ISLAMIC AZAD UNIVERSITY
SCIENCE AND RESEARCH BRANCH

NUMERICAL SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS AND NUMERICAL METHOD FOR INTEGRATION OF FUZZY INTEGRALS

Supervisor

Prof. Saeid Abbasbandy

Prof. Esmaeil Babolian

Advisers

Dr. Tofigh Allahviranloo

By

Majid Alavi

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY(Ph.D.)
IN APPLIED MATHEMATICS (NUMERICAL ANALYSIS)
2006

ISLAMIC AZAD UNIVERSITY
SCIENCE AND RESEARCH BRANCH
DEPARTMENT OF MATHEMATICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Numerical Solution of Fuzzy Differential Equations and Numerical Method For Integration of Fuzzy Integrals** " by **Majid Alavi** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

Dated: 2006

Research Supervisor: ______________________
Prof. Saeid Abbasbandy
Prof. Esmaeil Babolian

External Examiner: ______________________
Dr. Babak Asady

Examing Committee: ______________________
Dr. M. A. Fariborzi Araghi

____
Dr. A. Vahidi

____
Dr. F. Hosseinzadeh

ISLAMIC AZAD UNIVERSITY
SCIENCE AND RESEARCH BRANCH [1]

Date: **2006**

Author: **Majid Alavi**

Title: **Numerical Solution of Fuzzy Differential Equations and Numerical Method For Integration of Fuzzy Integrals**

Department: **Mathematics**

Degree: **Ph.D.** Convocation: ******** Year: **2006**

Signature of Author

[1] Poonak, Tehran

To My dears:

My Wife and My Son

and My Parents

Contents

Abstract

In this work we propose first a method for the simulation of uncertain dynamical systems. Uncertainty is taken into account by replacing crisp functions by fuzzy functions on the right hand side of differential equations. Crisp initial conditions are replaced with fuzzy ones. Due to the uncertainty incorporated in the model, the behavior of dynamical systems modelled in this way will generally not be unique. Rather, we obtain a large set of trajectories which are more or less compatible with the description of the system. We propose so-called *fuzzy reachable sets* for characterizing the fuzzy set of solutions to a fuzzy initial value problem. The main part of this study is devoted to the development of numerical methods for the approximation of reachable set of one dimensional fuzzy differential inclusions. An efficient algorithm to solve them in MAPLE has been devised, which is easy to implement.

Then using the embedding method, numerical procedure for solving fuzzy integral equations(FIEs) have been investigated. For this we use parametric form of fuzzy number and convert a linear fuzzy Fredholm integral equation to two linear systems of integral equations of the second kind in crisp case. We can use one of the numerical methods such as Nystrom and find the approximation solution of the system and hence obtain an approximate for fuzzy solution of the linear fuzzy Fredholm integral equations of the second kind. The proposed methods is illustrated by solving some numerical examples.

Originality

The following sections are proposed in this work:

Section 3.4: In this section we discuss property of α-reachable set in fuzzy differential inclusions

Section 3.5: We propose an easy algorithm (TRS) for solving fuzzy differential inclusions and represent an example.

Section 4.2: We define and discuss property center and width of fuzzy number in parametric form.

Section 4.4: We propose a method for convert a linear fuzzy Fredholm integral equation to two crisp linear system of integral equation of the second kind and apply a standard numerical method for solving them then combine these for finding solution of fuzzy integral.

Articles

1. Tuning of reachable set in one dimensional fuzzy differential inclusions, Chaos, Solitons and Fractals 26(2005)1337-1341.(ISI)

2. Numerical method for solving linear Fredholm fuzzy integral equations of the second kind, Chaos, Solitons and Fractals 31 (2007) 138-146.(ISI)

Acknowledgements

I would like to thank Prof. S. Abbasbandy and Prof. E. Babolian, my supervisors, for their many suggestions and constant support during this research. I am also thankful to Dr. T. Allahviranloo my adviser for his guidance through the early years of chaos and confusion.

I would like to thank Prof. G. R. Jahanshahloo, head of Mathematics Department for his useful helps

Of course, I am grateful to my parents and my family(M. Salavati , Emad) for their patience and *love*. Without them this work would never have come into existence(literary). I thank from Z. sadeghi and A. Alavi (my sister) for kindly behavior thought my education. I am also thank to Dr. B. Asady My external examiner and for useful helps in during this research.

I am appreciated Dr. M. A. Fariborzi Araghi my internal examiner.

Deep appreciation are expressed to my dear friends Dr. R. Ezzati, Dr. A. Jafarian, Dr. L. Alizadeh (for all the good and bad times we had together throughout the wholly my education.); I wish to thank the following: H. A. Salavati,; A. Mansory, Dr. A. Haidari (for special effects). Finally, I had the pleasure of Islamic Azad

University, Arak branch, for their scholarship makes research like this possible.

M. Alavi

October 19, 2006

Introduction

Fuzzy sets were introduced by Zadeh (1965) as a means of representing and mainpulating data that was not precise, but rather fuzzy. Fuzzy logic provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge based systems. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes, such as thinking and reasoning. The conventional approaches to knowledge representation lack the means for representing the meaning of fuzzy concepts. As a consequence, the approaches based on first order logic and classical probability theory do not provide an appropriate conceptual framework for dealing with the representation of commonsense knowledge, since such knowledge is by its nature both lexically imprecise and noncategorical.

The development of fuzzy logic was motivated in large measure by the need for a conceptual framework which can address the issue of uncertainty and lexical impression. Since 1965, when Zadeh published his pioneering paper, hundreds of examples have been supplied where the nature of uncertainty in the behavior of a given system possesses fuzzy rather than stochastic nature. Non-stationary fuzzy systems described by fuzzy processes look as their natural extension into the time domain. From different viewpoints they were carefully studied. Solution of differential equations provide a noteworthy example of time-dependent fuzzy sets. The term "fuzzy differential equation" was coined in 1978 by A. Kandel, W. j. Byatt; They were published much extended version of this note two years later. Since then, the theory of fuzzy differential equations seem to have split into two independent branches, where the first one relies upon the notion of Hukuhara derivative while the other dose not. The concept of integration of fuzzy functions was first introduced by Dubois and Prade [5] and investigated by Goetschel and Voxman [38], Kaleva [31], Matloka [26] and others. Wu and Ma [50] represent the first applications of fuzzy integration. They investigated the fuzzy Fredholm integral equation of the second kind(FFIE-2). Recently, numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method was introduced by Babolian *et al.* [6].

Chapter 1

Fuzzy logic

1.1 Introduction

The characteristic function of a crisp ordinary set assigns a value of either 0 or 1 to each individual in the universal set, thereby discriminating between members and non members of the crisp set under consideration. This function can be generalized such that values assigned to the element of the universal set fall within a specified range and indicate the membership grade of these elements in the set in under consideration. Large values denote higher degrees of membership. Such a function is called *membership function*, and the set defined by it a *fuzzy set*.

The most commonly used range of values of a membership function is the unit interval $[0, 1]$. In this case, each membership function maps elements of a given *universal set* X, which is always a crisp set, into real numbers in $[0, 1]$.

Two distinct notations are most commonly employed in the literature to denote membership functions. In one of them, the membership function of a fuzzy set A is denoted by μ_A, that is,

$$\mu_A : X \to [0, 1].$$

In the other one, the function is denoted by A and has, of course, the same form:

$$A : X \to [0, 1].$$

According to the first notation, the symbol of the fuzzy set A is distinguished from the symbol of its membership function μ_A. According to the second notation, this distinction is not made, but no ambiguity results from this double use of the same symbol. It is clear that A is completely determined by the set of tuples

$$A = \{(x, \mu_A(x)) | x \in X\}.$$

Frequently we will write simply $A(x)$ instead of $\mu_A(x)$. The family of all fuzzy (sub)sets in X is denoted by $\mathcal{F}(X)$. Fuzzy subsets of the real line are called *fuzzy quantities*. If $X = \{x_1, x_2, \cdots, x_n\}$ is a finite set and A is a fuzzy set in X then we often use the notation

$$A = \frac{\mu_1}{x_1} + \cdots + \frac{\mu_n}{x_n}$$

where the term

$$\frac{\mu_i}{x_i}, i = 1, \cdots, n$$

signifies that μ_i is the grade of membership of $x_i \in A$ and the plus sign represents union.

