2$ $\overline{m}=2$ $\sigma=1$ $\gamma=3$	$\underline{m}=2$ $\overline{m}=2$ $\sigma=2.5$ $\gamma=2.5$	$\underline{m}=2$ $\overline{m}=2$ $\sigma=1$ $\gamma=0.5$
1 $3-r$	$\underline{m}=1$ $\overline{m}=2$ $\sigma=0$ $\gamma=1$	$\underline{m}=1$ $\overline{m}=2$ $\sigma=0$ $\gamma=1$	$\underline{m}=1.75$ $\overline{m}=1.75$ $\sigma=1.25$ $\gamma=1.25$	$\underline{m}=0.66667$ $\overline{m}=2.33333$ $\sigma=1$ $\gamma=0$
$\mu-\theta\sqrt{-\ln r}$ $\mu+\theta\sqrt{-\ln r}$	$\underline{m}=\mu$ $\overline{m}=\mu$ $\sigma=a\ \theta$ $\gamma=a\ \theta$	$\underline{m}=\mu-c\ \theta$ $\overline{m}=\mu+c\ \theta$ $\sigma=(b-c)\ \theta$ $\gamma=(b-c)\ \theta$	$\underline{m}=\mu$ $\overline{m}=\mu$ $\sigma=d\ \theta$ $\gamma=d\ \theta$	$\underline{m}=\mu-q\ \theta$ $\overline{m}=\mu+q\ \theta$ $\sigma=(p-q)\ \theta$ $\gamma=(p-q)\ \theta$

Table 4.1. Numerical results of examples

As we see, for all fuzzy numbers there is no trapezoidal fuzzy number computed by method (3), also by this method a fuzzy number can only be approximated with a symmetric triangular fuzzy number. And as it seems, there is no guarantee to have an approximating fuzzy number with the same core as original fuzzy number for methods (2), (3) and (4) even the original fuzzy number is a trapezoidal one. By method (3) a real interval is approximated by a triangular fuzzy number. However by our method a crisp real interval is the nearest one to itself and can not be approximated with a triangular fuzzy number. Moreover to find the approximating fuzzy number one must check four conditions for method (2) and five conditions for method (4), however by our method it will be known by explicit relations (4.6.1).

4.7 Powers of a Trapezoidal Fuzzy Number

As an application of the nearest trapezoidal fuzzy number, we can find the nearest trapezoidal fuzzy number related to any power of a trapezoidal fuzzy number. Let $\tilde{u}$ be a nonnegative trapezoidal fuzzy number $(\underline{m}_u, \overline{m}_u, \sigma_u, \gamma_u)$. It is clear that if $\tilde{u}^n$ be the n^{th} power of $\tilde{u}$, then $\tilde{u}^n$ is a trapezoidal shaped fuzzy number where

$$\begin{cases} \underline{\tilde{u}^n}(r) = [\underline{m}_u - (1-r)\sigma_u]^n, \\ \overline{\tilde{u}^n}(r) = [\overline{m}_u + (1-r)\gamma_u]^n. \end{cases}$$

We can easily compute $Amb(\tilde{u}^n)$ and $Val(\tilde{u}^n)$, also the modal value of $\tilde{u}^n$ is the interval $[\underline{m}_u^n, \overline{m}_u^n]$.

Now let $\tilde{v}^{(n)} = (\underline{m}_v^{(n)}, \overline{m}_v^{(n)}, \sigma_v^{(n)}, \gamma_v^{(n)})$ be the nearest trapezoidal fuzzy number to $\tilde{u}^n$. Therefore,

$$\underline{m}_v^{(n)} = \underline{m}_u^n, \quad \overline{m}_v^{(n)} = \overline{m}_u^n, \tag{4.7.1}$$

and

$$\sigma_v^{(n)} = \frac{\underline{m}_u^n - (Val(\tilde{u}^n) - Amb(\tilde{u}^n))}{1 - 2I_1} \quad , \quad \gamma_v^{(n)} = \frac{Val(\tilde{u}^n) + Amb(\tilde{u}^n) - \overline{m}_u^n}{1 - 2I_1}. \tag{4.7.2}$$

