ISLAMIC AZAD UNIVERSITY
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ITERATIVE METHODS FOR FUZZY LINEAR SYSTEMS

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To My dears:

My Wife

and

My Parents

Contents

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Abstract

One of the major applications by using fuzzy number arithmetics is treating those linear systems, which their parameters are entirely or partially represented by fuzzy numbers. A general model for solving a $n \times n$ fuzzy linear system whose coefficients matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector, which use the embedding method and replace the original $n \times n$ linear system $AX = Y$ by a $2n \times 2n$ crisp linear system with a coefficients matrix S which may be singular even if A be nonsingular.

In this thesis some iterative methods for finding solution of $n \times n$ fuzzy linear system like *Conjugate Gradient* method and *Generalized Minimal Residual* (GMRES) Method, are given and we use *Fixed Point* and *Steepest Descent* Method for solving nonlinear equations. The following contents are organized with four major chapters. In chapter one, we state elementary definitions and results in fuzzy logic. In chapter two we define Fuzzy linear system(FSLE) and use direct method for solving fuzzy linear systems. In chapter three we solve fuzzy linear systems by iterative methods. chapter four is considered for solving fuzzy nonlinear equations by numerical methods.

Chapter 1

Fuzzy Logic

1.1 Introduction

Fuzziness is not a priori an obvious concept and demands some explanation. "Fuzziness" is what Black (NF 1937) calls "vagueness" when he distinguishes it from "generality" and from "ambiguity". Generalizing refers to the application of a symbol to a multiplicity of objects in the field of reference, ambiguity to the association of a finite number of alternative meanings having the same phonetic form. But, the fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol applies.

More specifically, let X be a field of reference, also called a universe of discourse or universe for short, covering a definite range of objects. Consider a subset $\tilde{A}$ where transition between membership and nonmembership is gradual rather than abrupt. This "fuzzy subset" obviously has no well-defined boundaries. Fuzzy classes of objects are often encountered in real life. For instance, $\tilde{A}$ may be the set of tall men in a community X. Usually, there are members of X who are definitely tall, others who are definitely not tall, but there exist also borderline cases. Traditionally, the grade of membership 1 is assigned to the objects that completely belong to $\tilde{A}$-here the men who are definitely tall, conversely the objects that do not belong to $\tilde{A}$ at all are assigned a membership value 0. Quite naturally, the grades of membership of the borderline cases lie between 0 and 1. The more an element or object x belongs to $\tilde{A}$, the closer to 1 is its grade of membership $\mu_{\tilde{A}}(x)$. The use of a numerical scale such as the interval [0, 1] allows a convenient representation of the gradation in membership. Precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or a group. Moreover, they are context-dependent. The grades of membership reflect an "ordering" of the objects in the universe, induced by the predicate associated with $\tilde{A}$; this "ordering", when it exists, is more important than the membership values themselves. The membership assessment of objects can sometimes be made easier by the use of a similarity measure with respect to an ideal element. Note that a membership value $\mu_{\tilde{A}}(x)$ can be interpreted as the degree of compatibility of the predicate associated with $\tilde{A}$ and the object x. For concepts such as "tallness", related to a physical measurement scale, the assignment of membership values will often be less controversial than for more complex and subjective concepts such as "beauty".

The above approach, developed by Zadeh (1964), provides a tool for modeling human-centered systems. As a matter of fact, fuzziness seems to pervade most human perception and thinking processes. Parikh (1977) has pointed out that no nontrivial first-order-logic-like observational predicate (i.e., one pertaining to perception) can be defined on an observationally connected space; the only possible observational predicates on such a space are not classical predicates but "vague" ones. Moreover, according to Zadeh (1973), one of the most important facets of human thinking is the ability to summarize information "into labels of fuzzy sets which bear an approximate relation to the primary data". Linguistic descriptions, which are usually summary descriptions of complex situations, are fuzzy in essence.

It must be noticed that fuzziness differs from imprecision. In tolerance analysis imprecision refers to lack of knowledge about the value of a parameter and is thus expressed as a crisp tolerance interval. This interval is the set of possible values of the parameters. Fuzziness occurs when the interval has no sharp boundaries, i.e., is a fuzzy set $\tilde{A}$. Then, $\mu_{\tilde{A}}(x)$ is interpreted as the degree of possibility (Zadeh, 1978) that x is the value of the parameter fuzzily restricted by $\tilde{A}$.

The word fuzziness has also been used by Sugeno (1977) in a radically different context. Consider an arbitrary object x of the universe X; to each nonfuzzy subset A of X is assigned a value $g_x(A) \in [0, 1]$ expressing the "grade of fuzziness" of the statement " x belongs to A". In fact this grade of fuzziness must be understood as a grade of certainty: according to the mathematical definition of g, $g_x(A)$ can be interpreted as the probability, the degree of subjective belief, the possibility, that x belongs to A. Generally, g is assumed increasing in the sense of set inclusion, but not necessarily additive as in the probabilistic case. The situation modeled by Sugeno is more a matter of guessing whether $x \in A$ rather than a problem of vagueness in the sense of Zadeh. The existence of two different points of view on "fuzziness" has been pointed out by MacVicar-Whelan (1977) and Skala. The monotonicity assumption for g seems to be more consistent with human guessing than does the additivity assumption. For instance, seeing a piece of Indian pottery in a shop, we may try to guess whether it is genuine or counterfeit; obviously, genuineness is a fuzzy concept. Hence x is the Indian pottery; A is the crisp set of genuine Indian artifacts; and $g_x(A)$ expresses, for instance, a subjective belief that the pottery is indeed genuine. The situation is slightly more complicated when we try to guess whether the pottery is old: actually, the set $\tilde{A}$ of old Indian pottery is fuzzy because "old" is a vague predicate.

1.2 Fuzzy Sets

Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X is often viewed as a characteristic function, μ_A from X to $\{0, 1\}$ such that

$$\mu_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in A. \end{cases}$$

$\{0, 1\}$ is called a valuation set.

If the valuation set is allowed to be the real interval $[0, 1]$, A is called a fuzzy set (Zadeh, 1965), and $\mu_A(x)$ is the grade of membership of x in A. Sometimes we denoted $\mu_A(x)$ by $A(x)$. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to

A. Clearly, A is a subset of X that has no sharp boundary. Hence A is completely characterized by the set of pairs

$$A = \{(x, \mu_A(x)) | x \in X\}$$

A more convenient notation was proposed by Zadeh [41]. When X is a finite set $\{x_1, ..., x_n\}$ a fuzzy set on X is expressed as

$$A = \frac{\mu_A(x_1)}{x_1} + \cdots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}.$$

When X is not finite, we write

$$A = \int_x \mu_A(x).$$

Definition 1.2.1. (Equality of fuzzy sets) Two fuzzy sets A and B are said to be equal (denoted $A = B$) if

$$\forall\ x \in X,\ \ \mu_A(x) = \mu_B(x).$$

Definition 1.2.2. (Support) The support of a fuzzy set A is the ordinary subset of X:

$$supp\ A = \{x \in X,\ \ \mu_A(x) > 0\}.$$

Definition 1.2.3. (Core) The core of a fuzzy set A is the set of all points with the membership degree one in A equal 1.

$$core A = \{x \in X |\ \mu_A(x) = 1\}.$$

Definition 1.2.4. (Crossover points) The elements of x such that $\mu_A(x) = \frac{1}{2}$ are the crossover points of A.

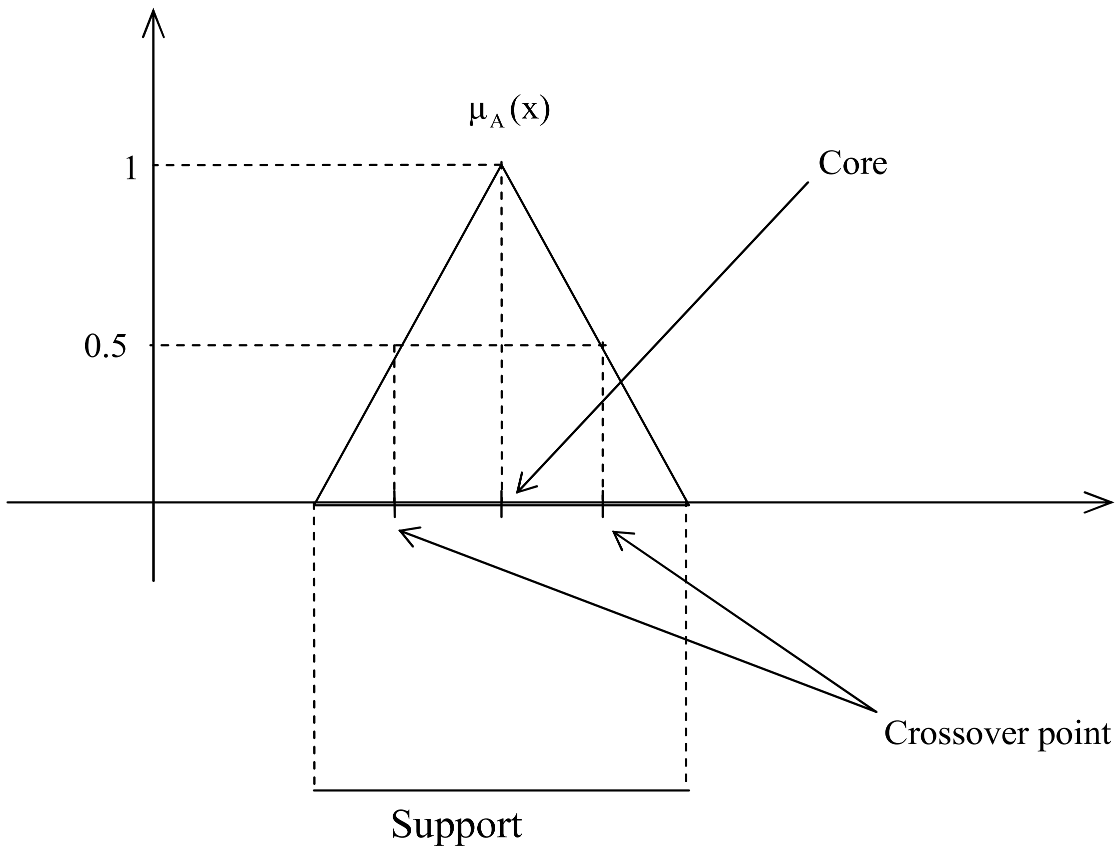

Figure 1.1: Graphical example of support, core, crossover point.

Definition 1.2.5. (Height of a fuzzy set) The height of A is $hgt(A) = sup_{x \in X}\ \mu_A(x)$, i.e., the least upper bound of $\mu_A(x)$.

Definition 1.2.6. (Normal fuzzy set) A is said to be normal if and only if $\exists\ x \in X,\ \mu_A(x) = 1$; this definition implies $hgt(A) = 1$.

Definition 1.2.7. (Empty fuzzy set) The empty set ϕ is defined as $\forall\ x \in X,\ \mu_\phi(x) = 0$; of course, $\forall x \in\ X,\ \mu_X(x) = 1$.

Example 1.2.1. *Let*

$$A = \frac{0.1}{7} + \frac{0.5}{8} + \frac{0.8}{9} + \frac{1.0}{10} + \frac{0.8}{11} + \frac{0.5}{12} + \frac{0.1}{13}.$$

A is a fuzzy set of integers approximately equal to 10.

Example 1.2.2. *Let*

$$\mu_A(x) = \frac{1}{1 + [\frac{1}{5}(x - 10)^2]}.$$

A is a fuzzy set of real numbers clustered around 10.

Definition 1.2.8. (Union and Intersection) The classical union ($\cup$) and intersection ($\cap$) of ordinary subsets of X can be extended by the following formulas, proposed by Zadeh [42]:

$$\forall x \in X, \qquad \mu_{A\cup B}(x) = max(\mu_A(x), \mu_B(x)), \tag{1.1}$$

$$\forall x \in X, \qquad \mu_{A\cap B}(x) = min(\mu_A(x), \mu_B(x)), \tag{1.2}$$

where $\mu_{A\cup B}$ and $\mu_{A\cap B}$ are respectively the membership functions of $A \cup B$ and $A \cap B$.

These formulas give the usual union and intersection when the valuation set is reduced to $\{0, 1\}$. Obviously, there are other extensions of $\cup$ and $\cap$ coinciding with the binary operators.

Definition 1.2.9. (Containment or subset) A is said to be included in B ($A \subseteq B$) if and only if

$$\forall x \in X, \quad \mu_A(x) \leq \mu_B(x).$$

When the inequality is strict, the inclusion is said to be strict and is denoted $A \subset B$.

Definition 1.2.10. (Complement) The complement $\overline{A}$ of A is defined by the membership function

$$\forall\ x \in X, \quad \mu_{\overline{A}}(x) = 1 - \mu_A(x). \tag{1.3}$$

When A is an ordinary subset of X, the pair $(A, \overline{A})$ is a partition of X provided that $A \neq \emptyset$ and $A \neq X$.

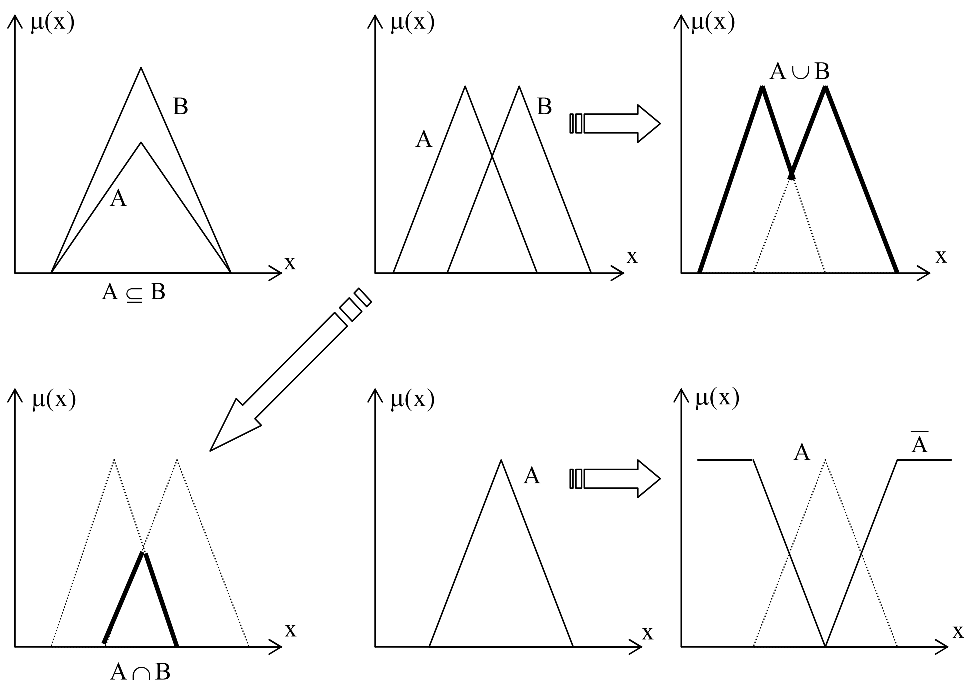

Figure 1.2: Graphical examples of containment, union, intersection and complement.

Definition 1.2.11. (Fuzzy partition) When A is a fuzzy set ($\neq \emptyset, \ \neq X$), the pair $(A, \overline{A})$ is called a fuzzy partition; more generally an m-tuple $(A_1, ..., A_m)$ of fuzzy sets $(\forall i,\ A_i \neq \emptyset \ \ and \ \ A_i \neq X)$ such that

$$\forall\ x \in X, \quad \sum_{i=1}^{m} \mu_{A_i}(x) = 1 \qquad (orthogonality) \tag{1.4}$$

is still called a fuzzy partition of X [28].

Definition 1.2.12. ($\alpha - Cuts$) When we want to exhibit an element $x \in X$ that typically belongs to a fuzzy set A, we may demand its membership value to be greater than some threshold $\alpha \in]0, 1]$. The ordinary set of such elements is the $\alpha - cut$ $[A]_\alpha$ of A, $[A]_\alpha = \{ \ x \in X, \ \mu_A(x) \geq \alpha \ \}$. One also defines the strong $\alpha - cut$ as $[A]_{\overline{\alpha}} = \{ \ x \in X, \ \mu_A(x) > \alpha \ \}$.