Example 1.1.1. *Suppose we want to define the set of natural numbers "close to 1".*

This can be expressed by

$$A = \frac{0.0}{-2} + \frac{0.3}{-1} + \frac{0.6}{0} + \frac{1.0}{1} + \frac{0.6}{2} + \frac{0.3}{3} + \frac{0.0}{4}.$$

Example 1.1.2. *Consider three fuzzy sets that represent the concepts a young, middle-aged, and old person. A reasonable expression of this concept of membership function* $A_1,\ A_2,\ and A_3$ *is shown in Figure 1.1. These functions are defined on the interval* $[0, 80]$ *as follows:*

$$A_1(x) = \begin{cases} 1 & where \quad x \leq 20 \\ \frac{(35-x)}{15} & where \quad 20 < x < 35 \\ 0 & where \quad x \geq 35 \end{cases} \tag{1.1}$$

$$A_2(x) = \begin{cases} 0 & where \quad x \leq 20 \ or \ \geq 60 \\ \frac{(x-20)}{15} & where \quad 20 < x < 35 \\ \frac{(60-x)}{15} & where \quad 45 < x < 60 \\ 1 & where \quad 35 \leq x \leq 45 \end{cases} \tag{1.2}$$

$$A_3(x) = \begin{cases} 0 & where \quad x \leq 45 \\ \frac{(x-45)}{15} & where \quad 45 < x < 60 \\ 1 & where \quad x \geq 60 \end{cases} \tag{1.3}$$

Figure 1.1: From left to right membership functions representing the concepts of a young: $A1$, middle-aged: $A2$, and old person: $A3$

Definition 1.1.1. Let A and B be fuzzy subsets of a classical set X. We say that A is a subset of B if $A(x) \leq B(x)$ for all $x \in X$.

Definition 1.1.2. Let A and B be fuzzy subsets of a classical set X. We say that A is equal to B , denoted $A = B$, if $A \subset B$ and $B \subset A$. In other word, $A = B$ if and only if $A(x) = B(x)$ for all $x \in X$.

Definition 1.1.3. (support) Let A be a fuzzy subset of X, the support of A, denoted $supp(A)$, is the crisp subset of X whose elements all have nonzero membership grades in A. i.e.,

$$supp(A) = \{x \in X | A(x) > 0\}$$

Definition 1.1.4. (normal fuzzy set) A fuzzy subset A of classical set X is called normal if there exists an $x \in X$ such that $A(x) = 1$. Otherwise A is subnormal.

Definition 1.1.5. (α-cut) An α-level set of a fuzzy set A of X is a non-fuzzy set denoted by $[A]_\alpha$ and is defined by

$$[A]_\alpha = \begin{cases} \{t \in X | A(x) \geq \alpha\} & if\ \alpha > 0 \\ cl(supp(A)) & if\ \alpha = 0 \end{cases}$$

where $cl(supp(A))$ denotes the closure of the support of A.

As an example , the following is a complete characterization of all α-cut for fuzzy sets $A_1,\ A_2,\ A_3$ given in Example(1.1.2),

$[A_1]_\alpha = [0, 35 - 15\alpha]$, $[A_2]_{\alpha,} = [15\alpha + 20, 60 - 15\alpha]$, $[A_3]_\alpha = [15\alpha + 45, 80]$ for all $\alpha \in [0, 1]$

Definition 1.1.6. (convex fuzzy set) A fuzzy set A of X is called convex if $[A]_\alpha$ is a convex subset of X for each $\alpha \in [0, 1]$.

To avoid confusion, we note that the definition of convexity for fuzzy sets does not mean that the membership function of a convex fuzzy set is a convex function. In fact, membership function of convex fuzzy sets are functions that are, according to standard definitions, concave and not convex. We now state a useful theorem that provides us with an alternative formulation of convexity of fuzzy sets. For the sake of simplicity, we restrict the theorem to fuzzy sets on $\mathbb{R}$, which are of primary interest in this text.

Theorem 1.1.3. *A fuzzy set A on $\mathbb{R}$ is convex iff*

$$A(\lambda x_1 + (1 - \lambda)x_2) \geq \min[A(x_1), A(x_2)] \tag{1.4}$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0, 1]$, where min denotes the minimum operator.

Proof 1.1.1. *Assume that A is convex and let $\alpha = A(x_1) \leq A(x_2)$. Hence $x_1, x_2 \in [A]_\alpha$ and, moreover, $\lambda x_1 + (1-\lambda)x_2 \in [A]_\alpha$ for any $\lambda \in [0,1]$ by the convexity of A. Consequently,*

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = A(x_1) = \min[A(x_1), A(x_2)].$$

Conversely assume that A satisfies (1.4). We need to prove that for any $\alpha \in [0,1]$, $[A]_\alpha$ is convex. Now for any $x_1, x_2 \in [A]_\alpha$(i.e., $A(x_1) \geq \alpha,\ A(x_2) \geq \alpha$), and for any $\lambda \in [0,1]$, by (1.4)

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)] \geq \min(\alpha, \alpha) = \alpha,$$

i.e., $\lambda x_1 + (1-\lambda)x_2 \in [A]_\alpha$. Therefore, $[A]_\alpha$ is convex for any $\alpha \in (0,1]$. Hence, A is convex.

Among the various type of fuzzy sets, of special significance are fuzzy sets that are defined on the set $\mathbb{R}$ of real numbers. Membership functions of these sets, which have the form

$$A : \mathbb{R} \to [0,1].$$

clearly have a quantitative meaning and may, under certain conditions, be viewed as fuzzy number that can be defined as follows.

Definition 1.1.7. A fuzzy number is a map $A : \mathbb{R} \to I = [0,1]$ which satisfies:

(i) A is upper semi-continuous, i.e.,

$$\{x | A(x) < t\} \;\; is \;\; open \;\; for\ all\ t \in \mathbb{R}.$$

(ii) $A(x) = 0$ outside some interval $[c, d] \subset \mathbb{R}$.

(iii) There exist real numbers a, b such that $c \leq a \leq b \leq d$ where

1. $A(x)$ is monotonic increasing on $[c, a]$.

2. $A(x)$ is monotonic decreasing on $[b, d]$.

3. $A(x) = 1,\ a \leq x \leq b$.

Definition 1.1.8. A quasi fuzzy number A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions

$$\lim_{t \to +\infty} A(t) = 0, \qquad \lim_{t \to -\infty} A(t) = 0.$$

Remark 1.1.1. Let A be a fuzzy number. The $[A]_\alpha$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\alpha \in [0, 1]$.

Definition 1.1.9. A fuzzy number A is called triangular fuzzy number with center a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{\alpha} & if \qquad a - \alpha \leq t \leq a \\ 1 - \frac{(t-a)}{\beta} & if \quad a \leq t \leq a + \beta \\ 0 & \qquad otherwise \end{cases}$$

and we use for it the notation $A = (a, \alpha, \beta)$. It can easily be verified that

$$[A]_r = [a - (1-r)\alpha, a + (1-r)\beta],\ \forall r \in [0, 1].$$

The support of A is $[a - \alpha, a + \beta]$.

Figure 1.2 represents the triangular fuzzy number $A = (6, 1, 2)$

Figure 1.2: Triangular fuzzy number.