4.8 The Nearest Trapezoidal Fuzzy Number to Multiplication of Two Trapezoidal Fuzzy Numbers

Let $\tilde{u} = (\underline{m}_u, \overline{m}_u, \sigma_u, \gamma_u), \tilde{v} = (\underline{m}_v, \overline{m}_v, \sigma_v, \gamma_v)$ be two nonnegative trapezoidal fuzzy numbers and $\tilde{w} = \tilde{u} \otimes \tilde{v}$, be the multiplication of the two fuzzy numbers $\tilde{u}$ and $\tilde{v}$ based on extension principle. We know that $\tilde{w}$ is a trapezoidal shape fuzzy number but not a trapezoidal one. Now using the nearest method we can approximate it by a trapezoidal fuzzy number $\tilde{y} = (\underline{m}_y, \overline{m}_y, \sigma_y, \gamma_y)$, where $\sigma_y = \underline{m}_y - l_y$, $\gamma_y = r_y - \overline{m}_y$ and

$$\begin{cases} l_y = \frac{3}{2}\sigma_u\sigma_v + 2[\sigma_u(\underline{m}_v - \sigma_v) + \sigma_v(\underline{m}_u - \sigma_u)] + 3(\underline{m}_u - \sigma_u)(\underline{m}_v - \sigma_v) - 2\underline{m}_u\underline{m}_v, \\ \underline{m}_y = \underline{m}_u\, \underline{m}_v, \\ \overline{m}_y = \overline{m}_u\, \overline{m}_v, \\ r_y = \frac{3}{2}\gamma_u\gamma_v - 2[\gamma_u(\overline{m}_v + \gamma_v) + \gamma_v(\overline{m}_u + \gamma_u)] + 3(\overline{m}_u + \gamma_u)(\overline{m}_v + \gamma_v) - 2\overline{m}_u\overline{m}_v. \end{cases}$$

Lemma 4.8.1. *The nearest fuzzy number $\tilde{y}$ to $\tilde{w} = \tilde{u} \otimes \tilde{v}$, is well defined, i.e.*

$$\gamma_y, \sigma_y \geq 0.$$

4.9 Numerical Examples

Example 4.9.1. *Let* $\mu_{\tilde{u}}(x) = \begin{cases} \frac{2}{(x-3)^2+1} - 1 & , \quad 2 \leq x \leq 4, \\ 0 & , \quad otherwise, \end{cases}$ *, and* $s(r) = r$.

The nearest linear approximation of $\tilde{u}$, is $\tilde{v} = ((6-\frac{3\pi}{2})r-3+\frac{3\pi}{2}, (-6+\frac{3\pi}{2})r+9-\frac{3\pi}{2})$, see Figure 4.3.

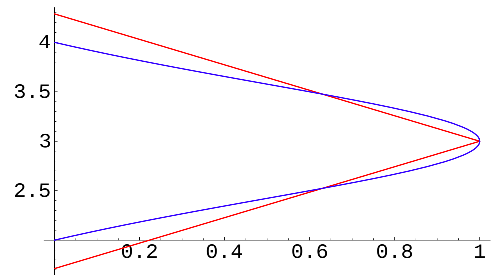


Figure 4.3 : The nearest linear approximation of fuzzy number $\tilde{u}$.

Example 4.9.2. *Let* $\tilde{u} = (3^r, 7-3^r)$, *and* $s(r) = r$.

The nearest quadratic approximation of $\tilde{u}$ is $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, such that

$$\underline{v}(r) = 1.03911 + 0.76644r + 1.19446r^2,$$

$$\overline{v}(r) = 5.96089 - 0.76644r - 1.19446r^2.$$

The functions $\tilde{u}$ and $\tilde{v}$ are shown in Figure 4.4.

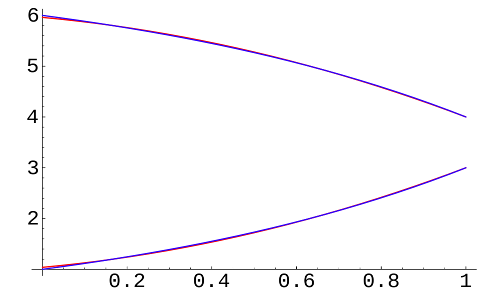


Figure 4.4 : The nearest quadratic approximation of fuzzy number $\tilde{u}$.

Example 4.9.3. *Let* $\tilde{u} = (2 + \ln\{(e-1)r+1\}, 4 - \ln\{(e-1)r+1\})$ *and* $s(r) = r$.