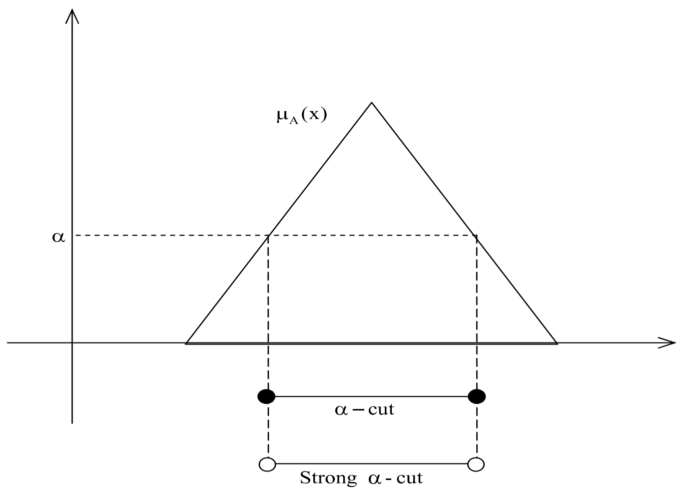

Figure 1.3: $\alpha - Cut \ \ and \ \ Strong \ \alpha - Cut$.

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its $\alpha - cuts$ according to the formula [22]

$$\mu_A(x) = \sup_{\alpha \in]0,1]} min(\alpha, \mu_{A_\alpha}(x)),$$

where

$$\mu_{A_\alpha}(x) = \begin{cases} 1, & x \in A, \\ 0, & otherwise. \end{cases}$$

It is easily showed that the following properties hold:

$$[A \cup B]_\alpha = [A]_\alpha \cup [B]_\alpha, \qquad [A \cap B]_\alpha = [A]_\alpha \cap [B]_\alpha.$$

However, $[\overline{A}]_\alpha = \overline{[[A]_{1-\alpha}]} \neq \overline{[[A]_\alpha]}$ if $(\alpha \neq \frac{1}{2})$. This result stems from the fact that generally there are elements that belong neither to $[A]_\alpha$ nor to $[\overline{A}]_\alpha$ ($[A]_\alpha \cup [\overline{A}]_\alpha \neq X$). Radecki [37] has defined level fuzzy sets of a fuzzy set A as the fuzzy sets $[\tilde{A}]_\alpha, \alpha \in]0,1[$, such that

$$[\tilde{A}]_\alpha = \{\ (x, \mu_A(x))|\ x \in [A]_\alpha\ \}.$$

The rational behind this definition is the fact that in practical applications it is sufficient to consider fuzzy sets defined in only one part of their support the most significant part in order to save computing time and computer memory storage. Radecki has developed an algebra of level fuzzy sets. However, $[\tilde{\overline{A}}]_\alpha$, the approximation of $\overline{A}$, cannot be obtained from $[\tilde{A}]_\alpha$, ($[\overline{\tilde{A}}]_\alpha \neq [\tilde{\overline{A}}]_\alpha$).

Definition 1.2.13. (Cardinality) When X is a finite set, the cardinality $|A|$ of a fuzzy set A on X is defined as:

$$|A| = \sum_{x \in X} \mu_A(x).$$

$|A|$ is sometimes called the power of A. $\|A\| = |A|/|X|$ is the relative cardinality. It can be interpreted as the proportion of elements of X that are in A, [23].

Definition 1.2.14. (Convexity) A fuzzy set A of X is called convex if $[A]_\alpha$ is a convex subset of X for each $\alpha \in [0,1]$,

Figure 1.4 represents a convex fuzzy set and a nonconvex fuzzy set.

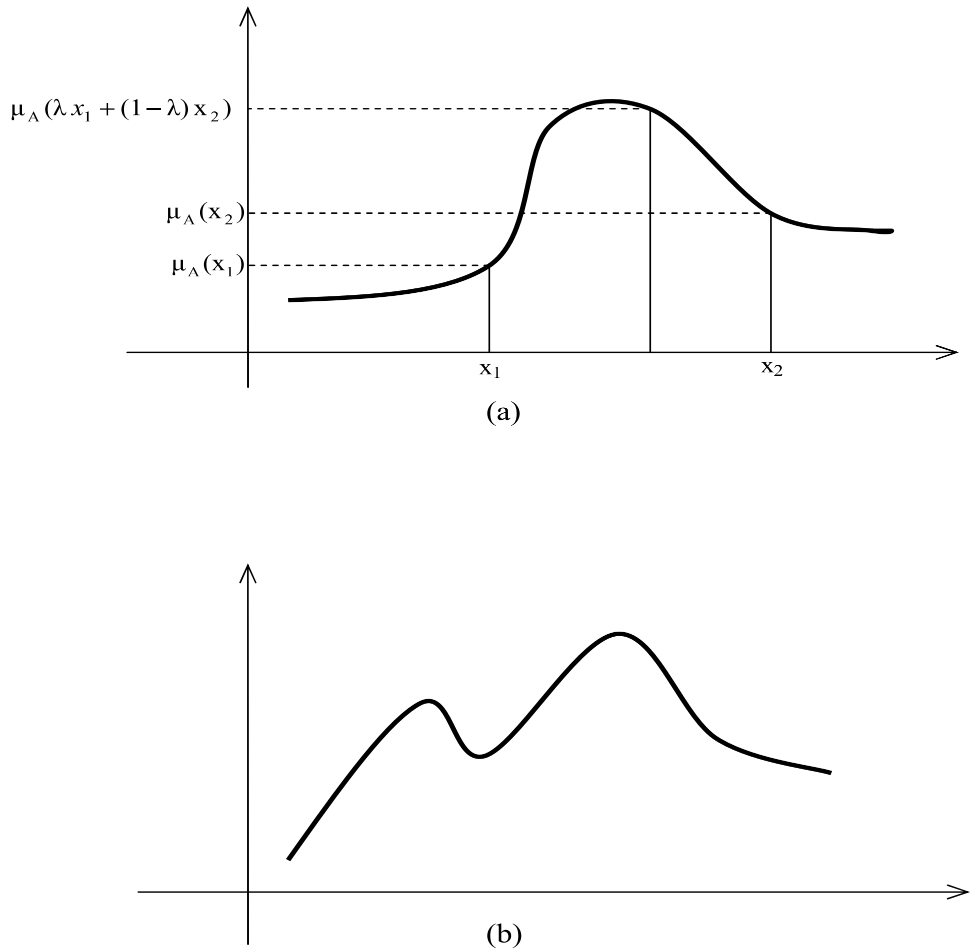

Figure 1.4:(a) Convex fuzzy set. (b) Nonconvex fuzzy set.

To avoid confusion, we note that the definition of convexity for fuzzy sets does not mean that the membership function of a convex fuzzy set is a convex function. In fact, membership function of convex fuzzy sets are functions that are, according to standard definitions, concave and not convex. We now state a useful theorem that provides us with an alternative formulation of convexity of fuzzy sets. For the sake of simplicity, we restrict the theorem to fuzzy sets on $\mathbb{R}$, which are of primary interest in this text.

Theorem 1.2.3. *A fuzzy set A on $\mathbb{R}$ is convex if and only if*

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_A(x_1), \mu_A(x_2)] \tag{1.5}$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0,1]$, where min denotes the minimum operator.

Proof. Assume that A is convex and let $\alpha = \mu_A(x_1) \leq \mu_A(x_2)$. Hence $x_1, x_2 \in [A]_\alpha$ and, moreover, $\lambda x_1 + (1-\lambda)x_2 \in [A]_\alpha$ for any $\lambda \in [0,1]$ by the convexity of A. Consequently,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = \mu_A(x_1) = \min[\mu_A(x_1), \mu_A(x_2)].$$

Conversely assume that A satisfies (1.5). We need to prove that for any $\alpha \in [0,1]$, $[A]_\alpha$ is convex. Now for any $x_1, x_2 \in [A]_\alpha$(*i.e.*, $\mu_A(x_1) \geq \alpha$, $\mu_A(x_2) \geq \alpha$), and for any $\lambda \in [0,1]$, by (1.5)

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_A(x_1), \mu_A(x_2)] \geq \min(\alpha, \alpha) = \alpha,$$

i.e., $\lambda x_1 + (1-\lambda)x_2 \in [A]_\alpha$. Therefore, $[A]_\alpha$ is convex for any $\alpha \in (0,1]$. Hence, A is convex. □

Among the various type of fuzzy sets, of special significance are fuzzy sets that are defined on the set $\mathbb{R}$ of real numbers. Membership functions of these sets, which have the form

$$A : \mathbb{R} \to [0,1],$$

clearly have a quantitative meaning and may, under certain conditions, be viewed as fuzzy number that can be defined as follows.

Definition 1.2.15. (Fuzzy number) A fuzzy number is a map $A : \mathbb{R} \to I = [0, 1]$ which satisfies:

(i) A is upper semi-continuous, i.e.,

$$\{x|A(x) < t\} \ \ is \ \ open \ \ for\ all\ t \in \mathbb{R}.$$

(ii) $A(x) = 0$ outside some interval $[c, d] \subset \mathbb{R}$.

(iii) There exist real numbers a, b, c, d such that $c \leq a \leq b \leq d$ where

1. $A(x)$ is monotonic increasing on $[c, a]$.
2. $A(x)$ is monotonic decreasing on $[b, d]$.
3. $A(x) = 1,\ a \leq x \leq b$.

Definition 1.2.16. (Quasi fuzzy number) A quasi fuzzy number A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions

$$\lim_{t\to+\infty} A(t) = 0, \qquad \lim_{t\to-\infty} A(t) = 0.$$

Remark 1.2.1. Let A be a fuzzy number. Then $[A]_\alpha$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\alpha \in [0, 1]$.

Definition 1.2.17. (Triangular fuzzy number) A fuzzy number A is called triangular fuzzy number with center a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{\alpha} & if \quad a - \alpha \leq t \leq a \\ 1 - \frac{(t-a)}{\beta} & if \quad a \leq t \leq a + \beta \\ 0 & otherwise \end{cases}$$

and we use for it the notation $A = (a, \alpha, \beta)$. It can easily be verified that

$$[A]_r = [a - (1-r)\alpha, a + (1-r)\beta], \ \forall r \in [0, 1].$$

The support of A is $[a - \alpha, a + \beta]$.

Figure 1.5 represents triangular fuzzy number $A = (6, 1, 2)$

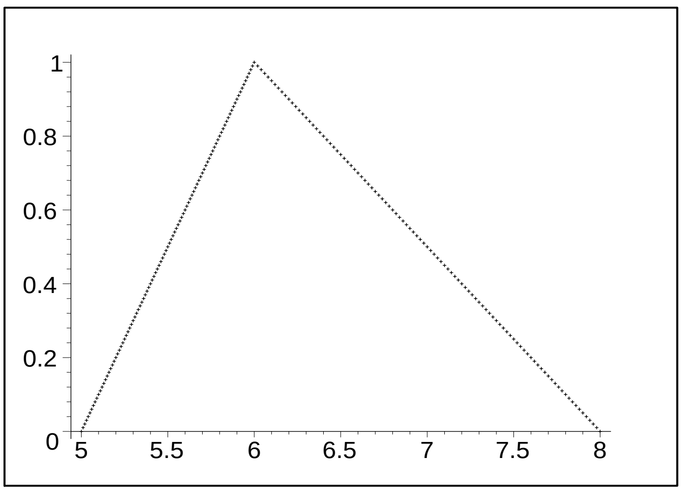

Figure 1.5: Triangular fuzzy number.

Definition 1.2.18. (Trapezoidal fuzzy number) A fuzzy number A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{\alpha} & if \quad a-\alpha \leq t \leq a \\ 1 & if \quad a \leq t \leq b \\ 1 - \frac{(t-b)}{\beta} & if \quad b \leq t \leq b+\beta \\ 0 & otherwise \end{cases}$$

and we use for it the notation $A = (a, b, \alpha, \beta)$. It can easily be shown that

$$[A]_r = [a - (1-r)\alpha, b + (1-r)\beta], \forall r \in [0,1].$$

The support of A is $[a - \alpha, b + \beta]$.

Figure 1.6 represent trapezoidal fuzzy number $A = (3, 5, 1, 2)$

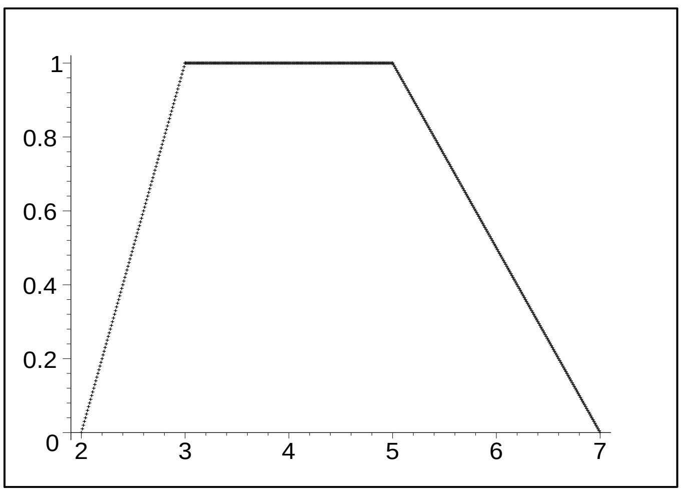

Figure 1.6: Trapezoidal fuzzy number.

Definition 1.2.19. We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, which satisfy the following requirements, [24]:

1. $\underline{u}(\alpha)$ is a bounded left continuous non-decreasing function over $[0, 1]$,

2. $\overline{u}(\alpha)$ is a bounded left continuous non-increasing function over $[0,1]$,

3. $\underline{u}(\alpha) \leq \overline{u}(\alpha), 0 \leq \alpha \leq 1$.

A crisp number u is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = u, 0 \leq \alpha \leq 1$. By appropriate definitions the fuzzy number space $\{\underline{u}(\alpha), \overline{u}(\alpha)\}$ becomes a convex cone.

Definition 1.2.20. Any fuzzy number A can be described as

$$A(t) = \begin{cases} L(\frac{a-t}{\alpha}) & if \quad t \in [a-\alpha, a] \\ 1 & if \quad t \in [a,b] \\ R(\frac{t-b}{\beta}) & if \quad t \in [b, b+\beta] \\ 0 & otherwise \end{cases}$$

where $[a,b]$ is the core of A,

$$L : [0,1] \to [0,1], \quad R : [0,1] \to [0,1]$$

are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $R(1) = L(1) = 0$. We call this fuzzy interval of LR-type and refer to it by

$$A = (a, b, \alpha, \beta)_{LR}.$$

The support of A is $(a-\alpha, b+\beta)$.

Definition 1.2.21. (Fuzzy point) Let A be a fuzzy number. If supp(A)$= \{x_0\}$ then A is called a fuzzy point and we use the notation $A = \overline{x_0}$.

Figure 1.7 represents fuzzy point $A = \overline{5}$

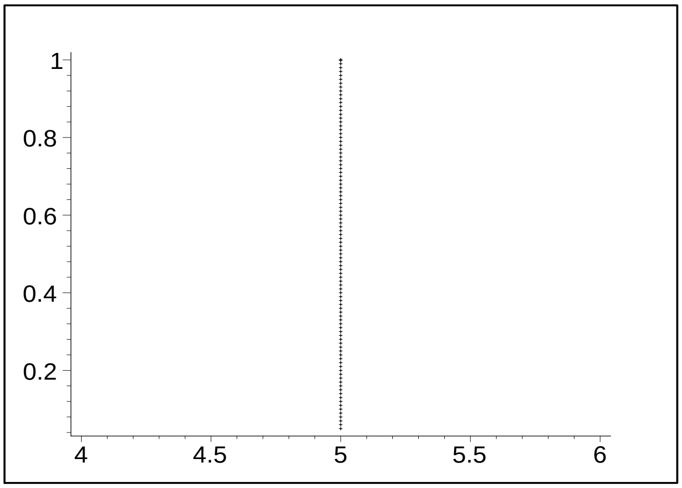

Figure 1.7: Fuzzy point $A = \overline{5}$.

Definition 1.2.22. The space E^n is all of fuzzy subsets U of $\mathbb{R}^n$ which satisfy the following conditions:

1. U is normal,
2. U is fuzzy convex,
3. U is upper semi-continuous,
4. $[U]_0$ is bounded subset of $\mathbb{R}^n$,

when $n = 1$, elements of E^1 are Fuzzy numbers.