Definition 1.1.10. A fuzzy number A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{\alpha} & if \quad a - \alpha \leq t \leq a \\ 1 & if \quad a \leq t \leq b \\ 1 - \frac{(t-b)}{\beta} & if \quad b \leq t \leq b + \beta \\ 0 & otherwise \end{cases}$$

and we use for it the notation $A = (a, b, \alpha, \beta)$. It can easily be shown that

$$[A]_r = [a - (1 - r)\alpha, b + (1 - r)\beta], \forall r \in [0, 1].$$

The support of A is $[a - \alpha, b + \beta]$.

Figure 1.3 represents the trapezoidal fuzzy number $A = (3, 5, 1, 2)$

Figure 1.3:Trapezoidal fuzzy number.

Definition 1.1.11. Any fuzzy number $A \in \mathcal{F}$ cab be described as

$$A(t) = \begin{cases} L(\frac{a-t}{\alpha}) & if \quad t \in [a-\alpha, a] \\ 1 & if \quad t \in [a, b] \\ R(\frac{t-b}{\beta}) & if \quad t \in [b, b+\beta] \\ 0 & otherwise \end{cases}$$

where $[a, b]$ is the core of A,

$$L : [0,1] \to [0,1], \quad R : [0,1] \to [0,1]$$

are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $R(1) = L(1) = 0$. We call this fuzzy interval of LR-type and refer to it by

$$A = (a, b, \alpha, \beta)_{LR}.$$

The support of A is $[a-\alpha, b+\beta]$.

Definition 1.1.12. Let A be a fuzzy number. If supp(A)= $\{x_0\}$ then A is called a fuzzy point and we use the notation $A = \overline{x_o}$.

Figure 1.4 represent fuzzy point $A = \overline{5}$

Figure 1.4: Fuzzy point $A = \overline{5}$

Definition 1.1.13. The space E^n is all of fuzzy subsets U of $\mathbb{R}^n$ which satisfyes the following conditions

1. U is normal,
2. U is fuzzy convex,
3. U is upper semi-continuous,
4. $[U]_0$ is bounded subset of $\mathbb{R}^n$,

when $n = 1$, elements of E^1 are Fuzzy numbers.

1.2 The extension principle

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to add, subtract, multiply and divide with fuzzy quantities. The process of doing these operations is called fuzzy arithmetic. We shall first introduce an important concept from fuzzy sets theory which is called the extension principle. We then use it to provide for this arithmetic operations on fuzzy numbers. In general, the extension principle pays a fundamental role in enabling us to extend any point operations to operations involving fuzzy sets. In the following we introduce this principle.

Definition 1.2.1. Assume X and Y are crisp sets and let f be a mapping from X to Y,

$$f : X \to Y$$

such that for each $x \in X, \quad f(x) = y \in Y$. Assume A is a fuzzy subset of X, using the extension principle, we can define $f(A)$ as a fuzzy subset of Y such that

$$f(A) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & if \quad f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

It should be noted that if f is strictly increasing (or strictly decreasing) $f(A)(y)$

can be write as

$$f(A)=\begin{cases} A(f^{-1}(y)) & if \quad y\in f(X)\\ 0 & otherwise\end{cases}$$

Example 1.2.1. *Let* $f(x)=\frac{1}{1+exp(-x)}$ *and* A *be a fuzzy number. Then*

$$f^{-1}(y)=\begin{cases} \ln(\frac{y}{1-y}) & if\ 0\le y\le 1\\ \\ 0 & otherwise\end{cases}$$

it follows that

$$f(A)(y)=\begin{cases} A(\ln(\frac{y}{1-y})) & if\ 0\le y\le 1\\ \\ 0 & otherwise\end{cases}$$

The extension principle can be generalized to n-place function.

Definition 1.2.2. Let $X_1, X_2, \cdots, X_n$ and Y be a family of sets. Assume f is a mapping from the cartesian product $X_1\times X_2\times\cdots\times X_n$ into Y, that is, for each n-tuple $(x_1, x_2, \cdots, x_n)$ such that $x_i\in X_i$, we have

$$f(x_1, x_2, \cdots, x_n)=y\in Y.$$

Let $A_1, A_2, \cdots, A_n$ be fuzzy subsets of $X_1, X_2, \cdots, X_n$, respectively; then the extension principle allows for the evaluation of $f(A_1, A_2, \cdots, A_n)$. In particular, $f(A_1, A_2, \cdots, A_n)=B$, where B is a fuzzy subset of Y such that

$$f(A_1, A_2, \cdots, A_n)(y)=\begin{cases} \sup\{\min\{A_1(x_1), \cdots, A_n(x_n)\}|x\in f^{-1}(y)\} & if\ f^{-1}(y)\neq\emptyset\\ \\ 0 & otherwise\end{cases}$$

For $n=2$ this relation is as follows,

$$f(A_1, A_2)(y)=\sup\{A_1(x_1), A_2(x_2)|f(x_1, x_2)=y\} \tag{1.5}$$

Example 1.2.2. *(extended addition) Let* $f : X \times X \to X$ *be defined as*

$$f(x_1, x_2) = x_1 + x_2,$$

i.e., f *is the addition operator. Suppose* A_1 *and* A_2 *are fuzzy subsets of* X*. Then using the extension principle we get*

$$f(A_1, A_2) = \sup_{x_1+x_2=y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = A_1 + A_2$.

Example 1.2.3. *(extended subtraction) Let* $f : X \times X \to X$ *be defined as*

$$f(x_1, x_2) = x_1 - x_2,$$

i.e., f *is the addition operator. Suppose* A_1 *and* A_2 *are fuzzy subsets of* X*. Then using the extension principle we get*

$$f(A_1, A_2) = \sup_{x_1-x_2=y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = A_1 - A_2$.

It is important that if A is a fuzzy number then

$$(A - A)(y) = \sup_{x_1-x_2=y} min\{A(x_1), A(x_2)\}, \quad y \in \mathbb{R}$$

is not equal to the fuzzy number $\overline{0}$, where $\overline{0}(t) = 1$ if $t = 0$ and $\overline{0}(t) = 0$ if $t \neq 0$. see figure 1.5

Figure 1.5: From left to right the membership function of $A - A$ and A

Example 1.2.4. *Let $f : X \times X \to X$ be defined as*

$$f(x_1, x_2) = \lambda x_1 + \lambda x_2, \quad \lambda \in \Re$$

suppose A_1 and A_2 are fuzzy subsets of X. Then using the extension principle we get

$$f(A_1, A_2) = \sup_{\lambda x_1 + \lambda x_2 = y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = \lambda A_1 + \lambda A_2$.

Example 1.2.5. *(extended multiplication) Let $f : X \times X \to X$ be defined as*

$$f(x_1, x_2) = x_1 x_2,$$

i.e., f is the multiplication operator. Suppose A_1 and A_2 are fuzzy subsets of X. Then using the extension principle we get

$$f(A_1, A_2) = \sup_{x_1 x_2 = y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = A_1 A_2$.

Example 1.2.6. *(extended division) Let $f : X \times X \to X$ be defined as*

$$f(x_1, x_2) = \frac{x_1}{x_2}, \qquad x_2 \neq 0$$

i.e., f is the division operator. Suppose A_1 and A_2 are fuzzy subsets of X. Then using the extension principle we get

$$f(A_1, A_2) = \sup_{\frac{x_1}{x_2}=y, x_2\neq 0} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = \frac{A_1}{A_2}$.

Definition 1.2.3. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be function from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$. Then f is said to be a fuzzy function (or mapping) and we use the notation

$$f : \mathcal{F}(X) \to \mathcal{F}(Y),$$

it should be noted , however, that a fuzzy function is not necessarily defined by zadeh's extension principle. It can be any function which maps a fuzzy set $A \in \mathcal{F}(X)$ into a fuzzy set $B := f(A) \in \mathcal{F}(Y)$.