The nearest cubic approximation of $\tilde{u}$ will be $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, such that

$$\underline{v}(r) = 2.01846 + 1.4866r - 0.64225r^2 + 0.13718r^3,$$

$$\overline{v}(r) = 3.98154 - 1.4866r + 0.64225r^2 - 0.13718r^3.$$

The functions $\tilde{u}$ and $\tilde{v}$ are shown in Figure 4.5.

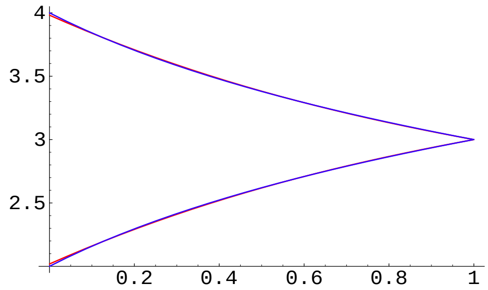


Figure 4.5 : The nearest cubic approximation of fuzzy number $\tilde{u}$.

Example 4.9.4. *Let* $s(r) = r$ *and* $\tilde{u} = (3, 4, 1, 1)$.

Therefore $\tilde{v}^{(2)} = (9, 16, \frac{11}{2}, \frac{17}{2})$, see Figure 4.6.

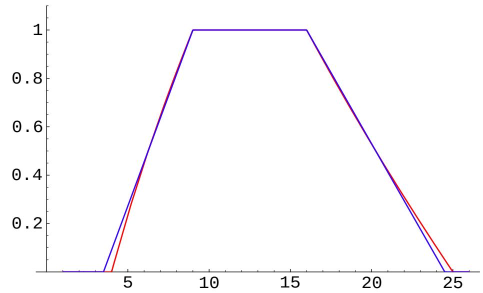


Figure 4.6 : The nearest trapezoidal fuzzy number to $\tilde{u}^2$

Example 4.9.5. *Let* $s(r) = r$ *and* $a \geq 0$.

If $\mu_{\tilde{u}}(x) = \chi_{\{a\}}(x)$ then $\mu_{\tilde{v}^{(n)}}(x) = \chi_{\{a^n\}}(x)$.

Example 4.9.6. *Let* $\tilde{u} = (2, 1, 1)$ *and* $\tilde{v} = (4, 2, 1)$.

Hence $\tilde{w} = \tilde{u} \otimes \tilde{v}$ and $\tilde{y} = (8, 7, \frac{13}{2})$, the nearest triangular fuzzy number, are shown in the Figure 4.7.

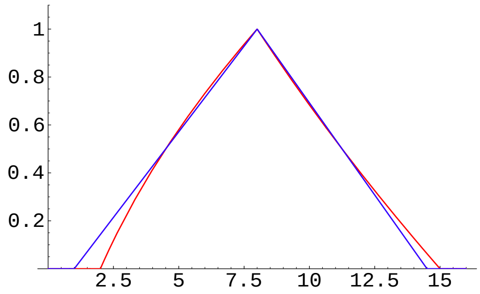


Figure 4.7 : The nearest triangular fuzzy number

Example 4.9.7. *Let* $\tilde{u} = (2, 3, 1, 1)$ *and* $\tilde{v} = (3, 4, 1, 2)$.

Hence $\tilde{w} = \tilde{u} \otimes \tilde{v}$ and $\tilde{y} = (6, 12, \frac{9}{2}, 11)$, the nearest trapezoidal fuzzy number, are shown in the Figure 4.8.

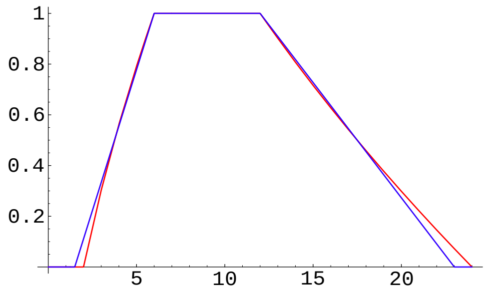


Figure 4.8 : The nearest trapezoidal fuzzy number

4.10 Conclusion

In this chapter we proposed a method to approximate a fuzzy number by a fuzzy number from different type. We introduced a distance D and near approximation of an arbitrary fuzzy number. Near approximation of a trapezoidal fuzzy number is itself and hence our distance D, is a metric on the set of all trapezoidal fuzzy numbers. In this chapter the nearest approximation of a fuzzy number was introduced and a method for computing it, was presented. We approximated any powers of a trapezoidal fuzzy number with a trapezoidal one, also the multiplication of two fuzzy numbers was approximated by a trapezoidal one.