1.3 The Extension Principle

The extension principle, introduced by Zadeh, is one of the most basic ideas of fuzzy set theory. It provides a general method for extending nonfuzzy mathematical concepts in order to deal with fuzzy quantities. Some illustrations are given including the notion of fuzzy distance between fuzzy sets. The extension principle is then systematically applied to real algebra: operations on fuzzy numbers are extensively developed. These operations generalize interval analysis and are computationally attractive. Although the set of real fuzzy numbers equipped with an extended addition or multiplication is no longer a group, many structural properties are preserved. Lastly, the extension principle is shown to be very useful for defining set-theoretic operations for higher order fuzzy sets.

Definition 1.3.1. (Extension principle) Let X be a cartesian product of universes, $X = X_1 \times \cdots \times X_r$ and $A_1, \cdots, A_r$ be r fuzzy sets in $X_1, \cdots, X_r$, respectively. The cartesian product of $A_1, \cdots, A_r$ is defined as

$$A_1 \times \cdots \times A_r = \int_{X_1 \times X_2 \times \cdots \times X_r} \min(\mu_{A_1}(x_1), \cdots, \mu_{A_r}(x_r))/(x_1, \cdots, x_r).$$

Let f be a mapping from $X_1 \times \cdots \times X_r$ to a universe Y such that $y = f(x_1, \cdots, x_r)$. The extension principle [43] allows us to induce from r fuzzy sets A_i a fuzzy set B on Y through f such that

$$\mu_B(y) = \begin{cases} \sup_{x_1, \cdots, x_r} \min(\mu_{A_1}(x_1), \cdots, \mu_{A_r}(x_r)) & (x_1, \cdots, x_r) \in f^{-1}(y), \\ 0 & f^{-1}(y) = \emptyset, \end{cases} \tag{1.6}$$

where $f^{-1}(y)$ is the inverse image of y. $\mu_B(y)$ is the greatest among the membership values $\mu_{A_1 \times \cdots \times \mu_{A_r}}(x_1, \cdots, x_r)$ of the realizations of y using r-tuples $(x_1, \cdots, x_r)$. The special case when $r = 1$ was already solved by Zadeh [43]. When f is one to one, (1.6) becomes $\mu_B(y) = \mu_A f^{-1}(y)$ when $f^{-1}(y) \neq \emptyset$.

Zadeh usually writes (1.6) as

$$B = f(A_1, \cdots, A_r) = \int_{X_1 \times \cdots \times X_r} \min(\mu_{A_1}(x_1), \cdots, \mu_{A_r}(x_r))/f(x_1, \cdots, x_r).$$

where the sup operation is implicit.

Denoting the image of $A_1, \cdots, A_r$, by $B = f(A_1, \cdots, A_r)$ the following proposition holds if f is continuous [35]:

$$[f(A_1, \cdots, A_r)]_\alpha = f([A_1]_\alpha, \cdots, [A_r]_\alpha)$$

$$iff \quad \forall y \in Y, \quad \exists x_1^*, \cdots, x_r^*, \quad \mu_B(y) = \mu_{A_1 \times \cdots \times A_r}(x_1^*, \cdots, x_r^*) \tag{1.7}$$

(the upper bound in (1.6) is attained).

Remark 1.3.1. While a discretization of the valuation set generally commutes with the extension of function f, this is not true for the discretization of the universe ($X_i = \mathbb{R}$).

1.4 Application of Extension Principle

Let X be a metric space equipped with the pseudometric d, i.e.,

1. d is a mapping from X^2 to $\mathbb{R}^+$;
2. $d(x, x) = 0 \;\; \forall x \in X$;
3. $d(x_1, x_2) = d(x_2, x_1) \;\; \forall x_1, x_2 \in X$;
4. $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3) \;\; \forall x_1, x_2, x_3 \in X$.

A fuzzy distance $\tilde{d}$ between fuzzy sets A and B on X is defined using (1.6) as

$$\forall \delta \in \mathbb{R}^+, \quad \mu_{\tilde{d}(A,B)}(\delta) = \sup_{\delta = d(u,v)} min(\mu_A(u), \mu_B(v)),$$

$\tilde{d}(A, B)$ models a distance between fuzzy "spots". When A and B are connected subsets of X, $\tilde{d}(A, B)$ is an ordinary interval whose extremities are respectively the shortest and greatest distance between a point of A and a point of B. $\tilde{d}$ is a mapping from E^2 to the set of fuzzy sets on $\mathbb{R}^+$ (i.e., positive real fuzzy sets). $\tilde{d}(A, B)$ can be interpreted as the fuzzy diameter of A and B. It is clear that we have $\tilde{d}(A, B) = \tilde{d}(B, A)$.

The question of knowing whether some triangular inequality like (4) still holds for $\tilde{d}$ is less straightforward. Let $[A]_\alpha$, $[B]_\alpha$, $[C]_\alpha$ be the $\alpha-cuts$ of three fuzzy sets A, B, C on X. Let us respectively denote by u, v, w any element of $[A]_\alpha$, $[B]_\alpha$, $[C]_\alpha$. The following inequalities hold:

$$\sup_{u,w} d(u, w) = d(u^*, w^*) \leq d(u^*, v) + d(v, w^*) \leq \sup_v (d(u^*, v) + d(v, w^*));$$
$$\sup_v (d(u^*, v) + d(v, w^*)) \leq \sup_{u,v,w} (d(u, v) + d(v, w));$$
$$\sup_{u,v,w} (d(u, v) + d(v, w)) \leq \sup_{u,v} d(u, v) + \sup_{v,w} d(v, w);$$
$$\inf_{u,v} d(u, v) + \inf_{v,w} d(v, w) \leq \inf_{u,v,w} (d(u, v) + d(v, w)).$$

Definition 1.4.1. (Compatibility of Two Fuzzy Sets)[44], Given a fuzzy set A on X, $\mu_A(x)$ is the grade of membership of x in A. We may also call it the degree of compatibility of the fuzzy value A with the nonfuzzy value x. The extension principle allows us to evaluate the compatibility of the fuzzy value A with another fuzzy value B, taken as a reference.

Let τ be this compatibility. τ is a fuzzy set on $[0, 1]$ since it is $\mu_A(x)$. Using (1.6),

$$\mu_\tau(u) = \sup_{x: u = \mu_A(x)} \mu_B(x) \qquad \forall u \in [0, 1], \tag{1.8}$$

or, using Zadeh's notation,

$$\tau = \mu_A(B) = \int_X \mu_A(x)/\mu_B(x).$$

An example of computation of $\mu_\tau(u)$ is pictured in Fig. 1.8. When μ_A is one to one, $\mu_\tau = \mu_B\ o\ \mu_A^{-1}$, where o is the composition of functions. When $A = B$, μ_τ is the identity function, $\mu_\tau(u) = u$. Remember that the converse proposition does not hold: A and B can be very different while $\mu_\tau(u) = u$.

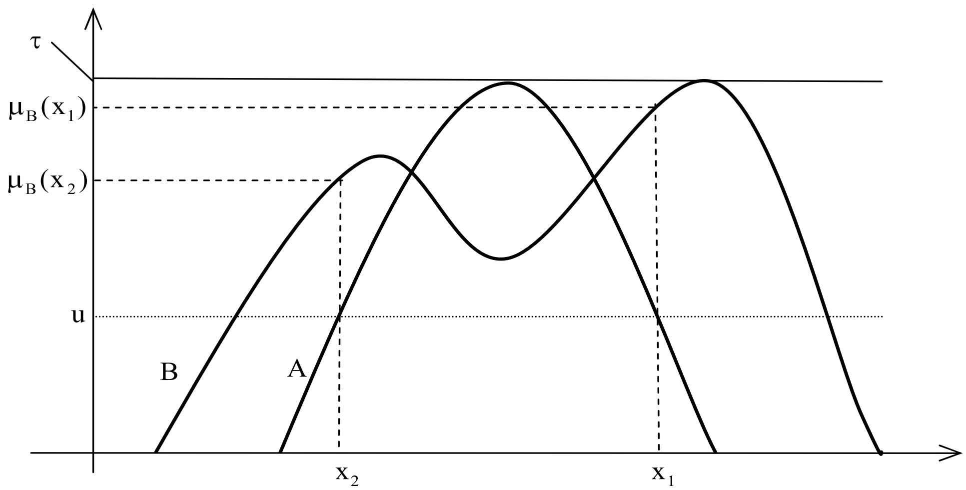

Figure 1.8

Hence τ is a normal fuzzy set if B is. To prove this, observe that if b is such that $\mu_B(b) = 1$ then $\mu_\tau(\mu_A(b)) = 1$. The converse proposition is obvious provided that the sup is attained in (1.8).

If μ_B has only one relative maximum b, $\mu_B(b) = hgt(B)$, then μ_B has only one relative maximum. This is obvious from Zadeh's form of the extension principle.

From now on B is assumed to have only one global maximum b. $\mu_A(b)$ is the mean value of τ, i.e., the compatibility degree of A with respect to B is "approximately

$\mu_A(b)$"; $\mu_A(b)$ can be considered as a scalar inclusion index somewhat like consistency; instead of choosing $hgt(A \cap B)$, we prefer here the membership value in A of the element that mostly belongs to B. Note that the mean value of τ is always less than $hgt(A)$.

1.4.1 Hausdorff Metric

Let x be a point in $\mathbb{R}^n$ and A nonempty subset of $\mathbb{R}^n$. Define the *distance* $d(x, A)$ from x to A by

$$d(x, A) = \inf\{\| x - a \| : a \in A\},$$

also define

$$d_H^*(B, A) = \sup\{d(b, A) : b \in B\}.$$

Therefore define the *Hausdorff distance* between nonempty subsets A and B of $\mathbb{R}^n$ by

$$d_H(A, B) = \max\{d_H^*(A, B), d_H^*(B, A)\}.$$

This is now symmetric in A and B. Consequently,

(a) $d_H(A, B) \geq 0$ with $d_H(A, B) = 0$ if and only if $\overline{A} = \overline{B}$

(b) $d_H(A, B) = d_H(B, A)$

(c) $d_H(A, B) \leq d_H(A, C) + d_H(C, B)$,

for any nonempty subsets A, B and C of $\mathbb{R}^n$. Restricting attention to the subspace κ^n and κ_c^n the above Hausdorff distance is a metric, the *Hausdorff metric.*

Definition 1.4.2. (continuity) Consider mapping F from a domain T in $\mathbb{R}^k$ into the metric space(κ_C^n, d_H). The usual definition of continuity of mapping between metric spaces applies here. A set valued mapping $F : \mathbb{R}^k \to \kappa_C^n$ is continuous at t_0 in T if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$d_H(F(t), F(t_0)) < \varepsilon \tag{1.9}$$

for all $t \in T$ with $\| t - t_0 \| < \delta$. Equivalently, this can be stated in terms of convergence of sequences, that is as

$$\lim_{t_n \to t_0} d_H(F(t_n), F(t_0)) = 0$$

for all sequences $\{t_n\}$ in T with $t_n \to t_0$.

Using the Hausdorff separation d_H^* and neighborhoods, observe that (1.9) is equivalent to both

$$d_H^*(F(t), F(t_0)) < \varepsilon \qquad and \qquad d_H^*(F(t_0), F(t)) < \varepsilon$$

In the first case F is said to be *upper − semicontinuous* and in second case F is said to be *lower − semicontinuous* at t_0. Clearly F is continuous at t_0 if and only if it is both upper-semicontinuous and lower-semicontinuous at t_0. A set valued mapping can be lower-semicontinuous without being upper-semicontinuous, and vice versa.

Example 1.4.1. *The set valued mapping F from $\mathbb{R}$ into defined by*

$$F(t) = \begin{cases} \{0\} & for \quad t = 0 \\ [0,1] & for \quad t \in \mathbb{R} - \{0\} \end{cases}$$

is lower-semicontinuous,but not upper-semicontinuous, at $t_0 = 0$. *On the other hand,F from* $\mathbb{R}$ *defined by*

$$F(t) = \begin{cases} [0,1] & for \ \ t = 0 \\ \{0\} & for \ \ t \in \mathbb{R} - \{0\} \end{cases}$$

is upper-semicontinuous, but not lower-semicontinuous, at $t_0 = 0$.

1.5 Operations on Fuzzy Numbers

Some previous works related to operations on fuzzy numbers are those of Jain [27], Nahmias [36], Mizumoto and Tanaka [32],[31] Baas and Kwakernaak [15].

1.5.1 Unary Operation

Let φ be a unary operation; the extension principle reduces to

$$\forall M \in E, \ \ \mu_{\varphi(M)}(z) = \sup_{z=\varphi(x)} \mu_M(x).$$

Opposite of a fuzzy number: $\varphi(x) = -x$ is denoted by $-M$ and is such that

$$\forall x \in \mathbb{R}, \ \ \mu_{-M}(x) = \mu_M(-x).$$

M and $-M$ are symmetrical with respect to the axis $x = 0$.

Inverse of a fuzzy number: $\varphi(x) = \frac{1}{x}$ is denoted by M^{-1} and is such that

$$\forall x \in \mathbb{R} - \{0\}, \ \ \mu_{M^{-1}}(x) = \mu_M(\frac{1}{x}).$$

Let us call a fuzzy number M positive (resp. negative) if its membership function is such that $\mu_A(x) = 0, \quad \forall x < 0$(resp. $\forall x > 0$). This is denoted by $M > 0$ (resp. $M < 0$).

If M is neither positive nor negative, M^{-1} is no longer convex, and generally does not vanish when $|x| \longrightarrow 0$ (see Fig. 1.9(b)). However, when M is positive or negative, M^{-1} is convex (Fig. 1.9(a)).

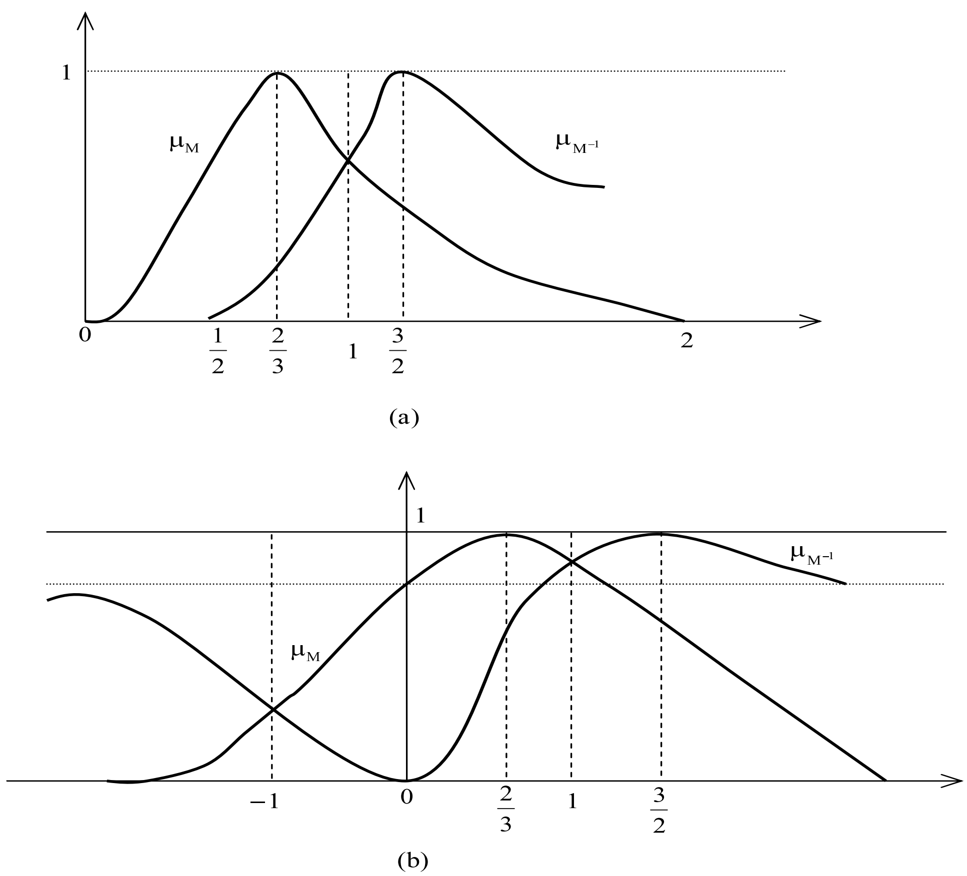

Figure 1.9

Scalar multiplication: $\mu_{\lambda.M(x)} = \mu_M(\frac{x}{\lambda}), \quad \forall\lambda \in \mathbb{R} - \{0\}$.