Definition 1.2.4. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets . A fuzzy mapping $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ is said to be monotonic increasing if from $A, A' \in \mathcal{F}(X)$ and $A \subset A'$ it follows that $f(A) \subset f(A')$.

Theorem 1.2.7. *Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets. Then every fuzzy mapping $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined by the extension principle is monotonic increasing.*

Proof 1.2.1. *Let $A, A' \in \mathcal{F}(X)$ such that $A \subset A'$. Then using the definition of sup-min extension principle we get*

$$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \leq \sup_{x \in f^{-1}(y)} A'(x) = f(A')(y)$$

for all $y \in Y$.

Let A and B be fuzzy numbers with $[A]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ and $[B]_\alpha = [\underline{b}(\alpha), \overline{b}(\alpha)]$. Then it can easily be shown that

$$[A + B]_\alpha = [\underline{a}(\alpha) + \underline{b}(\alpha), \overline{a}(\alpha) + \overline{b}(\alpha)]$$

$$[-A]_\alpha = [-\overline{a}(\alpha), -\underline{a}(\alpha)]$$

$$[A - B]_\alpha = [\underline{a}(\alpha) - \overline{b}(\alpha), \overline{a}(\alpha) - \underline{b}(\alpha)]$$

$$[\lambda A]_\alpha = [\lambda \underline{a}(\alpha), \lambda \overline{a}(\alpha)], \quad \lambda \geq 0$$

$$[\lambda A]_\alpha = [\lambda \overline{a}(\alpha), \lambda \underline{a}(\alpha)], \quad \lambda < 0$$

for all $\alpha \in [0, 1]$, i.e. any set of the extended sum of two fuzzy numbers is equal to the sum of their $\alpha - level$ sets. Now we show that this property is valid for any continuous function

Theorem 1.2.8. *[12]Let $f : X \to X$ be a continuous function and let A be fuzzy numbers then*

$$[f(A)]_\alpha = f([A]_\alpha)$$

where $f(A)$ is defined by the extension principle and

$$(f[A]_\alpha) = \{f(x) | x \in [A]_\alpha\}.$$

Theorem 1.2.9. *[12] Due to (1.5) it is well known that when f is continuous then*

$$[f(A,B)]_\alpha = f([A]_\alpha, [B]_\alpha)$$

for all $A, B \in E^n, \quad 0 \le \alpha \le 1$

Remark 1.2.1. If $[A]_\alpha = [\underline{a}, \overline{a}]$ and f is monotonic increasing then from the above theorem we get

$$[f(A)]_\alpha = f([A]_\alpha) = f([\underline{a}(\alpha), \overline{a}(\alpha)]) = [f(\underline{a}(\alpha)), f(\overline{a}(\alpha))].$$

Example 1.2.10. *Let $f(x,y) = xy$ and let $[A]_\alpha = [a_{1\alpha}, a_{2\alpha}]$ and $[B]_\alpha = [b_{1\alpha}, b_{2\alpha}]$ be two fuzzy numbers. Applying above theorem we get*

$$[AB]_\alpha = [A]_\alpha [B]_\alpha = [a_{1\alpha}b_{1\alpha}, a_{2\alpha}b_{2\alpha}],$$

hold if and only if A and B are both nonnegative, i.e. $A(x) = B(x) = 0$ for $x \le 0$. In general form we obtain a very complicated expression for the α level sets of the product AB

$$[AB]_\alpha = [m_\alpha, M_\alpha],$$

where

$$m_\alpha = \min\{a_{1\alpha}b_{1\alpha}, a_{1\alpha}b_{2\alpha}, a_{2\alpha}b_{1\alpha}, a_{2\alpha}b_{2\alpha}\},$$

$$M_\alpha = \max\{a_{1\alpha}b_{1\alpha}, a_{1\alpha}b_{2\alpha}, a_{2\alpha}b_{1\alpha}, a_{2\alpha}b_{2\alpha}\},$$

for $\alpha \in I = [0,1]$.

Chapter 2

Fuzzy analysis

2.1 Introduction

Real analysis and calculus of functions of a real variable form a highly development and very important theory for mathematics, and applied science. This theory involves concepts of convergence, limits, continuity, differentiation and integration. These techniques are applied to a vast number of areas: approximation, differential equations, etc. Central to implicit in these ideas are topological notions like open and closed sets, completeness, compactness and so on. When formulating a calculus of fuzzy-valued functions and a topology of fuzzy numbers, it is natural first to develop metrics on spaces of fuzzy sets and fuzzy numbers. Then definition of limit and convergence will allow the development of limit operations such as differentiation and integration, and formulation of a calculus and its applications

2.2 Compact convex subsets in $\mathbb{R}^n$

Attention will be focused on the following two spaces of nonempty subsets of $\mathbb{R}^n$:

1) κ^n consisting of all nonempty compact (closed and bounded) subset of $\mathbb{R}^n$,

2) κ_c^n considering of all nonempty compact convex subsets of $\mathbb{R}^n$.

Note the strict inclusion $\kappa_C^n \subset \kappa^n$.

Proposition 2.2.1. *κ^n and κ_c^n are closed under the operations addition and scalar multiplication.*

In fact, these two operations induce a linear structure on κ^n and κ_c^n with zero element $\{0\}$. The structure is that of a cone rather than a vector space because, in general, $A + (-1)A \neq \{0\}$.

Example 2.2.2. *Let $A = [0, 1]$ so that $(-1)A = [-1, 0]$, and so*

$$A + (-1)A = [0, 1] + [-1, 0] = [-1, 1].$$

Thus, adding -1 times a set does not constitute a natural operation of subtraction. Instead, define the Hukuhara difference $A -_h B$ of nonempty sets A and B, provided it exists, as the nonempty set C satisfying $A = B + C$.

From the preceding example,

$$[-1, 1] -_h [-1, 0] = [0, 1],$$

$$[-1,1] -_h [0,1] = [-1,0].$$

Clearly, $A -_h A = \{0\}$ for all nonempty sets A. An obvious necessary condition for the Hukuhara difference $A -_h B$ to exist is that some translate of B is a subset of $A, B + \{c\} \subseteq A$ for some $c \in \Re^n$. When it exists, $A -_h B$ is unique. However, the Hukuhara difference need not exist, as it is seen from the following example.

Example 2.2.3. $\{0\} -_h [0,1]$ *does not exist, since translate of* $[0,1]$ *can never belong to the singeltone set* $\{0\}$.

2.3 The Hausdorff Metric

Let x be a point in $\Re^n$ and A be a nonempty subset of $\Re^n$. Define the *distance* $d(x,A)$ from x to A by

$$d(x,A) = \inf\{\| x - a \| : a \in A\},$$

also define the Hausdorff seperation between A, B of $\Re^n$ by

$$d_H^*(B,A) = \sup\{d(b,A) : b \in B\}.$$

Therefore define the *Hausdorff distance* between nonempty subsets A and B of $\Re^n$ by

$$d_H(A,B) = \max\{d_H^*(A,B), d_H^*(B,A)\}.$$

This is now symmetric in A and B. Consequently,

(a) $d_H(A,B) \geq 0$ with $d_H(A,B) = 0$ if and only if $\overline{A} = \overline{B}$

(b) $d_H(A,B) = d_H(B,A)$

(c) $d_H(A,B) \leq d_H(A,C) + d_H(C,B)$,

for any nonempty subsets A, B and C of $\Re^n$. Restricting attention to the subspace κ^n and κ^n_c the above Hausdorff distance is a metric, the *Hausdorff metric.*

Definition 2.3.1. (continuity) Consider mapping F from a domain T in $\mathbb{R}^k$ into the metric space(κ^n_C, d_H). The usual definition of continuity of mapping between metric spaces is applied here. A set valued mapping $F : \mathbb{R}^k \to \kappa^n_C$ is continuous at t_0 in T if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$d_H(F(t), F(t_0)) < \varepsilon \tag{2.1}$$

for all $t \in T$ with $\| t - t_0 \| < \delta$. Equivalently, this can be stated in terms of convergence of sequences, that is as

$$\lim_{t_n \to t_0} d_H(F(t_n), F(t_0)) = 0$$

for all sequences $\{t_n\}$ in T with $t_n \to t_0$.