Bibliography

[1] Abbasbandy S., Interpolation of fuzzy data by complete splines, *J. Appl. Math. Comput.* **8** (2001) 587-594.

[2] Abbasbandy S. and Asady B., The nearest trapezoidal fuzzy number to a fuzzy quantity, *Appl. Math. Comput.* **156** (2004) 381-386.

[3] Abbasbandy S. and Asady B., Note on "A new approach for defuzzification", *Fuzzy Sets and Systems* **128** (2002) 131-132.

[4] Abbasbandy S. and Babolian E., Interpolation of fuzzy data by natural splines, *J. Appl. Math. Comput.* **5** (1998) 457-463.

[5] Asmuss S. V. and Sostak A. P., Extremal problem of approximation theory in fuzzy contex, *Fuzzy Sets and Systems* **105** (1999), 249-257.

[6] Cheney E. W., Introduction to Approximation Theory, McGraw-Hill Book, New York 1966.

[7] Delgado M., Vila M. A. and Voxman W., On a canonical representation of fuzzy numbers, *Fuzzy Sets and Systems* **93** (1998) 125-135.

[8] Diamond P., Time dependent differential inclusions, cocycle attractors and fuzzy differential equations, *IEEE Trans. On Fuzzy Systems* **7** (1999), 734-740.

[9] Diamond P. and Kloeden P., Metric spaces of fuzzy sets, Singapore: World Scientific, 1994.

[10] Dubois D. and Prade H., Fuzzy Sets and Systems: Theory and Application, Academic Press, New York, 1980.

[11] Dubois D. and Prade H., A unified view of ranking techniques for fuzzy numbers. Proc. 1999 IEEE International Conference on Fuzzy Systems, Seoul, 1328-1333.

[12] Dubois D. and Prade H., The mean value of a fuzzy number, *Fuzzy Sets and Systems* **24** (1987) 279-300.

[13] Dubois D. and Prade H., Operations on fuzzy numbers, *Intenat. J. Systems Sci.* **9** (1978) 613-626.

[14] D'Urso P. and Gastaldi T., An orderwise polynomial regression procedure for fuzzy data. *Fuzzy Sets and Systems* **130** (2002) 1-19.

[15] Fortemps P. and Roubens M., Ranking and defuzzification methods based on area compensation, *Fuzzy Sets and Systems* **82** (1996) 319-330.

[16] Grzegorzewski P., Nearest interval approximation of a fuzzy number, *Fuzzy Sets and Systems* **130** (2002) 321-330.

[17] Kaleva O., Interpolation of fuzzy data, *Fuzzy Sets and Systems* **61** (1994) 63-70.

[18] Kaleva O. and Seikkala S., On fuzzy metric spaces, *Fuzzy Sets and Systems* **12** (1984) 215-229.

[19] Lodwick W. A. and Santos J., Constructing consistent fuzzy surfaces from fuzzy data, *Fuzzy Sets and Systems* **135** (2003) 259-277.

[20] Lowen R., A fuzzy Lagrange interpolation theorem, *Fuzzy Sets and Systems* **34** (1990) 33-38.

[21] Ma Ming., Kandel A. and Friedman M., A new approach for defuzzification, *Fuzzy Sets and Systems* **111** (2000) 351-356.

[22] Ma M., Kandel A. and Friedman M., Correction to "A new approach for defuzzification", *Fuzzy Sets and Systems* **128** (2002) 133-134.

[23] Maleki H. R., Tata M. and Mashinchi M., Linear programming with fuzzy variables, *Fuzzy Sets and Systems* **109** (2000) 21-33.

[24] Mhaskar H. N. and Pai D. V., Fundamentals of Approximation Theory, CRC Press, 2000.

[25] Miguel L., De Compos Ibanes L. M. and Gonzalez-Munoz A., A subjective approach for ranking fuzzy numbers, *Fuzzy Sets and Systems* **29** (1989) 145-153.