Exponential of a Fuzzy number: Let $\varphi(x) = e^x$. $\varphi(M)$ is denoted e^M and is such that

$$\mu_{e^M}(x) = \begin{cases} \mu_M(lnx) & x > 0, \\ 0 & oterwise. \end{cases}$$

e^M is a positive fuzzy number. Moreover, $e^{-M} = (e^M)^{-1}$.

1.5.2 Extended Addition and Multiplication

Addition: Addition is an increasing operation. Hence, the extended addition ($\oplus$) of fuzzy numbers gives a fuzzy number. Note that $-(M \oplus N) = (-M) \oplus (-N)$. ($\oplus$) is commutative and associative but has no group structure. The identity of ($\oplus$) is the nonfuzzy number 0. But M has no symmetrical element in the sense of a group structure. In particular, $M \oplus (-M) \neq 0, \quad \forall M \in E - \mathbb{R}$.

Multiplication: Multiplication is an increasing operation on $\mathbb{R}^+$ and a decreasing operation on $\mathbb{R}^-$. Hence, the product of fuzzy numbers ($\odot$) that are all either positive or negative gives a positive fuzzy number. Note that $-(M) \odot N = -(M \odot N)$, so that the factors can have different signs. ($\odot$) is commutative and associative. The set of positive fuzzy numbers is not a group for ($\odot$): although $\forall M, \quad M \odot 1 = M$, the product $M \odot M^{-1} \neq 1$ as soon as M is not a real number. M has no inverse in the sense of a group structure.

1.5.3 Extended Subtraction

Subtraction is neither increasing nor decreasing. However, it is easy to check that $M \ominus N = M \oplus (-N), \ \forall (M, N) \in E^2$ so that $M \ominus N$ is a fuzzy number whenever M and N are.

1.5.4 Extended Division

Division is neither increasing nor decreasing. But, since $M \oslash N = M \odot (N^{-1}), \forall (M, N) \in E^2$, $M \oslash N$ is a fuzzy number when M and N are positive or negative fuzzy numbers. The division of ordinary fuzzy numbers can be performed similarly to multiplication, by decomposition.

1.5.5 Extended Max and Min

Max and min are increasing operations in $\mathbb{R}$. The maximum (resp. minimum) of n fuzzy numbers $M_1, \cdots, M_n$, denoted $\widetilde{max}(M_1, \cdots, M_n)$ (resp. $\widetilde{min}(M_1, \cdots, M_n)$), is a fuzzy number. The maximum (resp. minimum), $\widetilde{max}$ (resp. $\widetilde{min}$) is the dual operation with respect to union (resp. intersection) because $M_1 \cup \cdots \cup M_n$ (resp. $M_1 \cap \cdots \cap M_n$) is obtained by considering the nonfuzzy maximum (resp. minimum) of the n membership functions and $\widetilde{max}(M_1, \cdots, M_n)$ (resp. $\widetilde{min}(M_1, \cdots, M_n)$) is similarly obtained provided that we exchange the coordinate axes $0x$ and $0y$ and that we consider increasing and decreasing parts separately.

Let M, N, P be three fuzzy numbers (i.e., convex normalized fuzzy sets of $\mathbb{R}$). The following properties hold:

$\widetilde{max}$ and $\widetilde{min}$ are commutative and associative operations; they are mutually distributive,

$$\widetilde{min}(M, \widetilde{max}(N, P)) = \widetilde{max}[\widetilde{min}(M, N), \widetilde{min}(M, P)],$$

$$\widetilde{max}(M, \widetilde{min}(N, P)) = \widetilde{min}[\widetilde{max}(M, N), \widetilde{max}(M, P)];$$

absorption laws,

$$\widetilde{max}(M, \widetilde{min}(M, N)) = M, \qquad \widetilde{min}(M, \widetilde{max}(M, N)) = M;$$

De Morgan laws,

$$1 \ominus \widetilde{min}(M, N) = max(1 \ominus M, 1 \ominus N),$$

$$1 \ominus \widetilde{max}(M, N) = min(1 \ominus M, 1 \ominus N);$$

note that $1 \ominus M$ is the "dual" of $\overline{M}$: indeed, $1 \ominus (1 \ominus M) = M$:

idempotents, $\widetilde{max}(M, M) = M = \widetilde{min}(M, M)$.

Definition 1.5.1. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be a function from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$. Then f is said to be a fuzzy function (or mapping) and we use the notation

$$f : \mathcal{F}(X) \to \mathcal{F}(Y).$$

It should be noted , however, that a fuzzy function is not necessarily defined by Zadeh's extension principle. It can be any function which maps a fuzzy set $A \in \mathcal{F}(X)$ into a fuzzy set $B := f(A) \in \mathcal{F}(Y)$.

Definition 1.5.2. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets . A fuzzy mapping $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ is said to be monotonic increasing if from $A, A' \in \mathcal{F}(X)$ and $A \subset A'$ it follow that $f(A) \subset f(A')$.

Theorem 1.5.1. *Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets. Then every fuzzy mapping $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined by the extension principle is monotonic increasing.*

Proof. Let $A, A' \in \mathcal{F}(X)$ such that $A \subset A'$. Then using the definition of sup-min extension principle we get

$$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \leq \sup_{x \in f^{-1}(y)} A'(x) = f(A')(y)$$

for all $y \in Y$. $\square$

Let A and B be fuzzy numbers with $[A]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ and $[B]_\alpha = [\underline{b}(\alpha), \overline{b}(\alpha)]$. Then it can easily be shown that

$$[A + B]_\alpha = [\underline{a}(\alpha) + \underline{b}(\alpha), \overline{a}(\alpha) + \overline{b}(\alpha)]$$

$$[-A]_\alpha = [-\overline{a}(\alpha), -\underline{a}(\alpha)]$$

$$[A - B]_\alpha = [\underline{a}(\alpha) - \overline{b}(\alpha), \overline{a}(\alpha) - \underline{b}(\alpha)]$$

$$[\lambda A]_\alpha = [\lambda \underline{a}(\alpha), \lambda \overline{a}(\alpha)], \quad \lambda \geq 0$$

$$[\lambda A]_\alpha = [\lambda \overline{a}(\alpha), \lambda \underline{a}(\alpha)], \quad \lambda < 0$$

for all $\alpha \in [0, 1]$, i.e. any set of the extended sum of two fuzzy numbers is equal to the sum of their $\alpha - level$ sets.

Remark 1.5.1. If $[A]_\alpha = [\underline{a}(\alpha)]$ and f is monotonic increasing then from the above theorem we get

$$[f(A)]_\alpha = f([A]_\alpha) = f([\underline{a}(\alpha), \overline{a}(\alpha)]) = [f(\underline{a}(\alpha)), f(\overline{a}(\alpha))].$$

Example 1.5.2. *Let* $f(x, y) = xy$ *and let* $[A]_\alpha = [a_{1\alpha}, a_{2\alpha}]$ *and* $[B]_\alpha = [b_{1\alpha}, b_{2\alpha}]$ *be two fuzzy numbers. Applying above theorem we get*

$$[AB]_\alpha = [A]_\alpha[B]_\alpha = [a_{1\alpha}b_{1\alpha}, a_{2\alpha}b_{2\alpha}],$$

hold if and only if A *and* B *are both nonnegative, i.e.* $A(x) = B(x) = 0$ *for* $x \leq 0$. *In general form we obtain a very complicated expression for the* α *level sets of the product* AB

$$[AB]_\alpha = [m_\alpha, M_\alpha],$$

where

$$m_\alpha = \min\{a_{1\alpha}b_{1\alpha}, a_{1\alpha}b_{2\alpha}, a_{2\alpha}b_{1\alpha}, a_{2\alpha}b_{2\alpha}\},$$

$$M_\alpha = \max\{a_{1\alpha}b_{1\alpha}, a_{1\alpha}b_{2\alpha}, a_{2\alpha}b_{1\alpha}, a_{2\alpha}b_{2\alpha}\},$$

for $\alpha \in I$.

Chapter 2

Fuzzy Linear Systems

2.1 Introduction

The main advantage of fuzzy models is their ability to describe expert knowledge in a descriptive, human like way, in the form of simple rules using linguistic variables. The theory of fuzzy sets allows the existence of uncertainty to vagueness (or fuzziness) rather than due to randomness. When using fuzzy sets, accuracy is traded for complexity-fuzzy logic models do not need an accurate definition on many systems (in term of the parameters).

Simultaneous linear equations play a major role in representing various systems in natural science, engineering, and social domain. Since in many applications at least some of the system's parameters and measurements are represented by exert experience in terms of fuzzy rather than crisp numbers, it is immensely important to develop mathematical models and numerical procedures that would appropriately deal with those general fuzzy terms.

2.2 Fuzzy Linear System

Definition 2.2.1. The $n \times n$ linear system of equations

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1,\\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2,\\
&\vdots\\
&a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n,
\end{aligned}
\tag{2.1}
$$

where the coefficients matrix $A = (a_{ij}), 1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and $y_i \in E^1, 1 \leq i \leq n$ is called a fuzzy system of linear equations (FSLE).

Using the extension principle, the addition and the scalar multiplication of fuzzy numbers are defined by

$$(u+v)(x) = \sup_{x=s+t} min\{u(s), v(t)\},$$

$$(ku)(x) = u(x/k);\ \ k \neq 0,$$

for $u, v \in E^1,\ k \in \mathbb{R}$. Equivalently, for arbitrary $u = (\underline{u}, \overline{u}),\ v = (\underline{v}, \overline{v})$ and $k \in \mathbb{R}$, we may define the addition and the scalar multiplication as

$$(\underline{u+v})(\alpha) = \underline{u}(\alpha) + \underline{v}(\alpha),$$

$$(\overline{u+v})(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha),$$

$$(\underline{ku})(\alpha) = k\underline{u}(\alpha), \quad (\overline{ku})(\alpha) = k\overline{u}(\alpha), \quad k \geq 0,$$

$$(\underline{ku})(\alpha) = k\overline{u}(\alpha), \quad (\overline{ku})(\alpha) = k\underline{u}(\alpha), \quad k \leq 0.$$

Definition 2.2.2. A fuzzy number vector $(x_1, x_2, \cdots, x_n)^T$ given by $[x_i]_\alpha = (\underline{x_i}(\alpha), \overline{x_i}(\alpha)), 1 \leq i \leq n, 0 \leq \alpha \leq 1$, is called a solution of (2.1) if

$$min\{\sum_{j=1}^{n} a_{ij}u_j | u_j \in [x_j]_\alpha\} = \underline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y_i}, \tag{2.2}$$

$$max\{\sum_{j=1}^{n} a_{ij}u_j | u_j \in [x_j]_\alpha\} = \overline{\sum_{j=1}^{n} a_{ij}x_j} = \sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y_i}, \tag{2.3}$$

where $[x_j]_\alpha$ is α-level set of x_j.

For a particular i, if $a_{ij} > 0, \quad 1 \leq j \leq n$, we simply get

$$\sum_{j=1}^{n} a_{ij}\underline{x}_j = \underline{y}_i, \qquad \sum_{j=1}^{n} a_{ij}\overline{x}_j = \overline{y}_i. \tag{2.4}$$

In general, however, an arbitrary equation for either $\underline{y}_i$ or $\overline{y}_i$ may include a linear combination of $\underline{x}_j$ and $\overline{x}_j$'s. Consequently, in order to solve the system given by (2.1) one must solve a $2n \times 2n$ crisp linear system where the right-hand side column is the function vector $(\underline{y}_1, ..., \underline{y}_n, \overline{y}_1, ..., \overline{y}_n)^T$.

Let us now rearrange the linear systems of equation (2.2) and (2.3) so that the unknowns are $\underline{x}_i, \overline{x}_i, \ 1 \leq i \leq n$, and the right-hand side column is

$$Y = (\underline{y}_1, \cdots, \underline{y}_n, -\overline{y}_1, \cdots, -\overline{y}_n)^T.$$

We get the $2n \times 2n$ linear system

$$\begin{array}{l} s_{1,1}\underline{x}_1 + s_{1,2}\underline{x}_2 + \cdots + s_{1,n}\underline{x}_n + s_{1,n+1}(-\overline{x}_1) + \cdots + s_{1,2n}(-\overline{x}_n) = \underline{y}_1, \\ \vdots \\ s_{n,1}\underline{x}_1 + s_{n,2}\underline{x}_2 + \cdots + s_{n,n}\underline{x}_n + s_{n,n+1}(-\overline{x}_1) + \cdots + s_{n,2n}(-\overline{x}_n) = \underline{y}_n, \\ s_{n+1,1}\underline{x}_1 + \cdots + s_{n+1,n}\underline{x}_n + s_{n+1,n+1}(-\overline{x}_1) + \cdots + s_{n+1,2n}(-\overline{x}_n) = -\overline{y}_1, \\ \vdots \\ s_{2n,1}\underline{x}_1 + s_{2n,2}\underline{x}_2 + \cdots + s_{2n,n}\underline{x}_n + s_{2n,n+1}(-\overline{x}_1) + \cdots + s_{2n,2n}(-\overline{x}_n) = -\overline{y}_n, \end{array} \tag{2.5}$$

where s_{ij} are determined as flows:

$$a_{ij} \geq 0 \implies s_{ij} = s_{i+n,j+n} = a_{ij},$$
$$a_{ij} \leq 0 \implies s_{i+n,j} = s_{i,j+n} = -a_{ij},$$

and any s_{ij} which is not determined is zero. Using matrix notation we get

$$SX = Y, \tag{2.6}$$

where $S = (s_{ij}), \quad 1 \leq i, j \leq 2n$, and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \qquad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}. \tag{2.7}$$

Example 2.2.1. *Consider the* 2×2 *fuzzy linear system*

$$\begin{cases} x_1 - x_2 = y_1, \\ x_1 + 2x_2 = y_2. \end{cases}$$

The 4×4 *system is*

$$\begin{aligned} \underline{x}_1 \qquad\qquad + (-\overline{x}_2) &= \underline{y}_1, \\ \underline{x}_1 + 2\underline{x}_2 \qquad\qquad &= \underline{y}_2, \\ \underline{x}_2 + (-\overline{x}_1) \qquad &= -\overline{y}_1, \\ (-\overline{x}_1) + 2(-\overline{x}_2) &= -\overline{y}_2, \end{aligned}$$

i.e.

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now we have

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix},$$

where S_1 contains the positive entries of A,S_2 the absolute value of the negative entires of A and $A = S_1 - S_2$.

The linear system (2.6) is now a $2n \times 2n$ crisp linear system and can be uniquely solved for X, if and only if the matrix S is nonsingular. We therefore must answer two questions:

1. Is S nonsingular?

2. Do the components of the $2n$-dimensional solution vector X represent an n-dimensional solution fuzzy vector to the fuzzy system given by (2.1)?

If S is nonsingular, the answer to the second question is positive if and only if $(\underline{x}_i, \overline{x}_i)$ is a fuzzy number for all i.

The next example reveals the notable fact that S may be singular even if the original matrix A is not.

Example 2.2.2. *The matrix A of the linear fuzzy system*

$$\begin{cases} x_1 & - & x_2 & = & y_1, \\ x_1 & + & x_2 & = & y_2, \end{cases}$$

is nonsingular, while

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

is singular. In other words, a fuzzy linear system represented by a nonsingular matrix A may have no solution or an infinite number of solutions.

The next result eliminates the possibility of a unique fuzzy solution, whenever the crisp system is not uniquely solved, i.e. whenever A is singular.