Using the Hausdorff separation d^*_H and neighborhoods, we observe that (2.1) is equivalent to both

$$d^*_H(F(t), F(t_0)) < \varepsilon \qquad and \qquad d^*_H(F(t_0), F(t)) < \varepsilon$$

In the first case, F is said to be $upper-semicontinuous$ and in second case F is said to be $lower-semicontinuous$ at t_0. Clearly F is continuous at t_0 if and only if it is both upper-semicontinuous and lower-semicontinuous at t_0. A set valued mapping can be lower-semicontinuous without being upper-semicontinuous, and vice versa.

Example 2.3.1. *The set valued mapping F from $\mathbb{R}$ into $[0,1]$ defined by*

$$F(t) = \begin{cases} \{0\} & for \;\; t = 0 \\ [0,1] & for \;\; t \in \mathbb{R} - \{0\} \end{cases}$$

is lower-semicontinuous,but not upper-semicontinuous, at $t_0 = 0$. On the other hand,F from $\mathbb{R}$ defined by

$$F(t) = \begin{cases} [0,1] & for \;\; t = 0 \\ \{0\} & for \;\; t \in \mathbb{R} - \{0\} \end{cases}$$

is upper-semicontinuous, but not lower-semicontinuous, at $t_0 = 0$.

Let $B(R^k)$ and $B(\kappa_C^n)$ denote the σ-algebras of Borel subsets of (R^k) and (κ_C^n, d_H) respectively. Adopting the usual definition of Borel measurability of a mapping between metric spaces, if T is a subset of $\mathbb{R}$, a mapping $F : T \to \kappa_C^n$ is measurable if

$$\{t \in T : F(t) \in \beta\} \in B(\mathbb{R}^k) \;\; for \;\; all \; \beta \in B(\kappa_C^n).$$

Let $T \subseteq \mathbb{R}$ there is a definition of differentiation of F which involves Hukuhara difference quotients of F.

Suppose for all $\Delta t > 0$ sufficiently small the Hukuhara differences

$$F(t_0 + \Delta t) -_h F(t_0), \quad F(t_0) -_h F(t_0 - \Delta t) \tag{2.2}$$

exist with $t_0 + \Delta t,\; t_0 - \Delta t \in T$. Then F is said to be Hukuhara differentiable at $t_0 \in T$ if there exists an $F'(t_0) \in \kappa_C^n$, called the Hukuhara derivative of F at t_0, such that

$$\lim_{\Delta t \to 0^+} d_H((F(t_0 + \Delta t) -_h F(t_0))/\Delta t, F'(t_0)) = 0,$$

$$\lim_{\Delta t \to 0^+} d_H((F(t_0) -_h F(t_0 - \Delta t))/\Delta t, F'(t_0)) = 0.$$

Since a Hukuhara difference $A -_h B$ only exists if a translate of B is contained A, it follows from (2.2) that:

Proposition 2.3.2. *If $F : T \to \kappa_C^n$ with $T \subseteq \mathbb{R}$ is Hukuhara differentiable at $t_0 \in T$, then $diam(F(t))$ is nondecreasing at t_0.*

Let $F : [0,1] \to \kappa_C^n$ and let $S(F)$ denotes the set of integrable selectors of F over $[0,1]$. Then the Aumann integral of F over $[0,1]$ is defined as

$$\int_0^1 F(t)dt = \{\int_0^1 f(t)dt : \ f \in S(F)\}.$$

If $S(F) \neq \phi$, then the Aumman integral exists and F is said to be Aumman integrable. Say that F is integrably bounded on $[0,1]$ if there exists an integrable function $g : [0,1] \to \mathbb{R}^1$ such that

$$\| F(t) \| \leq g(t) \qquad for \ \ almost \ all \quad t \in [0,1].$$

If such an F has measurable selectors, then they are also integrable and $S(F)$ is nonempty.

Theorem 2.3.3. *[32] If $F : [0,1] \to \kappa_C^n$ is measurable and integrably bounded then it is Aumann integrable over each $[a,s] \subset [0,1]$ with $\int_a^s F(t)dt \in \kappa_C^n$ for all $s \in [a,1]$.*

Theorem 2.3.4. *[32] If $F, G : [0,1] \to \kappa_C^n$ are Aumann integrable with $\int_0^1 F(t)dt = \int_0^1 G(t)dt$, then $F(t) = G(t)$ for almost all $t \in [0,1]$.*

Theorem 2.3.5. *[32] Suppose $F_i, F : [0,1] \to \kappa_C^n, i = 1, 2, \cdots$ be measureable and uniformly integrably bounded. If $F_i(t) \to F(t)$ for all $t \in [0,1]$ as $i \to \infty$ then*

$$\lim_{i\to\infty} A_i = \lim_{i\to\infty} \int_0^1 F_i(t)dt = A = \int_0^1 F(t)dt$$

Theorem 2.3.6. *[32] If $G, F : [0,1] \to \kappa_C^n$ are integrable, then so also is*

$$d_H(F(t), G(t)) : [0,1] \to \mathbb{R}^1$$

and

$$d_H(\int_0^1 F(t)dt, \int_0^1 G(t)dt) \leq \int_0^1 d_H(F(t), G(t))dt.$$

2.4 Metric on E^n

The most commonly used metrics on E^n involve the Hausdorff distance between the level sets of the fuzzy sets. They are metric spaces applied to function $\phi : I = [0,1] \to \mathbb{R}^+$ defined by $\phi(\alpha) = d_H([U]_\alpha, [V]_\alpha)$ for $\alpha \in I$, where $U, V \in E^n$. The supremum metric d_∞ on E^n is defined by

$$d_\infty(U, V) = \sup\{d_H([U]_\alpha, [V]_\alpha) : \alpha \in I\}$$

for all $U, V \in E^n$ and is obviously a metric on E^n. Here the supremum need not be attained and cannot be replaced by the maximum.

Example 2.4.1. *Let $U, V \in E^1$ be defined on level sets by*

$$[U]_\alpha = [V]_\alpha = [0,1] \ \ for \ \ 0 \leq \alpha \leq \frac{1}{2},$$

$$[U]_\alpha = \{0\}, \ [v]_\alpha = [0, 2(1-\alpha)] \ for \ \ \frac{1}{2} < \alpha \leq 1,$$

therefore

$$\phi(\alpha) = \begin{cases} 0 & for \ \ 0 \leq \alpha \leq \frac{1}{2}, \\ 2(1-\alpha) & for \ \ \frac{1}{2} < \alpha \leq 1. \end{cases}$$

Then $\sup\{\phi(\alpha) : \alpha \in I\} = 1$, *but this is not attained.*

Theorem 2.4.2. *[32] (E^n, d_∞) is a complete metric space.*

Here, mapping F from a domain T in $\mathbb{R}^n$ into the space E^n of fuzzy sets on $\mathbb{R}^n$ is considered.

Definition 2.4.1. (continuity) A fuzzy set valued mapping $F : T \to E^n$ is said to be continuous at $t_0 \in T$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$d_\infty(F(t), F(t_0)) < \varepsilon$$

for all $t \in T$ with $\| t - t_0 \| < \delta$.