[26] Moon B. S., A curve smoothing method by using fuzzy sets, *Fuzzy Sets and Systems* **96** (1998) 353-358.

[27] Oussala M., On the compatibility between defuzzification and fuzzy arithmetic operations,*Fuzzy Sets and Systems* **128** (2002) 247-260.

[28] Rivlin T. J., An Introduction to the Approximation of Functions, Blaisdell, Waltham, Mass., 1969.

[29] Roubens M., Inequality constraints between fuzzy numbers and their use in mathematical programming, in: R. Slowinski, J. Teghem (Eds.), Stochastic Versus Fuzzy Approachs To Multiobjective mathematical programming Under Uncertainty, Kluwer Academic Publishers, Dordrecht, (1991) 321-330.

[30] Seikkala S., On the fuzzy initial value problem, *Fuzzy Sets and Systems* **24** (1987) 319-330.

[31] Teran P. and Lopez-Diaz M., Approximation of mappings with values which are upper semicontinuous functions, *J. Approx. Theory* **113** (2001) 245-265.

[32] Voxman W., Some remarks on distance between fuzzy numbers, *Fuzzy Sets and Systems* **100** (1998) 353-365.

[33] Wang X. and Kerre E. E., Reasonable properties for the ordering of fuzzy quantities (I), *Fuzzy Sets and Systems* **118** (2001) 375-385.

[34] Wang X. and Kerre E. E., Reasonable properties for the ordering of fuzzy quantities (II), *Fuzzy Sets and Systems* **118** (2001) 387-405.

[35] Yager R. R., A procedure for ordering fuzzy subsets of the unit interval, *Information Sciences* **24** (1981) 143-161.

[36] Yao J. S. and Wu K., Ranking fuzzy numbers based on decomposition principle and signed distance, *Fuzzy Sets and Systems* **116** (2000) 275-288.

[37] Yazdany Peraei E., Maleki H. R. and Mashinchi M., A method for solving a fuzzy linear programming, *J. Appl. Math. Comput.* **8** (2001) 347-356.

[38] Yen J. and Langari R., Fuzzy logic intelligence, control, and information, 2^{nd} Indian Reprint, Pearson Education Inc., 2004.

[39] Yoon K. P., A probabilistic approach to rank complex fuzzy numbers, *Fuzzy Sets and Systems* **80** (1996) 167-176.

[40] Zadeh L. A., Fuzzy Sets, *Inform. and Control* **8** (1965) 338-353.

[41] Zimmermann H. J., Fuzzy Set Theory and Its Applications, 2^{nd} Edition, Kluwer Academic, Boston, 1991.

Appendix

Proof of relations (4.4.1)-(4.4.3): Let

$$\alpha = \int_0^1 s(r)[\overline{u}(r) + \underline{u}(r) - \overline{v}(r) - \underline{v}(r)]dr = Val(\tilde{u}) - Val(\tilde{v}),$$

and we want to find $\min |\alpha|$.

Defining

$$\alpha^+ = \begin{cases} \alpha, & \alpha \geq 0; \\ 0, & \alpha < 0, \end{cases} \quad , \quad \alpha^- = \begin{cases} 0, & \alpha \geq 0; \\ -\alpha, & \alpha < 0. \end{cases}$$

We have $\alpha^+, \alpha^- \geq 0$, also

$$|\alpha| = \alpha^+ + \alpha^-,$$

and

$$\alpha = \alpha^+ - \alpha^-.$$

Thus we obtain the following linear programming.

$$\begin{cases} \min z_1 = \alpha^+ + \alpha^- \\ s.t. \\ \alpha^+ - \alpha^- + Val(\widetilde{v}) = Val(\widetilde{u}), \\ \alpha^+, \alpha^- \geq 0. \end{cases}$$

We obtain relation (4.4.2) in a similar way. Now let

$$\theta = \max \ \{|\overline{u}(1) - \overline{v}(1)|, |\underline{u}(1) - \underline{v}(1)|\}.$$

Thus

$$\begin{cases} \theta \geq |\overline{u}(1) - \overline{v}(1)|, \\ \theta \geq |\underline{u}(1) - \underline{v}(1)|, \end{cases}$$

and easily we obtain relation (4.4.3).