Theorem 2.2.3. *The matrix S is nonsingular if and only if the matrices $A = S_1 - S_2$ and $S_1 - S_2$ are both nonsingular.*

Proof. By adding the $(n+i)$ th row of S from its i th row for $1 \leq i \leq n$ we obtain

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} \longrightarrow \begin{bmatrix} S_1 + S_2 & S_2 + S_1 \\ S_2 & S_1 \end{bmatrix} = \acute{S}_1.$$

Next, we subtract the j th column of S to its $(n+j)$ th column for $1 \leq j \leq n$ to obtain

$$\acute{S}_1 = \begin{bmatrix} S_1 + S_2 & S_2 + S_1 \\ S_2 & S_1 \end{bmatrix} \longrightarrow \begin{bmatrix} S_1 + S_2 & 0 \\ S_2 & S_1 - S_2 \end{bmatrix} = \acute{S}_2.$$

Clearly, $|S| = |\acute{S}_1| = |\acute{S}_2| = |S_1 + S_2||S_1 - S_2| = |S_1 + S_2||A|$. Therefore $|S| \neq 0$ if and only if $|A| \neq 0$ and $|S_1 + S_2| \neq 0$, which concludes the proof. □

Corollary 1. If a crisp linear system does not have a unique solution, the associated fuzzy linear system does not have one either.

In order to solve the linear fuzzy system (2.1) we must now calculate S^{-1} (whenever it exists). The next result is taken from the theory of block matrices and provides the structure of S^{-1}.

Theorem 2.2.4. *If S^{-1} exists it must have the same structure as S, i.e.*

$$S^{-1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}.$$

Proof. Let s_{ij} denote the entry of S in the i-th row and the j-th column. If t_{ij} denotes the entry of S^{-1} at the same location then

$$t_{ij} = \frac{(-1)^{i+j}|S_{ji}|}{|S|},$$

where S_{ji} is the matrix obtained by removing the j-th row and the i-th column of S. Consider now for example the entries $t_{i,n+j}$ and $t_{n+i,j}$ of S^{-1} for some $1 \le i, j \le n$. The associated matrices are $S_{n+j,i}$ and $S_{j,n+i}$, respectively. It can be easily shown that $s_{n+j,i}$ can be obtain from $S_{j,n+i}$ by interchanging rows and columns an even number of times. Therefore

$$t_{i,n+j} = (-1)^{i+n+j}\frac{|S_{n+j,i}|}{|S|} = (-1)^{i+n+j}\frac{|S_{j,n+i}|}{|S|} = t_{n+i,j}.$$

Similarly, $t_{ij} = t_{n+i,n+j}$ for arbitrary i and j and thus S^{-1} must have the same structure like S. □

In order to calculate E and D we write

$$SS^{-1} = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} D & E \\ E & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

and get

$$S_1D + S_2E = I, \qquad S_2D + S_1E = 0$$

. By adding and then subtracting the corresponding sides of $S_1D+S_2E = I, \;\; S_2D+ S_1E = 0$ we obtain

$$D + E = (S_1 + S_2)^{-1}, \qquad D - E = (S_1 - S_2)^{-1},$$

and consequently

$$D = \frac{1}{2}[(S_1 + S_2)^{-1} + (S_1 - S_2)^{-1}],$$
$$E = \frac{1}{2}[(S_1 + S_2)^{-1} - (S_1 - S_2)^{-1}].$$

Assuming that S (i.e. $S_1 + S_2$ and $S_1 - S_2$) is nonsingular we obtain

$$X = S^{-1}Y.$$

The solution vector is thus unique but may still not be an appropriate fuzzy vector. The following result provides necessary and sufficient conditions for the unique solution vector to be a fuzzy vector, given arbitrary input fuzzy vector Y.

Theorem 2.2.5. *if S^{-1} is nonnegative and Y is an arbitrary fuzzy vector then the unique solution X of $X = S^{-1}Y$ is a fuzzy vector.*

Proof. Let $S^{-1} = (t_{ij}) > 0, \;\; 1 \leq i, j \leq 2n$ then

$$\underline{x}_i = \sum_{j=1}^{n} t_{ij}\underline{y}_i - \sum_{j=1}^{n} t_{i,n+j}\overline{y}_j, \qquad 1 \leq i \leq n, \tag{2.8}$$

$$-\overline{x}_i = \sum_{j=1}^{n} t_{n+i,j}\underline{y}_j - \sum_{j=1}^{n} t_{n+i,n+j}\overline{y}_j, \qquad 1 \leq i \leq n. \tag{2.9}$$

Due to the particular structure of S^{-1} we can replace (2.9) by

$$\overline{x}_i = -\sum_{j=1}^{n} t_{i,n+j}\underline{y}_j + \sum_{j=1}^{n} t_{i,j}\overline{y}_j, \quad 1 \leq i \leq n, \tag{2.10}$$

and by subtracting (2.8) from (2.10) we get

$$\overline{x}_i - \underline{x}_i = \sum_{j=1}^{n} t_{ij}(\overline{y}_j - \underline{y}_j) + \sum_{j=1}^{n} t_{i,n+j}(\overline{y}_j - \underline{y}_j), \quad 1 \leq i \leq n.$$

Thus, if Y is arbitrary input vector which represents a fuzzy vector, i.e. $\overline{y}_i - \underline{y}_i \geq 0, \ \ 1 \leq i \leq n$. $\square$

We now restrict the discussion to the triangular fuzzy numbers , i.e. $\underline{y}_i(\alpha), \overline{y}_i(\alpha)$ and consequently $\underline{x}_i(\alpha), \overline{x}_i(\alpha)$ are all linear functions of α. Having calculated X which solves $SX = Y$ we now define the fuzzy solution to the original system given by (2.1).

Definition 2.2.3. Let $X = \{(\underline{x}_i(\alpha), \overline{x}_i(\alpha)), \ \ 1 \leq i \leq n\}$ denote the unique solution of $SX = Y$. The fuzzy number vector $U = \{(\underline{u}_i(\alpha), \overline{u}_i(\alpha)), \ \ 1 \leq i \leq n\}$ defined by

$$\underline{u}_i(\alpha) = min\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), \underline{x}_i(1)\},$$

$$\overline{u}_i(\alpha) = max\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), \underline{x}_i(1)\}.$$

is called the fuzzy solution of $SX = Y$.

The use of $\underline{x}_i(1)$ is to eliminate the possibility of fuzzy numbers whose associated triangular possess an angle grater than $90°$. If $(\underline{x}_i(\alpha), \overline{x}_i(\alpha)), \ \ 1 \leq i \leq n$, are all fuzzy numbers then $\underline{u}_i(\alpha) = \underline{x}_i(\alpha), \ \overline{u}_i(\alpha) = \overline{x}_i(\alpha), 1 \leq i \leq n$ and U is called strong fuzzy solution. Otherwise, U is weak fuzzy solution.

Example 2.2.6. *Consider the 2×2 fuzzy linear system*

$$\begin{cases} x_1 - x_2 = (\alpha, 2-\alpha), \\ x_1 + 3x_2 = (4+\alpha, 7-2\alpha). \end{cases}$$

The extended 4×4 matrix is

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

and the solution is

$$X = \begin{bmatrix} \underline{x}_1(\alpha) \\ \underline{x}_2(\alpha) \\ -\overline{x}_1(\alpha) \\ -\overline{x}_2(\alpha) \end{bmatrix} = S^{-1}Y = \begin{bmatrix} 1.125 & -0.125 & 0.375 & -0.375 \\ -0.375 & 0.375 & -1.125 & 0.125 \\ 0.375 & -0.375 & 1.125 & -0.125 \\ -0.125 & 0.125 & -0.375 & 0.375 \end{bmatrix} \begin{bmatrix} \alpha \\ 4+\alpha \\ \alpha-2 \\ 2\alpha-7 \end{bmatrix},$$

i.e.

$$\underline{x}_1(\alpha) = 1.375 + 0.625\alpha, \qquad \overline{x}_1(\alpha) = 2.875 - 0.875\alpha,$$

$$\underline{x}_2(\alpha) = 0.875 + 0.125\alpha, \qquad \overline{x}_2(\alpha) = 1.375 - 0.375\alpha.$$

Here $\underline{x}_1 \leq \overline{x}_1$, $\underline{x}_2 \leq \overline{x}_2$; $\underline{x}_1$, $\underline{x}_2$ are monotonic decreasing functions. Therefore the fuzzy solution is $x_1 = (\underline{x}_1, \overline{x}_1)$, $x_2 = (\underline{x}_2, \overline{x}_2)$ and it is strong fuzzy solution.

Example 2.2.7. *Consider the 3×3 fuzzy linear system*

$$\begin{cases} x_1 + x_2 - x_3 = (\alpha, 2-\alpha), \\ x_1 - 2x_2 + x_3 = (2+\alpha, 3), \\ 2x_1 + x_2 + 3x_3 = (-2, -1-\alpha), \end{cases}$$

with

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 3 \end{bmatrix},$$

and

$$Y = (\alpha, 2+\alpha, -2, \alpha-2, -3, 1+\alpha)^T.$$

The solution vector of $SX = Y$ *is*

$$X = \begin{bmatrix} -2.31 + 3.62\alpha \\ -0.62 - 0.77\alpha \\ 1.08 - 2.15\alpha \\ -4.69 + 3.38\alpha \\ 1.62 - 0.23\alpha \\ 2.92 - 1.85\alpha \end{bmatrix},$$

i.e.

$$x_1 = (-2.31 + 3.62\alpha, 4.69 - 3.38\alpha),$$

$$x_2 = (-0.62 - 0.77\alpha, -1.62 + 0.23\alpha),$$

$$x_3 = (1.08 - 2.15\alpha, -2.92 + 1.85\alpha).$$

The fact that x_2 *and* x_3 *are not fuzzy numbers is obvious. The fuzzy solution in this* *case is a week solution given by*

$$u_1 = (-2.31 + 3.62\alpha, 4.69 - 3.38\alpha),$$

$$u_2 = (-1.62 + 0.23\alpha, -0.62 - 0.77\alpha),$$

$$u_3 = (-2.92 + 1.85\alpha, 1.08 - 2.15\alpha).$$

2.2.1 A Dual Fuzzy Linear System

Usually, there is no inverse element for an arbitrary fuzzy number $u \in E^1$, i.e. there exists no element $v \in E^1$ such that

$$u + v = 0.$$

Actually, for all non-crisp fuzzy numbers $u \in E^1$ we have

$$u + (-u) \neq 0.$$

Therefore, the fuzzy linear system of equations

$$AX = BX + Y,$$

cannot be equivalently replaced by the fuzzy linear system

$$(A - B)X = Y,$$

which had been investigated. In the sequel, we will called the fuzzy linear system

$$AX = BX + Y,$$

where $A = (a_{ij}),\ B = (b_{ij}),\ \ 1 \leq i, j \leq n$ are crisp coefficient matrices and Y a fuzzy vector, a dual fuzzy linear system.

Theorem 2.2.8. *Let $A = (a_{ij}),\ B = (b_{ij}),\quad 1 \le i, j \le n$, be nonnegative matrices. The dual linear system has a unique fuzzy solution if and only if the inverse of $A - B$ exists and has only nonnegative entries.*

Proof. We know the dual fuzzy linear system

$$\sum_{i=1}^{n} a_{ji} x_i = \sum_{i=1}^{n} b_{ji} x_i + y_j,$$

is equivalent to (since $a_{ji} \ge 0$ and $b_{ji} \ge 0$ for all i, j)

$$\sum_{i=1}^{n} a_{ji} \underline{x}_i = \sum_{i=1}^{n} b_{ji} \underline{x}_i + \underline{y}_j,$$

$$\sum_{i=1}^{n} a_{ji} \overline{x}_i = \sum_{i=1}^{n} b_{ji} \overline{x}_i + \overline{y}_j.$$

It follows, that

$$\sum_{i=1}^{n} (a_{ji} - b_{ji}) \underline{x}_i = \underline{y}_j,$$

$$\sum_{i=1}^{n} (a_{ji} - b_{ji}) \overline{x}_i = \overline{y}_j.$$

If $(A - B)^{-1}$ exists, the equations have unique solutions $(\underline{x}_i)_1^n$ and $(\overline{x}_i)_1^n$; and clearly if $(A - B)^{-1} \ge 0$ for all i, j $(\underline{x}_i, \overline{x}_i)$ is a fuzzy number. $\square$

The following theorem guarantees the existence of a fuzzy solution for a general case. Consider dual fuzzy linear system, and transform its $n \times n$ coefficient A and B into $2n \times 2n$ matrices. Define matrices $S = (s_{ij}), T = (t_{i,j});\quad 1 \le i, j \le 2n$ by

$$a_{ij} \ge 0 \Longrightarrow s_{ij} = a_{ij}, \qquad s_{i+n,j+n} = a_{ij},$$

$$a_{ij} < 0 \Longrightarrow s_{i,j+n} = -a_{ij}, \qquad s_{i+n,j} = -a_{ij},$$

$$b_{ij} \geq 0 \Longrightarrow t_{ij} = b_{ij}, \qquad t_{i+n,j+n} = b_{ij},$$

$$b_{ij} < 0 \Longrightarrow t_{i,j+n} = -b_{ij}, \qquad t_{i+n,j} = -b_{ij},$$

with all the remaining s_{ij}, t_{ij} taken to be zero.

Theorem 2.2.9. *The dual fuzzy linear system has a unique fuzzy solution if the inverse matrix $S - T$ exists and is nonnegative.*

Proof. Using the form of Eq.(2.5), we obtain that the system $AX = BX + Y$ is equivalent to the system

$$SX = TX + Y,$$

where X, Y are given by Eq.(2.7). Consequently,

$$(S - T)X = Y,$$

and a solution exists if $S - T$ is nonsingular. If in addition $(S - T)^{-1} \geq 0$, then by virtue of Theorem 2.2.5 the solution X provides a fuzzy solution. $\square$

Example 2.2.10. *Consider the 2×2 dual fuzzy linear system*

$$\begin{cases} x_1 + 3x_2 = 3x_1 + 2x_2 + (1+\alpha, 4-2\alpha), \\ 2x_1 + x_2 = x_1 + 5x_2 + (3+\alpha, 5-\alpha), \end{cases}$$

with

$$S - T = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix},$$

and

$$Y = (1+\alpha, 3+\alpha, -4+2\alpha, -5+\alpha)^T$$

and

$$(S-T)^{-1} = \begin{bmatrix} -0.5714 & -0.1429 & 0 & 0 \\ -0.1429 & -0.2857 & 0 & 0 \\ 0 & 0 & 0.5714 & 0.1429 \\ 0 & 0 & 0.1429 & 0.2857 \end{bmatrix},$$

The solution vector of $(S-T)X = Y$ *is*

$$X = \begin{bmatrix} -1.1667 - 0.8333\alpha \\ -1.3333 - 0.6667\alpha \\ 3.0001 - 1.2857\alpha \\ 2.0001 - 0.5715\alpha \end{bmatrix},$$

i.e.

$$x_1 = (1.1667 + 0.8333\alpha, -3.0001 + 1.2857\alpha),$$

$$x_2 = (1.3333 + 0.6667\alpha, -2.0001 + 0.5715\alpha).$$

The fact that x_1 *and* x_2 *are not fuzzy numbers is obvious. The fuzzy solution in this case is a week solution given by*

$$u_1 = (-3.0001 + 1.2857\alpha, -1.0001 - 0.7143\alpha),$$

$$u_2 = (-2.0001 + 0.5715\alpha, -1 - 0.4186\alpha).$$

2.2.2 LU Decomposition Method for FSLE

Theorem 2.2.11. *Let A be an $n \times n$ matrix with all nonzero leading principal minors. Then A has a unique factorization:*

$$A = LU,$$

where L is unit lower triangular and U is upper triangular, [19].

In order to decompose of S, we must find matrices L and U such that $S = LU$, which

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}.$$

L_{11} and L_{22} are lower triangular matrices, U_{11} and U_{22} are upper triangular matrices. Now we suppose that $A = S_1 - S_2$ has LU decomposition. We have

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix},$$

then

$$S_1 = L_{11}U_{11}, \tag{2.11}$$

$$S_2 = L_{11}U_{12} \Rightarrow U_{12} = L_{11}^{-1}S_2,$$

$$S_2 = L_{21}U_{11} \Rightarrow L_{21} = S_2U_{11}^{-1},$$

$$S_1 = L_{21}U_{12} + L_{22}U_{22}.$$

Now we can write

$$S_1 - S_2S_1^{-1}S_2 = L_{22}U_{22}. \tag{2.12}$$

From (2.11) and (2.12) if S_1 and $S_1 - S_2S_1^{-1}S_2$ both have LU decomposition, then S has LU decomposition.

Theorem 2.2.12. *Let S be an $n \times n$ symmetric positive definite matrix then there exists a unique lower triangular matrix L with positive diagonal entries such that*

$$S = LL^T,$$

[26].