Remark 2.4.1. Upper and lower semi-continuity of a fuzzy set valued mapping $F : T \to E^n$ is defined level setwise uniformly in $\alpha \in I$. Say that is upper semi-continuous at $t_0 \in T$ if for every $\varepsilon > 0$ there exists a $\delta(t_0, \varepsilon) > 0$ such that

$$d_H^*([F(t)]_\alpha, [F(t_0)]_\alpha) < \varepsilon$$

for $\alpha \in I$ and $t \in T$ with $\| t - t_0 \| < \delta$, and lower-semicontinuous at t_0 if for every $\varepsilon > 0$ there exists a $\delta(t_0, \varepsilon) > 0$ such that

$$d_H^*([F(t_0)]_\alpha, [F(t)]_\alpha) < \varepsilon$$

for $\alpha \in I$ and $t \in T$ with $\| t - t_0 \| < \delta$

2.5 Differentiation

One of the useful definitions of differentiation of a fuzzy set valued mapping of a single real variable is based on Hukuhara difference quotients be defined as follows

Definition 2.5.1. A mapping $F : T \to E^n$ is Hukuhara differentiaable at $t_0 \in T \subseteq \mathbb{R}$ if for some $h_0 > 0$ the Hukuhara differences

$$F(t_0 + \Delta t) -_h F(t_0), \quad F(t_0) -_h F(t_0 - \Delta t) \tag{2.3}$$

exist in E^n for all $0 < \Delta t < h_0$ and if there exists an $F'(t_0) \in E^n$ such that

$$\lim_{\Delta t \to 0^+} d_\infty((F(t_0 + \Delta t) -_h F(t_0))/\Delta t, F'(t_0)) = 0,$$

and

$$\lim_{\Delta t \to 0^+} d_\infty((F(t_0) -_h F(t_0 - \Delta t))/\Delta t, F'(t_0)) = 0.$$

The fuzzy set $F'(t_0)$ is called Hukuhara derivative of F at t_0.

Note that $U -_h V = W \in E^n$ is defined on level sets, that is as $[U -_h V]_\alpha = [U]_\alpha -_h [V]_\alpha = [W]_\alpha$ for all $\alpha \in I$. In view of the definition of the metric d_∞ all the level set mapping $[F(.)]_\alpha$ are Hukuhara differentiable at t_0 with Hukuhara derivatives $[F'(t_0)]_\alpha$ for each $\alpha \in I$ when $F : T \to E^n$ is Hukuhara differentiable at t_0 with Hukuhara derivative $F'(t_0)$.

Essentially, the Hukuhara difference in (2.3) may not exist. From Proposition (2.2.1), it follows:

Theorem 2.5.1. *[32] If $F : T \to E^n$ are Hukuhara differentiable at $t_0 \in T \subseteq \mathbb{R}$, then $diam([F(t)]_\alpha)$ is nondecreasing in t at t_0 for each $\alpha \in I$.*

Theorem 2.5.2. *[32] If $F : T \to E^n$ is Hukuhara differentiable at $t_0 \in T \subseteq \mathbb{R}$, then*

(i) *its derivative $F'(t_0)$ is unique*

(ii) *it is continuous at t_0.*

Theorem 2.5.3. *[32] If $F, G : T \to E^n$ is Hukuhara differentiable at $t_0 \in T \subseteq \mathbb{R}$, then $F + G$ and cF, for all $c \in \mathbb{R}$, are Hukuhara differentiable at t_0 and*

$$(F + G)'(t_0) = F'(t_0) + G'(t_0), \quad (cF)'(t_0) = cF'(t_0).$$

Theorem 2.5.4. *If $F : T \to E^n$ are Hukuhara differentiable at $t_0 \in T \subseteq \mathbb{R}$ with derivative $F'(t_0) : T \to E^n$ continuous at each $t_0 \in T$. Then for all $t_0, t_1 \in T$ with $t_0 < t_1$*

$$d_\infty(F(t_1), F(t_0)) \leq |t_1 - t_0| \max_{t_0 \leq t \leq t_1} \parallel F'(t) \parallel .$$

Remark 2.5.1. If $F : T \to E^1$ is Hukuhara differentiable at $t_0 \in T \subseteq \mathbb{R}$ and $[F(t)]_\alpha = [f_\alpha(t), g_\alpha(t)]$ for each $\alpha \in I$ and Hukuhara derivative $F'(t_0)$ has level sets

$$[F'(t_0)]_\alpha = [f'_\alpha(t_0), g'_\alpha(t_0)] \tag{2.4}$$

for each $\alpha \in I$. The converse need not apply, that is the differentiation of the end point mapping f_α, g_α need not apply the Hukuhara differentiability of F since the intervals in (2.4) need not be the level sets of a fuzzy set in E.

2.6 Integration

Integration of a fuzzy set valued mapping of a single real variable is defined level setwise in term of the Aumann integrals of its level set mappings. Without loss of generality, restrict attention to mappings defined on the unit interval $[0,1]$, that is, $F : [0,1] \to E^n$. A mapping $F : [0,1] \to E^n$ is said to be integrably bounded if there exists an integrable function $h : [0,1] \to \mathbb{R}$ such that

$$\| F(t) \| \leq h(t) \quad for\ all \quad t \in [0,1].$$

Since $\| F(t) \| = \| [F(t)]_0 \| \geq \| [F(t)]_\alpha \|$ for all $\alpha \in I$, each level set valued mapping $[F(.)]_\alpha : [0,1] \to \kappa_C^n$ is integrably bounded as defined in theorem (2.3.3). By theorem (2.8.1) each level set mapping $[F(.)]_\alpha : [0,1] :\to \kappa_n^c$ is Aumann integrable on $[0,1]$. If there exists a fuzzy set $U \in E^n$ such that $[U]_\alpha = \int_0^1 [F(t)]_\alpha dt$ for all$\alpha \in I$, say that F is integrable over $[0,1]$ and call U, which is denoted by $\int_0^1 F(t)dt$, its integral over

$[0,1]$. Thus, for each α

$$[\int_0^1 F(t)dt]_\alpha = \int_0^1 [F(t)]_\alpha dt.$$

Theorem 2.6.1. *[32]If $F, G : [0,1] \to E^n$ are integrable, then $F + G$ and λF for any $\lambda \in \mathbb{R}$ are integrable with*

$$\int_0^1 (F(t) + G(t))dt = \int_0^1 F(t)dt + \int_0^1 G(t)dt$$

$$\int_0^1 \lambda F(t)dt = \lambda \int_0^1 F(t)dt$$

Theorem 2.6.2. *[32] If $F : [0,1] \to E^n$ is integrable with level sets $[F(t)]_\alpha = [f_\alpha(t), g_\alpha(t)]$, then $f_\alpha, g_\alpha : [0,1] \to \mathbb{R}$ are integrable and*

$$[\int_0^1 F(t)dt]_\alpha = \int_0^1 [F(t)]_\alpha dt = [\int_0^1 f_\alpha(t)dt, \int_0^1 g_\alpha(t)dt] \tag{2.5}$$

for $\alpha \in I$. Conversely, if $f_\alpha, g_\alpha : [0,1] \to \mathbb{R}$ are integrable for each $\alpha \in I$, then F is integrable and (2.5) holds

Theorem (2.3.2) has a counterpart for fuzzy set valued mappings as follow

Theorem 2.6.3. *[32] Let $F_i, F : [0,1] \to E^n, i = 1, 2, ...$be strongly measurable and uniformly integrably bounded. If $F_i \to F(t)$ in the d_∞ metric as $i \to \infty$ for all $t \in [0,1]$, then*

$$\lim_{i\to\infty} U_i = \lim_{i\to\infty} \int_0^1 F_i(t)dt = U = \int_0^1 F(t)dt.$$

Chapter 3

Fuzzy differential equation

3.1 Introduction

Evolutionary mechanisms of many important dynamical processes are often modeled by differential equation, which need to be solved for specific initial conditions. For the most basic situation involving order autonomous ordinary differential equation in $\mathbb{R}^n$ this initial value problem takes the form

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad x(0) = x_0, \tag{3.1}$$

for which it is required to find a solution, that is a differentiable function $x = x(t)$ with $x(0) = x_0$, that satisfies the equation in some interval (t_1, t_2) for some $t_1 < 0 < t_2$, that may be infinite. When the mapping f on the right hand side of DE is continuous, an initial value problem is equivalent to solving the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds$$

for a continuous function $x = x(t)$. Here we interpret the right hand side of DE in (3.1) as a mapping $f : E^n \to E^n$ with the solutions taking values $X(t) \in E^n$ for all appropriate t. Traditionally the d_∞ metric on E^n and the Hukuhara derivative have been used for fuzzy DE.