$\square$

Proof of Lemma 4.4.2: From properties of regular reducing function we have $I_0 = \frac{1}{2}$, and for every $j > 0$ there exists a $\psi \in (0,1)$ such that

$$I_{j+1} = \int_0^1 r^{j+1} s(r) dr = \psi \int_0^1 r^j s(r) dr = \psi I_j < I_j.$$

$\square$

In [36] Yao and Wu defined a metric on fuzzy numbers. In their definition we can have two different triangular fuzzy numbers where their distance is zero but they are not equal. They defined $D(\tilde{u}, \tilde{v}) = \frac{1}{2} \int_0^1 (\underline{u}(r) + \overline{u}(r) - \underline{v}(r) - \overline{v}(r)) dr$ and say $\tilde{u} \approx \tilde{v}$ if $D(\tilde{u}, \tilde{v}) = 0$. Now let

$$\mu_{\tilde{u}}(x) = \begin{cases} x-2 & 2 \leq x \leq 3 \\ \frac{6-x}{3} & 3 \leq x \leq 6 \\ 0 & otherwise. \end{cases}, \mu_{\tilde{v}}(x) = \begin{cases} \frac{x-1}{3} & 1 \leq x \leq 4 \\ 5-x & 4 \leq x \leq 5 \\ 0 & otherwise, \end{cases}$$

$$\tilde{u} = (3,1,3) \quad , \quad \tilde{v} = (4,3,1)$$

$$\underline{u}(r) = 2 + r \quad , \quad \overline{u}(r) = 6 - 3r \quad , \quad \underline{v}(r) = 1 + 3r \quad , \quad \overline{v}(r) = 5 - r$$

$$D(\tilde{u}, \tilde{v}) = \frac{1}{2} \int_0^1 [(8-2r) - (6+2r)] dx = 0.$$

Therefore $\tilde{u} \approx \tilde{v}$ but $\tilde{u} \neq \tilde{v}$. Now let $s(r) = r$. In our method we have

$$D_s(\tilde{u}, \tilde{v}) = |Val(\tilde{u}) - Val(\tilde{v})| + |Amb(\tilde{u}) - Amb(\tilde{v})| + d_H([\tilde{u}]^1, [\tilde{v}]^1)$$

$$Val(\tilde{u}) = \int_0^1 r(8-2r) dr \quad , \quad Amb(\tilde{v}) = \int_0^1 r(4-4r) dr \quad , \quad [\tilde{u}]^1 = 3$$

$$Val(\tilde{v}) = \int_0^1 r(6+2r) \quad , \quad Amb(\tilde{v}) = \int_0^1 r(4-4r) dr \quad , \quad [\tilde{v}]^1 = 4$$

$$D_s(\tilde{u},\tilde{v}) = \left|\int_0^1 r(2-4r)dr\right| + |0| + |3-4| = \left|\frac{1}{3}\right| + |1| = \frac{4}{3}.$$

By our definition $\tilde{u}$ is not equal to $\tilde{v}$.

For single modal fuzzy numbers we have $d_H(\tilde{u},\tilde{v}) = |[\tilde{u}]^1 - [\tilde{v}]^1|$. Also in [15] Fortemps and Roubens use enclosed area $C(\tilde{u} \geq \tilde{v})$ to define ordering. Their work also have this unefficiency that we can find two triangular fuzzy numbers that $C(\tilde{u} \geq \tilde{v}) = C(\tilde{u} \leq \tilde{v}) = 0$ but $\tilde{u} \neq \tilde{v}$ [36]. In almost all ranking methods we can find two different triangular fuzzy numbers $\tilde{u}, \tilde{v}$ which $\tilde{u} \approx \tilde{v}$ but actually $\tilde{u} \neq \tilde{v}$. In our work two triangular fuzzy numbers $\tilde{u}, \tilde{v}$ are nearest to each others if and only if

$$\tilde{u} = \tilde{v}.$$

We introduced a distance between fuzzy numbers, in general D_s defined in Definition 4.3.1 is a pseudo-metric for $\mathcal{F}(\mathbb{R})$. In $\mathcal{TRF}(\mathbb{R})$ (set of triangular fuzzy numbers) and $\mathcal{TZF}(\mathbb{R})$ (set of all trapezoidal fuzzy numbers) even in $\mathcal{TF}(\mathbb{R})$ (set of all triangular and trapezoidal fuzzy numbers), D_s is a meteric.