So if S be a symmetric positive definite matrix we have

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix},$$

then

$$S_1 = L_{11}L_{11}^T, \tag{2.13}$$

$$S_2 = L_{11}L_{21}^T \Rightarrow L_{21}^T = L_{11}^{-1}S_2,$$

$$S_2 = L_{21}L_{11}^T \Rightarrow L_{21} = S_2(L_{11}^T)^{-1},$$

$$S_1 = L_{21}U_{12} + L_{22}U_{22},$$

and hence

$$S_1 - S_2S_1^{-1}S_2 = L_{22}L_{22}^T. \tag{2.14}$$

For using this method the matrices S_1 and $S_1 - S_2S_1^{-1}S_2$ should be symmetric positive definite.

Example 2.2.13. *Consider the 2×2 fuzzy system*

$$\begin{cases} x_1 - x_2 = (-7 + 2\alpha, -3 - 2\alpha), \\ x_1 + 3x_2 = (19 + 4\alpha, 27 - 4\alpha). \end{cases}$$

The extended 4×4 matrix is

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

and

$$S_1 = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix},$$

$$S_1 - S_2 S_1^{-1} S_2 = \begin{bmatrix} 1 & 0.3333 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.3333 \\ 0 & 2.6666 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0.3333 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0.3333 \\ 0 & 0 & 0 & 2.6666 \end{bmatrix}.$$

Now the solution is

$$x_1 = (\underline{x}_1(\alpha), \overline{x}_1(\alpha)) = (1 + \alpha, 3 - \alpha),$$

$$x_2 = (\underline{x}_2(\alpha), \overline{x}_2(\alpha)) = (6 + \alpha, 8 - \alpha).$$

Example 2.2.14. *Consider the* 3×3 *fuzzy system*

$$2x_1 + x_2 + 3x_3 = (11 + 8\alpha, 27 - 8\alpha),$$

$$4x_1 + x_2 - x_3 = (-23 + 10\alpha, -5 - 8\alpha),$$

$$-x_1 + 3x_2 + x_3 = (10 + 5\alpha, 27 - 12\alpha).$$

The extended 6×6 *matrix is*

$$S = \begin{bmatrix} 2 & 1 & 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 & 3 & 1 \end{bmatrix},$$

and

$$S_1 = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & -17 \end{bmatrix},$$

$$S_1 - S_2 S_1^{-1} S_2 = \begin{bmatrix} 2.0000 & 1.0000 & 3.0000 \\ 4.0000 & 1.0000 & -0.3592 \\ -0.3592 & 3.0000 & 1.0000 \end{bmatrix},$$

$$= \begin{bmatrix} 1.0000 & 0 & 0 \\ 0.5000 & 1.0000 & 0 \\ -0.1764 & -6.3592 & 1.0000 \end{bmatrix} \begin{bmatrix} 2.0000 & 1.0000 & 3.0000 \\ 0 & -0.5000 & -1.6529 \\ 0 & 0 & -10.2422 \end{bmatrix},$$

and hence $S = LU$ *that*

$$L = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 2.0000 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 3.0000 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0.3529 & 0.5000 & 1.0000 & 0 \\ 0 & 0 & 0 & 0.1764 & 6.3529 & 1.0000 \end{bmatrix},$$

$$U = \begin{bmatrix} 2.0000 & 1.0000 & 3.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 6.0000 & 0 & 0 & 0 \\ 0 & 0 & 17.0000 & 6 & 0 & 0 \\ 0 & 0 & 0 & 2.0000 & 1.0000 & 3.0000 \\ 0 & 0 & 0 & 0 & 0.5000 & 1.8529 \\ 0 & 0 & 0 & 0 & 0 & 10.2422 \end{bmatrix}.$$

Now the solution is

$$x_1 = (\underline{x}_1(\alpha), \overline{x}_1(\alpha)) = (-4 + 2\alpha, -1 - \alpha),$$

$$x_2 = (\underline{x}_2(\alpha), \overline{x}_2(\alpha)) = (1 + \alpha, 5 - 3\alpha),$$

$$x_3 = (\underline{x}_3(\alpha), \overline{x}_3(\alpha)) = (6 + \alpha, 8 - \alpha).$$

Example 2.2.15. *Consider the* 3×3 *fuzzy system*

$$\begin{cases} 4x_1 + 2x_2 - x_3 = (-27 + 7\alpha, -7 - 13\alpha), \\ 2x_1 + 7x_2 + 6x_3 = (1 + 15\alpha, 40 - 24\alpha), \\ -x_1 + 6x_2 + 10x_3 = (26 + 18\alpha, 47 - 33\alpha). \end{cases}$$

The extended 6×6 *matrix is*

$$S = \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 1 \\ 2 & 7 & 6 & 0 & 0 & 0 \\ 0 & 6 & 10 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 2 & 7 & 6 \\ 1 & 0 & 0 & 0 & 6 & 10 \end{bmatrix},$$

and

$$S_1 = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 7 & 6 \\ 0 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 2.0000 & 0 & 0 \\ 1.0000 & 2.4495 & 0 \\ 0 & 2.4495 & 2.0000 \end{bmatrix} \begin{bmatrix} 2.0000 & 1.0000 & 0 \\ 0 & 2.4495 & 2.4495 \\ 0 & 0 & 2.0000 \end{bmatrix},$$

$$S_1 - S_2 S_1^{-1} S_2 = \begin{bmatrix} 3.7500 & 2.0000 & -0.1250 \\ 2.0000 & 7.0000 & 6.0000 \\ -0.1250 & 6.0000 & 9.6458 \end{bmatrix}$$

$$= \begin{bmatrix} 1.9365 & 0 & 0 \\ 1.0328 & 2.4358 & 0 \\ -0.0645 & 2.4906 & 1.8544 \end{bmatrix} \begin{bmatrix} 1.9365 & 1.0328 & -0.0645 \\ 0 & 2.4358 & 2.4906 \\ 0 & 0 & 1.8544 \end{bmatrix},$$

and hence $S = LL^T$ *that*

$$L = \begin{bmatrix} 2.0000 & 0 & 0 & 0 & 0 & 0 \\ 1.0000 & 2.4495 & 0 & 0 & 0 & 0 \\ 0 & 2.4495 & 2.0000 & 0 & 0 & 0 \\ 0 & 0 & 0.5000 & 1.9365 & 0 & 0 \\ 0 & 0 & 0 & 1.0328 & 2.4358 & 0 \\ 0.5000 & 0.2041 & 0.2500 & 0.0645 & 2.4956 & 1.8544 \end{bmatrix}.$$

Now the solution is

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (-5 + \alpha, -2 - 2\alpha),$$

$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (-1 + \alpha, 2 - 2\alpha),$$

$$x_3 = (\underline{x}_3(r), \overline{x}_3(r)) = (3 + \alpha, 5 - \alpha).$$

Chapter 3

Iterative Methods for Solving Fuzzy Linear Equations

3.1 Introduction

Systems of fuzzy linear equations arise in a large number of areas, both directly in modeling physical situations and indirectly in the numerical solution of other mathematical models. These applications occur in virtually all areas of the physical, biological and social sciences. Because of the widespread importance of fuzzy linear systems, we do much research on numerical solution of these systems and some of these are defined, analyzed and illustrated in this chapter.

The most common type of problem is to solve a square linear system

$$Ax = b,$$

of moderate order, with coefficients that are mostly nonzero and crisp matrix with n columns and n rows with the fuzzy vector b. For such systems the matrix S in (2.6) must generally be stored and there are limitations in most computers. With the rapid decrease in the cost of computer memory, quite large fuzzy linear systems can be accommodated on some machines, the practical upper limits on the order will be of size 100 to 1000. The algorithms based on direct methods; for solving such systems are defined in Section 2. As we noted it is direct method in the theoretical sense that if rounding errors are ignored, then the exact answer is found in a finite number of steps.

A second important type of problem is to solve $Ax = b$ when A is square, sparse, with large order. A sparse matrix is one in which most entries are zero. Such systems arise in variety of ways, but we restrict our development to those for which there is a simple, known pattern for the nonzero entries. Because of the large order of most sparse fuzzy systems, sometimes as large as 10^5 or more, the fuzzy linear systems cannot usually be solved by direct method such as LU decomposition method. Iteration methods are the preferred method of solution and these are introduced by Allahviranloo in [7] and [8]. X. Wang et al. presented an iterative algorithm for dual linear system of the form $X = AX + U$, where A is real $n \times n$ matrix, the unknown vector X and the constant U are all vectors consisting of n fuzzy numbers [40].

We use iterative conjugate gradient method for solving symmetric positive definite FSLE ($FSLE^+$) in this chapter. Numerical examples are given. And in the sequel we use iterative generalized minimal residual (GMRES) method for solving fuzzy linear system and present it's algorithm.

3.2 Conjugate Gradient (CG) Method for $FSLE^{+}$

Theorem 3.2.1. *A symmetric matrix is positive definite if and only if all its eigenvalues are positive [20].*

Theorem 3.2.2. *If $S_1 + S_2$ and $S_1 - S_2$ are symmetric positive definite then S is symmetric positive definite.*

Proof. Obviously S is symmetric. Let λ be the eigenvalue of

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix},$$

and X be the corresponding eigenvector, hence

$$\begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \lambda \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

and then

$$\begin{cases} S_1X_1 + S_2X_2 = \lambda X_1, \\ S_2X_1 + S_1X_2 = \lambda X_2. \end{cases}$$

By addition and subtraction of the above relations we have

$$\begin{cases} (S_1 + S_2)(X_1 + X_2) = \lambda(X_1 + X_2), \\ (S_1 - S_2)(X_1 - X_2) = \lambda(X_1 - X_2). \end{cases}$$

The above relations show that λ, is the eigenvalue of $S_1 + S_2$ and $S_1 - S_2$, which concludes the proof. □

Theorem 3.2.3. *If $S_1 - S_2$ is a symmetric positive definite and tridiagonal matrix then $S_1 + S_2$ is symmetric positive definite.*

Proof. Let

$$S_1 - S_2 = \begin{bmatrix} g_1 & f_2 & & & \\ f_2 & g_2 & f_3 & & \\ & \ddots & \ddots & \ddots & \\ & & f_{n-1} & g_{n-1} & f_n \\ & & & f_n & g_n \end{bmatrix},$$

where $g_1, g_2, \ldots, g_n$ are positive. Now we have

$$det(S_1 - S_2 - \lambda I) = (g_n - \lambda)P_{n-1}(\lambda) - f_n^2 P_{n-2}(\lambda) = det(S_1 + S_2 - \lambda I),$$

where $P_k(\lambda)$ is the characteristic polynomial of $k \times k$ principle submatrix of $S_1 - S_2$, which concludes the proof. □

Now we can use some spatial iterative method like Conjugate Gradient Method to solve FSLE when $S_1 + S_2$ and $S_1 - S_2$ are symmetric positive definite. The Conjugate Gradient method is an effective method for symmetric positive definite systems. This method proceeds by generating vector sequences of iterates (*i.e*, successive approximations to the solutions), residuals corresponding to the iterates, and search directions used in updating the iterates and residuals. Although the length of these sequences can become large, only a small number of vectors needs to be kept in memory. In every iteration of the method, two inner products are performed in order to compute update scalars that are defined to make the sequences satisfy certain orthogonality conditions. On a symmetric positive definite linear system these imply that the distance to the true solution is minimized in some norm.

The iterates X^i are updated in each iteration by a multiple (α_i) of the search direction vector P^i, i.e.

$$X^i = X^{i-1} + \alpha_i P^i.$$

Corresponding residuals $R^i = Y - AX^i$ are updated as

$$R^i = R^{i-1} - q^i \quad where \quad q^i = AP^i.$$

The choice $\alpha = \alpha_i = R^{i^T} R^i / P^{i^T} AP^i$ minimizes $R^{i^T} A^{-1} R^i$ over all possible choices for α.

The search directions are updated using the residuals,

$$P^i = R^i + \beta_{i-1} P^{i-1},$$

where the choice $\beta_i = R^{i^T} R^i / R^{i-1^T} R^{i-1}$ ensures that P^i and AP^{i-1} are orthogonal [16].

In the absence of roundoff errors the conjugate gradient method should converge in no more than n iterations. In fact, it can be shown that the error at every step decreases [20]. Specifically it can be proved that:

Theorem 3.2.4.

$$\| X - X^k \|_2 < \| X - X^{k-1} \|_2,$$

where X is the exact solution.

Example 3.2.5. *Consider the* 3×3 *fuzzy system*

$$\begin{cases} 4x_1 + 2x_2 - x_3 = (-27 + 7r, -7 - 13r), \\ 2x_1 + 7x_2 + 6x_3 = (1 + 15r, 40 - 24r), \\ -x_1 + 6x_2 + 10x_3 = (26 + 18r, 47 - 33r). \end{cases}$$

The matrixes $S_1 + S_2$ *and* $S_1 - S_2$ *are symmetric positive definite so the extended matrix*

$$S = \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 1 \\ 2 & 7 & 6 & 0 & 0 & 0 \\ 0 & 6 & 10 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 2 & 7 & 6 \\ 1 & 0 & 0 & 0 & 6 & 10 \end{bmatrix},$$

is symmetric positive definite.

Now the exact solutions are

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (-5 + r, -2 - 2r),$$

$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (-1 + r, 2 - 2r),$$

$$x_3 = (\underline{x}_3(r), \overline{x}_3(r)) = (3 + r, 5 - r).$$

The exact and approximated solutions are plotted and compared in Fig. 3.1.

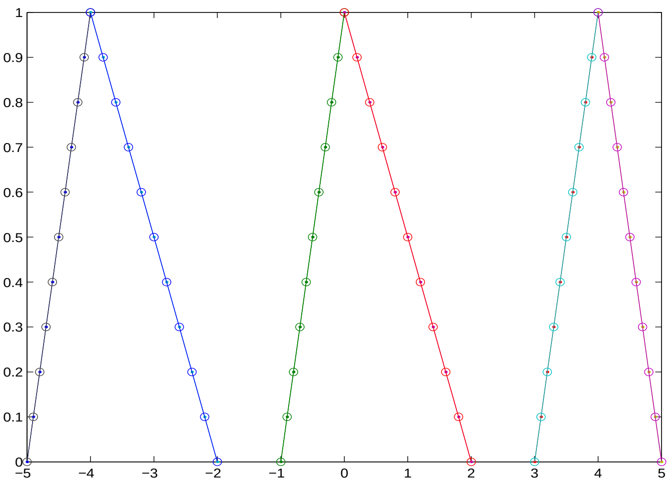

Figure 3.1: The Hausdrff distance of solutions with $\epsilon = 10^{-8}$ is 8.6153e-014.

Example 3.2.6. *Consider the 4×4 fuzzy system*

$$\begin{cases} 2x_1 - 2x_2 = (-16 + 8r, -2 - 6r), \\ -2x_1 + 2x_2 - 2x_3 = (2r, 12 - 14r), \\ -2x_2 + 2x_3 = (-6 + 12r, 16 - 10r), \\ 2x_4 = (12 + 2r, 18 - 4r). \end{cases}$$

The matrixes $S_1 + S_2$ and $S_1 - S_2$ are symmetric positive definite so the extended *matrix*

$$S = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

is symmetric positive definite.

Now the exact solutions are

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (-5 + r, -2 - 2r),$$

$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (-1 + r, 2 - 2r),$$

$$x_3 = (\underline{x}_3(r), \overline{x}_3(r)) = (3 + r, 5 - r).$$

The exact and approximated solutions are plotted and compared in Fig. 3.2.

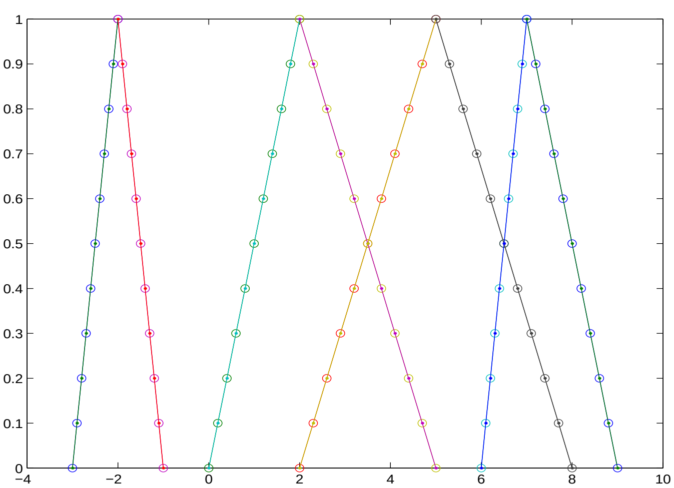

Figure 3.2: The Hausdrff distance of solutions with $\epsilon = 10^{-8}$ is 3.5527e-015.