Due to Eq.(2.4) we may replace Eq.(3.1) by the equivalent system

$$\begin{cases} x_1'(t) = f_1(t,x), & x_1(t_0) = x_{10} \\ x_2'(t) = f_2(t,x), & x_2(t_0) = x_{20} \end{cases}$$

where

$$f_{1\alpha}(t,x) = \min\{f(t,x) : x \in [x_{1\alpha}, x_{2\alpha}]\},$$

$$f_{2\alpha}(t,x) = \max\{f(t,x) : x \in [x_{1\alpha}, x_{2\alpha}]\}.$$

Example 3.1.1. *Let $A \in E^1$ has level sets $[A]_\alpha = [a_{1\alpha}, a_{2\alpha}]$ for $\alpha \in I$ and suppose that a solution $X : [0,T] \to E^1$ of the fuzzy DE. Suppose*

$$X'(t) = AX \tag{3.2}$$

on E^1 has level sets $[X(t)]_\alpha = [x_{1\alpha}(t), x_{2\alpha}(t)]$ for $\alpha \in I$ and $t \in [0,T]$. This fuzzy DE is equivalent to the coupled system of ordinary DE:

$$x_{1\alpha}'(t) = \min\{a_{1\alpha}x_{1\alpha}(t), a_{1\alpha}x_{2\alpha}(t), a_{2\alpha}x_{1\alpha}(t), a_{2\alpha}x_{2\alpha}(t)\},$$

$$x_{2\alpha}'(t) = \max\{a_{1\alpha}x_{1\alpha}(t), a_{1\alpha}x_{2\alpha}(t), a_{2\alpha}x_{1\alpha}(t), a_{2\alpha}x_{2\alpha}(t)\},$$

for $\alpha \in I$.

Since by remark (2.7.1) the Hukuhara derivative $X'(t)$ has level sets $[X'(t)]_\alpha = [x_{1\alpha}'(t), x_{2\alpha}'(t)]$ for $\alpha \in I$ and $t \in [0,T]$ and by Extension Principle, the fuzzy set $f(X(t)) = AX(t)$ has level sets

$$[AX(t)]_\alpha = [m_\alpha(t), M_\alpha(t)]$$

where

$$m_\alpha(t) = \min\{a_{1\alpha}x_{1\alpha}(t), a_{1\alpha}x_{2\alpha}(t), a_{2\alpha}x_{1\alpha}(t), a_{2\alpha}x_{2\alpha}(t)\},$$

$$M_\alpha(t) = \max\{a_{1\alpha}x_{1\alpha}(t), a_{1\alpha}x_{2\alpha}(t), a_{2\alpha}x_{1\alpha}(t), a_{2\alpha}x_{2\alpha}(t)\}$$

for $\alpha \in I$ *and* $t \in [0, T]$.

The existence and uniqueness of such a solution of an initial value problem(3.1), which is often written as $x = x(t, x_0)$, is important for both mathematical and modeling considerations. Classical existence and uniqueness theorems assume that the mapping f is continuous and that a Lipschitz condition

$$d_\infty(f(t,x), f(t,y)) \leq L d_\infty(x,y), \quad x, y \in E^1$$

is satisfied for some $L > 0.$, these conditions are given by Kaleva [31].

3.2 Fuzzy differential inclusion

Knowledge about differential equation is often incomplete or vague. For example, parameter values functional relationships, or initial conditions, may not be known precisely. Fuzzy differential equation (FDE) was first formulated by Kaleva [31] and Seikkala [44] in the form of time dependent. Kaleva had formulated fuzzy differential equation(FDE), in term of Hukuara derivative, [31]. Recently, Buckley and Feuring have given a very general formulation of fuzzy first-order initial value problem, [13]. They first find the crisp solution, fuzzify it and then check to see if it satisfies the FDE. If $x(t)$ is the solution to a FDE, then using Hukuhara derivative suffers a grave disadvantage in so far as $x(t)$ has the property that $\text{diam}(x(t))$ is nondecreasing in t, [47]. This renders the FDEs, unsuitable for modeling and simulation. To overcome this difficulty, first, Aubin [16] and then Hüllermeier [7] introduced the notion of the fuzzy differential inclusions(FDI) relation. Hüllermeier [7] introduced a numerical algorithm to solve general FDIs. Diamond has extended some theoretic notions such as periodicity, Lyapunov stability and attraction [34, 33]. Recently Majumdar is devised a numerical algorithm, which is easy to implement and named *Crystalline* algorithm, for the one dimensional FDIs, [20]. In this chapter we are going to propose a new numerical method for computing approximations of the set of all solutions to a FDI, which is named "Tuning of reachable set(TRS)". We briefly review FDIs, present our method (TRS algorithm) to solve FDIs. One way to model uncertainty and vague in a dynamical system is to replace functions and initial values in the problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{3.3}$$

by set-valued functions and initial sets which leads to a differential inclusion

$$\begin{cases} \dot{x}(t) \in F(t, x(t)), \\ x(0) \in X_0, \end{cases} \tag{3.4}$$

where $F : [0, T] \times \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ is a set-valued function and $X_0 \subset \mathbb{R}$ is compact and convex. A function $x : [0, T] \rightarrow \mathbb{R}$ is a solution to (3.4), if it is an absolutely continuous function and satisfies (3.4) almost everywhere. Let χ denotes the set of all solutions to (3.4). Particularly, we are interested in the so-called *reachable sets*

$$X(t) := \{x(t) | x \in \chi\}.$$

The reachable set $X(t)$ is the set of possible solution of (3.3) at time $t \in J = [0, T]$. A reasonable generalization of this approach which takes vagueness into account is to replace sets by fuzzy sets, i.e., (3.4) becomes a fuzzy differential inclusion [16, 45, 46]

$$\begin{cases} \dot{x} \in F(t, x(t)), \\ x(0) \in X_0, \end{cases} \tag{3.5}$$

with a fuzzy function $F : J \times \mathbb{R} \rightarrow F(\mathbb{R})$ and a fuzzy set $X_0 \subset F(\mathbb{R})$, where $F(\mathbb{R})$ is the set of all fuzzy subsets of $\mathbb{R}$. Also $\dot{x}(t)$ is the usual crisp derivative of the crisp differentiable function $x(t)$ with respect to t. In this chapter, we propose a numerical method for finding reachable set $X(t)$ that is based on the theoretical considerations of the following theorems.

Theorem 3.2.1. *[7] Suppose the fuzzy function $F : J \times \mathbb{R} \rightarrow E$ to be continuous in t and satisfy a Lipschitz condition*

$$d_\infty(F(t, x), F(t, y)) \leq L|x - y|$$

on $J \times \mathbb{R}$ with a Lipschitz constant $L > 0$. Consider the set $\widetilde{\chi}$ of solutions of (3.5). The reachable set $X(t)$ associated with χ is a normal, upper semi-continue and compactly supported fuzzy set for all $t \in J$. If F is also concave, i.e.,

$$\alpha F(t, x) + \beta F(t, y) \subset F(t, \alpha x + \beta y)$$

for any numbers $\alpha, \beta > 0$, $\alpha + \beta = 1$, then $X(t) \in E$.