3.3 Generalized minimal residual (GMRES) method for FSLE

The Generalized Minimal Residual Method is an extension of CG (which is only applicable to symmetric positive definite systems) to nonsymmetric systems. Like CG, it generates a sequence of orthogonal vectors, but in the absence of symmetry this can no longer be done with short recurrences; instead, all previously computed vectors in the orthogonal sequence have to be retained. For this reason, "restarted" versions of the method are used.

In the conjugate gradient method, the residuals form an orthogonal basis for the space $span\{r^{(0)}, Sr^{(0)}, S^2r^{(0)}, \ldots\}$. In GMRES, this basis is formed explicitly:

$$
\begin{aligned}
&\underline{w}^{(i)} = S_1\underline{v}^{(i)} + S_2\overline{v}^{(i)} \\
&\overline{w}^{(i)} = S_2\underline{v}^{(i)} + S_1\overline{v}^{(i)} \\
&\textbf{for} \quad k = 1, 2, \ldots, i \\
&\quad \underline{w}^{(i)} = \underline{w}^{(i)} - < \underline{w}^{(i)}, \underline{v}^{(i)} > \underline{v}^{(k)} \\
&\quad \overline{w}^{(i)} = \overline{w}^{(i)} - < \overline{w}^{(i)}, \overline{v}^{(i)} > \overline{v}^{(k)} \\
&\textbf{end} \\
&\underline{v}^{(i+1)} = \underline{w}^{(i)} / \|\underline{w}^{(i)}\|_2 \\
&\overline{v}^{(i+1)} = \overline{w}^{(i)} / \|\overline{w}^{(i)}\|_2
\end{aligned}
\tag{3.1}
$$

where $<,>$ is used for inner product.

We can recognize that (3.1) is a modified Gram-Schmidt orthogonalization. Now if we applied this orthogonalization to the Krylov sequence $\{S_1^k \underline{r}^{(0)}\}$ and $\{S_2^k \overline{r}^{(0)}\}$ the orthogonalization will be called the "Arnoldi method" [9]. The inner product coefficients $< \underline{w}^{(i)}, \underline{v}^{(i)} >$, $< \overline{w}^{(i)}, \overline{v}^{(i)} >$, $\|\underline{w}^{(i)}\|_2$ and $\|\overline{w}^{(i)}\|_2$ are stored in a Hessenberg matrix.

The GMRES iterates are constructed $X^{(i)} = (\underline{x}^{(i)}, \overline{x}^{(i)})$, as

$$
\underline{x}^{(i)} = \underline{x}^{(0)} + \underline{y}_1\underline{v}^{(1)} + \underline{y}_2\underline{v}^{(2)} + \ldots + \underline{y}_i\underline{v}^{(i)},
$$

$$
\overline{x}^{(i)} = \overline{x}^{(0)} + \overline{y}_1\overline{v}^{(1)} + \overline{y}_2\overline{v}^{(2)} + \ldots + \overline{y}_i\overline{v}^{(i)},
$$

where the coefficient $Y_k = (\underline{y}_k, \overline{y}_k)$ have been chosen to minimize the norms $\|Y - SX^{(i)}\|_2$. The GMRES algorithm has the property that this residual norm can be computed without the iterate having been formed. Thus, the expensive action of forming the iterate can be postponed until the residual norm is deemed small enough.

The pseudocode for the restarted GMRES(m) algorithm is given in algorithm 1.

The most popular form of GMRES is based on the modified Gram-Schmidt procedure, and uses restarts to control storage requirements. If no restarts are used, GMRES (like any other orthogonalizing Krylov-subspace method) will converge in no more than n steps (assuming exact arithmetic). Of course this is of no practical value when n is large; moreover, the storage and computational requirements in the absence of restarts are prohibitive. Indeed, the crucial element for successful application of GMRES(m) revolves around the decision of when to restart; that is the choice of m. Unfortunately, there exist examples for which the method stagnates and convergence takes place only at the n-th step. For such systems, any choice of m less than n fails to converge.

Saad and Schultz [38] have proven several useful results. In particular, they showed that if the coefficient matrix is real and nearly positive definite, then a reasonable value for m may be selected. Implication of the choice of m are discussed below.

A common implementation of GMRES is suggested by Saad and Schultz [38] and relies on using modified Gram-Schmidt orthogonalization. Householder transformations, which are relatively costly but stable, have also been proposed. The householder approach results in a three-fold increase in work; however, convergence may be better, especially for ill-conditioned systems [39]. The major drawback of GMRES is that the amount of work and storage required per iteration rises linearly with the iteration count. Unless one is fortunate enough to obtain extremely fast convergence, the cost will rapidly become prohibitive. The usual way to overcome this limitation is by restarting the iteration. After a chosen number (m) of iterations, the accumulated data are cleared and the intermediate results are used as the initial data for the next m iterations. This procedure is repeated until convergence is achieved. The difficulty is in choosing an appropriate value for m. If m is too small, GMRES(m) may converge slowly, or fail to converge at all. A value of m that is larger than necessary involves excessive work and use more storage. Unfortunately, there are no definite rules governing the choice of m.

Algorithm 1: The GMRES(m) method for FSLE

$X^{(0)} = (\underline{x}^{(0)}, \overline{x}^{(0)})$ *is an initial guess*

$(*)$ *Compute* $R_0 = (\underline{r}_0, \overline{r}_0)$ *from* $\underline{r}_0 = \underline{y} - S_1\underline{x}^{(0)} - S_2\overline{x}^{(0)}$

and $\overline{r}_0 = \overline{y} - S_2\underline{x}^{(0)} - S_1\overline{x}^{(0)}$

$\beta = \|\underline{r}\|_2 + \|\overline{r}\|_2$

$\underline{v}^{(1)} = \underline{r}_0/\beta$ *and* $\overline{v}^{(1)} = \overline{r}_0/\beta$

Define the $2(m+1) \times 2(m)$ *matrix* $\widetilde{H}_m = \{h_{i,j}\}$

for $j = 1, 2, ..., 2m$

Compute $W_j = (\underline{w}, \overline{w})$ *from* $\underline{w} = S_1\underline{v}^{(1)} + S_2\overline{v}^{(1)}$ *and* $\overline{w} = S_2\underline{v}^{(1)} + S_1\overline{v}^{(1)}$

for $i = 1, ..., j$

$h_{i,j} =< \underline{w}, \underline{v}^{(i)} > + < \overline{w}, \overline{v}^{(i)} >$

$\underline{w} = \underline{w} - h_{i,j}\underline{v}^{(i)}$ *and* $\overline{w} = \overline{w} - h_{i,j}\overline{v}^{(i)}$

end

$h_{j+1,j} = \|\underline{w}\|_2 + \|\overline{w}\|_2$

$\underline{v}^{(j+1)} = \underline{w}/h_{j+1,j}$ *and* $\overline{v}^{(j+1)} = \overline{w}/h_{j+1,j}$

end

Compute Y_m *the minimizer of* $\|\beta e_1 - \widetilde{H}_m Y\|_2$

and $\underline{x}_m = \underline{x}_0 + \underline{v}_m\underline{y}_m$ *and* $\overline{x}_m = \overline{x}_0 + \overline{v}_m\overline{y}_m$

Restart : *Compute* $\underline{r}_m = \underline{y} - S_1\underline{x}^{(m)} - S_2\overline{x}^{(m)}$ *and* $\overline{r}_m = \overline{y} - S_2\underline{x}^{(m)} - S_1\overline{x}^{(m)}$

if satisfied then stop

Else set $\underline{x}_0 = \underline{x}_m$ *and* $\overline{x}_0 = \overline{x}_m$, *go to the* $(*)$.

Example 3.3.1. *Consider the* 3×3 *fuzzy system*

$$\begin{cases} 2x_1 + x_2 + 3x_3 = (11 + 8r, 27 - 8r), \\ 4x_1 + x_2 - x_3 = (-23 + 10r, -5 - 8r), \\ -x_1 + 3x_2 + x_3 = (10 + 5r, 27 - 12r). \end{cases}$$

The extended 6×6 *matrix is*

$$S = \begin{bmatrix} 2 & 1 & 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 1 & 0 & 0 & 0 & 3 & 1 \end{bmatrix}.$$

Now the exact solution is

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (-4 + 2r, -1 - r),$$

$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (1 + r, 5 - 3r),$$

$$x_3 = (\underline{x}_3(r), \overline{x}_3(r)) = (6 + r, 8 - r).$$

The exact and obtained solutions with GMRES iterative method are plotted and compared in Fig. 3.3.

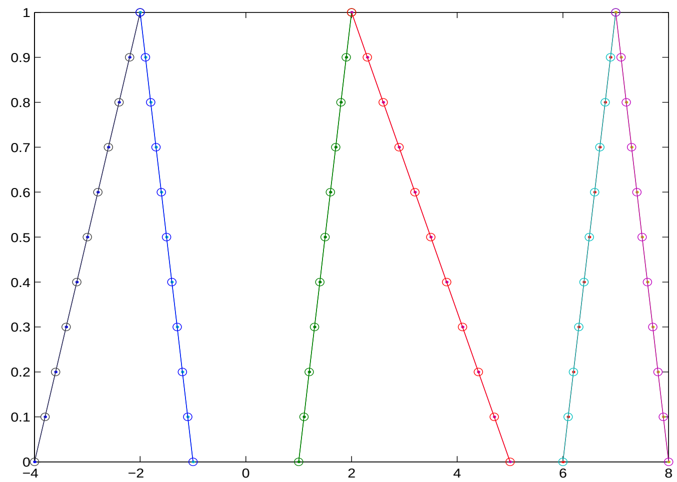

Figure 3.3: The Hausdrff distance of solutions with m=2 and $\epsilon = 10^{-8}$ *is 3.5527e-013.*

Example 3.3.2. *Consider the* 5×5 *fuzzy system*

$$\begin{cases} x_1 + 2x_3 + 3x_4 - 2x_5 = (-29 + 10r, 2 - 21r), \\ 2x_3 + 3x_4 - 3x_5 = (-40 + 11r, -8 - 23r), \\ 4x_1 + 5x_2 + x_3 - x_4 + 4x_5 = (23 + 27r, 90 - 40r), \\ 2x_1 + 6x_2 + 3x_5 = (8 + 20r, 66 - 38r), \\ x_1 - 6x_4 - 3x_5 = (-23 + 31r, 21 - 13r). \end{cases}$$

The extended 10×10 *matrix is*

$$S = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 3 \\ 4 & 5 & 1 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 2 & 6 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & 5 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 2 & 6 & 0 & 0 & 3 \\ 0 & 0 & 0 & 6 & 3 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now the exact solution is

$$x_1 = (\underline{x}_1(r), \overline{x}_1(r)) = (1 + r, 3 - r),$$

$$x_2 = (\underline{x}_2(r), \overline{x}_2(r)) = (-2 + 2r, 5 - 5r),$$

$$x_3 = (\underline{x}_3(r), \overline{x}_3(r)) = (4 + r, 7 - 2r),$$

$$x_4 = (\underline{x}_4(r), \overline{x}_4(r)) = (-6 + r, -1 - 4r),$$

$$x_5 = (\underline{x}_5(r), \overline{x}_5(r)) = (6 + 2r, 10 - 2r).$$

The exact and obtained solutions with GMRES iterative method are plotted and compared in Fig. 3.4.

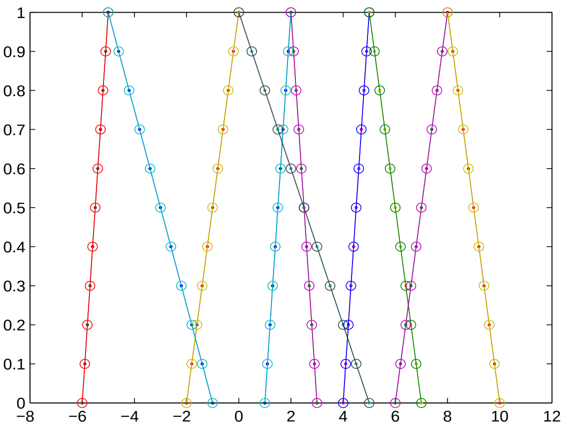

Figure 3.4: The Hausdrff distance of solutions with m=4 and $\epsilon = 10^{-8}$ *is 3.5527e-015.*

Chapter 4

Iterative methods for solving fuzzy nonlinear equations

4.1 Introduction

The numerical solution of consistent system of algebraic nonlinear equations $F(x) = 0$, arise quite often in engineering and the natural sciences. Many engineering design problems that must satisfy specified constraints can be expressed as a nonlinear equalities or inequalities. The concept of fuzzy numbers and arithmetic operation with these numbers were first introduced and investigated by [17, 45, 30, 29, 21, 36]. One of the major applications of fuzzy number arithmetic is nonlinear equations whose parameters are all or partially represented by fuzzy numbers [18, 25, 33]. Standard analytical techniques like Buckley and Qu method, [10, 11, 12, 13], can not suitable for solving the equations such as $(i) \quad ax^5 + bx^4 + cx^3 + dx - e = f$,

$(ii) \quad x - \sin(x) = g$,

where x, a, b, c, d, e, f and g are fuzzy numbers. We therefore need to develop the numerical methods to find the roots of such equations. Abbasbandy and Asady [1], investigated Newton's method for solving a fuzzy nonlinear system as

$$F(x) = c,$$

whose all parameters are fuzzy. The advantage of the Newton's method is it's speed of convergence once a sufficiently accurate approximation is known. A weakness of this method is that an accurate initial approximation to the solution is needed to ensure convergence. Here, we consider these equations, in general, as

$$F(x) = 0.$$

The Steepest Descent method converges only linearly to the solution, but it will usually converge even for poor initial approximations.
In this section we propose Fixed point and Steepest Descent methods for solving fuzzy nonlinear equations and illustrate some examples and conclusions.

4.2 The fixed point method

Now our aim is to obtain a solution for fuzzy nonlinear equation $F(x) = x$. Without any loss of generality, assume that x is positive, then the parametric form is as follows:

$$\begin{cases} \underline{F}(\underline{x}, \overline{x}, r) = \underline{x}(r), \\ \\ \overline{F}(\underline{x}, \overline{x}, r) = \overline{x}(r), \end{cases} \qquad \forall r \in [0, 1].$$

For all $r \in [0,1]$, the above system is equivalent to

$$\mathbf{F}(X(r)) = X(r),$$

where

$$X(r) = (\underline{x}(r), \overline{x}(r)),$$

and

$$\mathbf{F}(\underline{x}(r), \overline{x}(r)) = (\underline{F}(\underline{x}, \overline{x}, r), \overline{F}(\underline{x}, \overline{x}, r)).$$

Definition 4.2.1. A function G from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ has a fixed point at $P \in D$ if $G(P) = P$.

Theorem 4.2.1. *Let $D = \{(x_1, x_2, ..., x_n)^t |\ a_i \le x_i \le b_i,\ i = 1, 2, ..., n\}$ for some collection of constants $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$. Suppose G is a continues function from $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with the property that*

$$G(x) = (g_1(x), g_2(x), \dots, g_n(x)) \in D,$$

whenever $x \in D$. Then G has a fixed point in D[14].

Suppose, in addition, that G has continuous partial derivatives and a constant $K < 1$ exists with

$$|\frac{\partial g_i(x)}{\partial x_j}| \le \frac{K}{n}, \qquad whenever \;\; x \in D,$$

for each $j = 1, 2, ..., n$ and each component function g_i. Then the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by an arbitrarily selected $x^{(0)}$ in D and generated by

$$x^{(k)} = G(x^{(k-1)}), \qquad for\ each\ k \ge 1,$$

converges to the unique fixed point $P \in D$ and

$$\|x^{(k)} - P\|_\infty \leq \frac{K^k}{1-K}\|x^{(1)} - x^{(0)}\|_\infty.$$

In fact this theorem will be used to show that G has unique fixed point in D for x on its domain. Note that G having a unique fixed point in D dose not imply that the solution to the original system is unique on this domain. To approximate fixed point P, we choose $x^{(0)}$, and the sequence of vectors generated by

$$x^{(k)} = G(x^{(k-1)}), \tag{4.1}$$

converges to the unique solution of the system. For initial guess, one can use the trapezoidal fuzzy number

$$x_0 = (\underline{x}(1), \overline{x}(1), \underline{x}(1) - \underline{x}(0), \overline{x}(0) - \overline{x}(1)),$$

and in parametric form

$$\underline{x}_0(r) = \underline{x}(1) + (\underline{x}(1) - \underline{x}(0))(r-1), \quad \overline{x}_0(r) = \overline{x}(1) + (\overline{x}(0) - \overline{x}(1))(1-r).$$

Definition 4.2.2. Let $(\underline{x}(r), \overline{x}(r))$ denotes the obtained solution by iterating of (4.1), the fuzzy number $U(r) = (\underline{u}(r), \overline{u}(r))$ defined by

$$\underline{u}(r) = min\{\underline{x}(r), \overline{x}(r), \underline{x}(1), \overline{x}(1)\},$$

$$\overline{u}(r) = max\{\underline{x}(r), \overline{x}(r), \underline{x}(1), \overline{x}(1)\},$$

is called the obtained solution of (4.1). If $(\underline{x}(r), \overline{x}(r))$ is a fuzzy number then $\underline{u}(r) = \underline{x}(r)$, $\overline{u}(r) = \overline{x}(r)$, and then U is called a strong fuzzy obtained solution. Otherwise, U is a weak fuzzy obtained solution.

Example 4.2.2. *Consider the fuzzy nonlinear equation*

$$(0.222, 0.222, 0.111, 0.111)x^2 + (0.200, 0.200, 0.100, 0.100) = x.$$

Without any loss of generality, assume that x is positive, then the parametric form of this equation is as follows

$$\begin{cases} (0.111 + 0.111r)\underline{x}^2(r) & + & (0.100 + 0.100r) = \underline{x}(r), \\ (0.333 - 0.111r)\overline{x}^2(r) & + & (0.300 - 0.100r) = \overline{x}(r). \end{cases}$$

To obtain initial guess we use above system for $r = 0$ and $r = 1$, therefore

$$\begin{cases} 0.222\ \underline{x}^2(1) + 0.200 = \underline{x}(1), \\ 0.222\ \overline{x}^2(1) + 0.200 = \overline{x}(1), \end{cases} \begin{cases} 0.111\ \underline{x}^2(0) + 0.100 = \underline{x}(0), \\ 0.333\ \overline{x}^2(0) + 0.300 = \overline{x}(0), \end{cases}$$

consequently $\underline{x}(0) = 0.101$, $\overline{x}(0) = 0.338$ and $\underline{x}(1) = \overline{x}(1) = 0.210$. Therefore initial guess is $x_0 = (0.210, 0.210, 0.109, 0.128)$. The component equations then become

$$\begin{cases} \underline{x}^{(k)}(r) = (0.111 + 0.111r)(\underline{x}^{(k-1)}(r))^2 + (0.100 + 0.100r), \\ \overline{x}^{(k)}(r) = (0.333 - 0.111r)(\overline{x}^{(k-1)}(r))^2 + (0.300 - 0.100r). \end{cases}$$

Because the parametric functions are bounded and continuous for $r \in [0, 1]$ we obtain the solution after 10 iterations; which the maximum error would be less than 10^{-5}. For more details see Figure 4.1. Let $\underline{x}(0) = 8.908$, $\overline{x}(0) = 2.665$ and $\underline{x}(1) = \overline{x}(1) = 4.295$. Hence $x_0 = (4.295, 4.295, 1.630, 4.613)$ be a weak initial guess. The component equations then become

$$\begin{cases} (\underline{x})^{(k)}(r) = [((\underline{x})^{(k-1)}(r) - (0.100 + 0.100r))/(0.111 + 0.111r)]^{\frac{1}{2}}, \\ (\overline{x})^{(k)}(r) = [((\overline{x})^{(k-1)}(r) - (0.300 - 0.100r))/(0.333 - 0.111r)]^{\frac{1}{2}}. \end{cases}$$

Now parametric functions are bounded and continuous for $r \in [0,1]$ we obtain the weak solution after 15 iterations; which the maximum error would be less than 10^{-5}. For more details see Figure 4.2.

Now parametric functions are bounded and continuous for $r \in [0,1]$ we obtain the weak solution after 15 iterations; which the maximum error would be less than 10^{-5}. For more details see Figure 4.2. Now parametric functions are bounded and continuous for $r \in [0,1]$ we obtain the weak solution after 15 iterations; which the maximum error would be less than 10^{-5}. For more details see Figure 4.2. Now parametric functions are bounded and continuous for $r \in [0,1]$ we obtain the weak solution after 15 iterations; which the maximum error would be less than 10^{-5}. For more details see Figure 4.2. Now parametric functions are bounded and continuous for $r \in [0,1]$ we obtain the weak solution after 15 iterations; which the maximum error would be less than 10^{-5}. For more details see Figure 4.2. Now parametric functions are bounded and continuous for $r \in [0,1]$ we obtain the weak solution after 15 iterations; which the maximum error would be less than 10^{-5}. For more details see Figure 4.2. Now parametric functions are bounded and continuous for $r \in [0,1]$ we obtain the weak solution after 15 iterations; which the maximum error would be less than 10^{-5}. For more details see Figure 4.2.

4.3 The steepest descent method

$$\begin{cases} \underline{F}(\underline{x},\overline{x},r)=0, & \\ & \forall r\in[0,1] \\ \overline{F}(\underline{x},\overline{x},r)=0. & \end{cases}$$

The method of Steepest Decent determines a local minimum for a multivariable function of the form $G:\mathbb{R}^n\to\mathbb{R}$. Let the function G defined by

$$G(\underline{x},\overline{x},r)=[\underline{F}(\underline{x},\overline{x},r)+\overline{F}(\underline{x},\overline{x},r)]^2$$

Finding a local minimum for an arbitrary function G from $\mathbb{R}^2$ into $\mathbb{R}$, for all $0\le r\le 1$ can be intuitively described as follows:

1. Evaluate G at an initial approximation $X_0(r)=(\underline{x}_0(r),\overline{x}_0(r))$ for all $0\le r\le 1$.

2. Determine a direction from $X_0(r)=(\underline{x}_0(r),\overline{x}_0(r))$ that results in a decrease in the value of G for all $0\le r\le 1$.

3. Move an approximate amount in this direction and call the new value $X_1(r)=(\underline{x}_1(r),\overline{x}_1(r))$ for all $0\le r\le 1$.

4. Repeat steps 1 through 3 with $X_0(r)$ replaced by $X_1(r)$ for all $0\le r\le 1$.

Definition 4.3.1. For $G:\mathbb{R}^2\to\mathbb{R}$, the gradient of G at $X(r)=(\underline{x}(r),\overline{x}(r))$ is denoted $\nabla G(X)$ and defined by

$$\nabla G(X)=(\frac{\partial G}{\partial\underline{x}},\frac{\partial G}{\partial\overline{x}}).$$

The gradient has important property, the direction of greatest decrease in the value of G at X is the direction given by $-\nabla G(X)$. Now the object is to reduce $G(X(r))$ to it's minimal value of zero, so an appropriate choice for $X_p(r) = (\underline{x}_p(r), \overline{x}_p(r))$ is

$$\underline{x}_p(r) = \underline{x}_{p-1}(r) - \lambda(p-1)\nabla G,$$

$$\overline{x}_p(r) = \overline{x}_{p-1}(r) - \lambda(p-1)\nabla G.$$

For some constant $\lambda(p)$; $p = 0, 1, 2, ...$ and $0 \leq r \leq 1$.

One choose for $\lambda(p)$ can be as follows

$$2\lambda(p) = \frac{F(X^p).W(X^p).W^T(X^p).F(X^p)}{W(X^p).W^T(X^p).F(X^p).W(X^p).W^T(X^p).F(X^p)} = $$
$$= \frac{F^{(p)}.W^{(p)}.W^{T(p)}.F^{(p)}}{W^{(p)}.W^{T(p)}.F^{(p)}.W^{(p)}.W^{T(p)}.F^{(p)}}$$

where

$$F(X) = [\underline{F}(\underline{x}, \overline{x}, r), \overline{F}(\underline{x}, \overline{x}, r)]^T$$

for $0 \leq r \leq 1$,

and

$$W(X) = \frac{\partial F}{\partial X} = \begin{bmatrix} \frac{\partial \underline{F}}{\partial \underline{x}} & \frac{\partial \underline{F}}{\partial \overline{x}} \\ \frac{\partial \overline{F}}{\partial \underline{x}} & \frac{\partial \overline{F}}{\partial \overline{x}} \end{bmatrix},$$

is the Jacobian matrix of F[14].

For initial guess, one can use the fuzzy number

$$x_0 = (\underline{x}(0), \underline{x}(1), \overline{x}(0)),$$

and in parametric form

$$\underline{x}_0(r) = \underline{x}(0) + (\underline{x}(1) - \underline{x}(0))r, \quad \overline{x}_0(r) = \overline{x}(0) + (\overline{x}(0) - \overline{x}(1))r.$$

Example 4.3.1. *Consider the fuzzy nonlinear equation*

$$(3,4,5)x^2 + (1,2,3)x = (1,2,3).$$

Without any loss of generality, assume that x is positive, then the parametric form of this equation is as follows

$$\begin{cases} (3+r)\underline{x}^2(r) & + & (1+r)\underline{x}(r) & = & (1+r), \\ (5-r)\overline{x}^2(r) & + & (3-r)\overline{x}(r) & = & (3-r). \end{cases}$$

or equivalently

$$\begin{cases} (3+r)\underline{x}^2(r) & + & (1+r)\underline{x}(r) & - & (1+r) = 0, \\ (5-r)\overline{x}^2(r) & + & (3-r)\overline{x}(r) & - & (3-r) = 0. \end{cases}$$

To obtain initial guess we use above system for $r = 0$ and $r = 1$, therefore

$$\begin{cases} 4\underline{x}^2(1) & + & 2\underline{x}(1) & = & 2, \\ 4\overline{x}^2(1) & + & 2\overline{x}(1) & = & 2, \end{cases} \quad \text{and} \quad \begin{cases} 3\underline{x}^2(0) & + & \underline{x}(0) & = & 1, \\ 5\overline{x}^2(0) & + & \overline{x}(0) & = & 3. \end{cases}$$

Consequently $\underline{x}(0) = 0.434$, $\overline{x}(0) = 0.681$ and $\underline{x}(1) = \overline{x}(1) = \frac{1}{2}$. Therefore initial guess is $x_0 = (0.434, 0.5, 0.681)$. After 10 iterations, we obtain the solution which the maximum error would be less than 10^{-5}. For more details see Figure 4.3. Now suppose x is negative, we have

$$\begin{cases} (3+r)\overline{x}^2(r) & + & (3-r)\underline{x}(r) & - & (1+r) = 0, \\ (5-r)\underline{x}^2(r) & + & (1+r)\overline{x}(r) & - & (3-r) = 0. \end{cases}$$

For $r = 0$, we have, $\underline{x}(0) \simeq -0.629$ and $\overline{x}(0) \simeq -0.98$, hence $\underline{x}(0) > \overline{x}(0)$, therefore negative root does not exist.

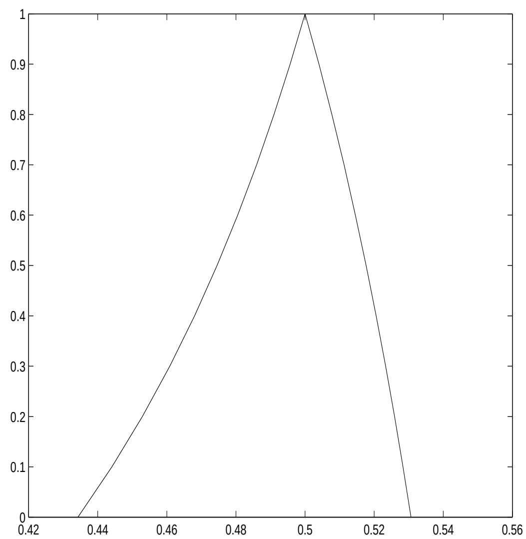

Figure 4.3 : Solution of Steepest Descent Method

Example 4.3.2. *Consider fuzzy nonlinear equation*

$$(1,2,3)x^3 + (2,3,4)x^2 + (3,4,5) = (5,8,13).$$

Without any loss of generality, assume that x *is positive, then parametric form of this equation is as follows*

$$\begin{cases} (1+r)\underline{x}^3(r) & + & (2+r)\underline{x}^2(r) & + & (3+r) & = & (5+3r), \\ (3-r)\overline{x}^3(r) & + & (3-r)\overline{x}^2(r) & + & (5-r) & = & (13-5r), \end{cases}$$

or equivalently

$$\begin{cases} (1+r)\underline{x}^3(r) & + & (2+r)\underline{x}^2(r) & - & (2+2r) = 0, \\ (3-r)\overline{x}^3(r) & + & (4-r)\overline{x}^2(r) & - & (8-4r) = 0. \end{cases}$$

By solving the above system for $r = 0$ *and* $r = 1$*, we obtain the initial guess* $x_0 = (0.76, 0.91, 1.06)$. *If we apply ten iterations from Steepest Descent method, the maximum error is less than* 10^{-5}, *Figure 4.4.*

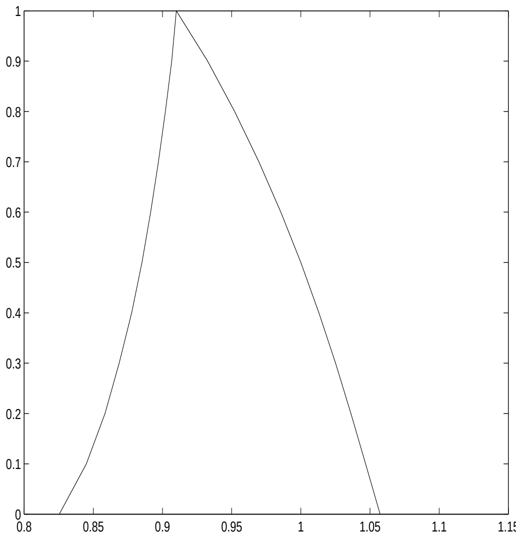

Figure 4.4 : Positive Solution of Steepest Descent Method

Conclusions

In this work Some iterative methods for finding solution of $n \times n$ fuzzy linear system like Conjugate Gradient Method and Generalized Minimal Residual (GMRES) Method, are given, several numerical examples show the efficiency of implemented algorithm also we use Fixed Point and Steepest Descent Methods for solving nonlinear equations and the numerical examples show the efficiency of implemented algorithm. We can work on full fuzzy linear or nonlinear systems in the future, perhaps this is not easy because the coefficient matrix and the right hand side vector in these systems are fuzzy.

Originality

The following sections are proposed in this work:

Section 3.2: In this section we use conjugate gradient method for solving fuzzy linear systems.

Section 3.3: In this section we use generalized minimal residual method for solving fuzzy linear systems.

Section 4.2: In this section we use fixed point method for solving fuzzy nonlinear equations.

Section 4.3: In this section we use steepest descent method for solving fuzzy nonlinear equations.

Articles

1. Conjugate gradient method for fuzzy symmetric positive definite system of linear equations, *Journal of Applied Mathematics and Computation*, 171 (2005) 1184-1191 (ISI).

2. LU decomposition method for solving fuzzy system of linear equations, *Journal of Applied Mathematics and Computation*, 175 (2006) 823-833 (ISI).

3. Steepest descent method for solving fuzzy system of linear equations, *Journal of Applied Mathematics and Computation*, 174 (2006) 669-675 (ISI).

4. Steepest descent method for solving fuzzy nonlinear equations, *Journal of Applied Mathematics and Computation*, 172 (2006) 633-643 (ISI).

5. Generalized minimal residual (GMRES) method for a fuzzy linear system, *Proceedings of the* 6^{th} *Iranian Conference on Fuzzy Systems and* 1^{st} *Islamic World Conference on Fuzzy Systems*, (2006) 71-80.

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