We call a function $x_\alpha : J \rightarrow \Re$ an α-solution to (3.5), if it is absolutely continuous and satisfies

$$\begin{cases} \dot{x}(t) \in F_\alpha(t, x(t)), \\ x(0) \in [X_0]_\alpha, \end{cases} \tag{3.6}$$

almost everywhere on J, where $F_\alpha(t, x(t))$ is the α-cut of the fuzzy set $F(t, x(t))$. The set of all α-solution to (3.6) is denoted χ_α and the α-reachable set $X_\alpha(t)$ is defined as $X_\alpha(t) := \{x(t) : x \in \chi_\alpha\}$. In this chapter, we tune a fix $\alpha \in (0, 1]$ and $t \in J = [0, T]$.

Remark 3.2.1. For $A = \chi_\lambda \in E^1$ where $\lambda > 0$ the fuzzy differential equation (3.2) becomes

$$X'(t) = -\lambda X(t), \quad X(0) = X_0, \tag{3.7}$$

the system of ordinary reduces to

$$x'_{1\alpha}(t) = -\lambda x'_{2\alpha}(t), \qquad x'_{2\alpha}(t) = -\lambda x'_{1\alpha}(t)$$

for $\alpha \in I$, with solution corresponding to a symmetric triangular fuzzy number initial value $X_0 \in E^1$ with $[X_0]_\alpha = [x_{01\alpha}, x_{02\alpha}] = [-x_{01(1-\alpha)}, x_{01(1-\alpha)}] = (1-\alpha)[-x_{01}, x_{01}]$ for $\alpha \in I$ given by

$$x_{1\alpha}(t) = \frac{1}{2}(x_{01\alpha} - x_{02\alpha})\exp(t) + \frac{1}{2}(x_{01\alpha} + x_{02\alpha})\exp(-t)$$

$$x_{2\alpha}(t) = \frac{1}{2}(x_{02\alpha} - x_{01\alpha})\exp(t) + \frac{1}{2}(x_{01\alpha} + x_{02\alpha})\exp(-t)$$

for $\alpha \in I$ and $t \geq 0$. Note that, since $x_{01\alpha} = x_{02\alpha} = a$ then

$$X(t)_\alpha = (1-\alpha)\exp(\lambda t)[-a, a], \qquad 0 \leq \alpha \leq 1$$

and $diam[X(t)]_0 = (2)\exp(\lambda t)$ is unbounded as $t \to \infty$ so the solution becomes fuzzier with increasing time demonstrating that this interpretation does not really generalize the crisp case. However, if (3.7) is interpreted as a family of differential inclusion

$$x'_\alpha(t) \in -\lambda x_\alpha(t), x_\alpha(0) \in X_\alpha := (1-\alpha)[-1,1], \quad 0 \le \alpha \le 1 \tag{3.8}$$

a much more intuitive outcome emerges. Since $-\lambda x_\alpha = \{-\lambda x_\alpha\}$ is a singleton set in κ_C^1, (3.8) becomes

$$x'_\alpha(t) = -\lambda x_\alpha(t), \quad x_\alpha(0) \in X_\alpha := (1-\alpha)[-1,1], \quad 0 \le \alpha \le 1$$

which has solution set $X_\alpha(t)$ on $[0,t]$ comprising the functions

$$x_\alpha(t) = x_\alpha(0)\exp(-\lambda t), \quad x_\alpha(0) \in X_\alpha.$$

Consequently, $X_\alpha(t) = (1-\alpha)\exp(-\lambda t)[-1,1]$. So $X(t)$ is interpreted to be symmetric triangular fuzzy number and $diam(suppX(t)) \to 0$ as $t \to \infty$.

3.3 Approximation of reachable sets

The discussion in previous section made clear that the behavior of a fuzzy dynamical system, characterize by the "fuzzy funnel" $\{X(t)|\ 0 \le t \le T\}$, can be described "levelwise" by the α-level sets $\{X_\alpha(t)|\ 0 \le t \le T\}$ for $\alpha \in (0,1]$. For a certain value α, such a "crisp funnel" is defined by the reachable sets of a General initial value problem (GIVP). Thus, we can characterize the set of solution to a Fuzzy initial value problem (FIVP) by characterizing the corresponding of course, it is not possible to compute the α-level sets for all values α within the interval $(0,1]$. Therefore, our first approximation step is to characterize a fuzzy reachable set $X(t)$ by a finite set of (crisp) reachable sets

$$\{[X]_\alpha(t) | \alpha \in A = \{\alpha_1, \alpha_2, ..., \alpha_n\} \subset (0, 1]\}$$

, and, hence, a "fuzzy funnel" by a finite set of "crisp funnel" by a finite set of "crisp funnels". The membership function of the fuzzy set $\widetilde{X}(t)$ is approximated by means of

$$\mu(x) = \max\{\alpha_k | x \in X_{\alpha_k}(t), 1 \leq k \leq m\}.$$

In [1] we have shown that this approximation makes sense: Under certain conditions, we can compute approximations of any degree of accuracy.

After having reduced the problem of computing fuzzy reachable sets to one of computing "crisp" reachable sets, we now turn to the question of how to characterize such sets. For computing approximations of reachable sets $X(t)$, it is necessary to define a "discrete version" of (3.5). A simple first order discretization of (3.5) is given by

$$\frac{y_{i+1} - y_i}{\Delta t} \in F(t_i, x_i), \tag{3.9}$$

where $0 = t_0 < t_1 < ... < t_N = T$ is a grid with stepsize $h = T/N = t_i - t_{i-1}$ $(i = 1, ..., N)$. As a solution to difference inclusion (3.9), we define any continuous and piecewise linear function

$$Y^N(t) = y_i + \frac{1}{\Delta t}(t - t_i)(y_{i+1} - y_i) \quad (t_i \leq t \leq t_{i+1}, i = 0, ..., N-1),$$

where $(y_0, \ldots, y_N)$satisfies (3.9). Furthermore , let S^N denotes the set of all such solutions [32].

The following Euler scheme is the set-valued generalization of (3.9):

$$Y(t + \Delta t) = \bigcup_{y \in Y(t)} y + \Delta t F(t, y), \qquad (3.10)$$

where Y_i^N is the reachable set associated with (3.10) at time t_i, i.e.,

$$Y_i^N = \{x \in \mathbb{R} | x = y^N(t_i) \ \ for some\ y^N \in S^N\}.$$

Now, the question is whether $Y_N^N \to X(T)$ for $N \to \infty$.

A closed-valued and continuous function $R : [0, T] \to 2^{\mathbb{R}^n}$ is called an R-solution of the initial value problem

$\dot{x} \in F(t, x(t))$, almost everywhere on $[0, T]$, $\qquad x(0) = x_0$

if $R(0) = 0$ and

$$\lim_{\Delta t \to 0} \frac{1}{\Delta} d_H(R(t + \Delta t), \bigcup_{x \in R(t)} x + \Delta t F(t, x)) = 0$$

uniformly in $t \in [0, T]$ [1]

3.4 Numerical Approximation method

This subsection presents a new numerical method for solving one dimensional fuzzy differential inclusions. One way to model uncertainty and vague in a dynamical system is to replace functions and initial values in the problem given a set $X(t)$ of possible state at time t, what does the set $X(t + \Delta t)$ look like? Since the solution (3.5), $x_\alpha(t)$

is absolutely continuous on $[0, T]$, then $x_\alpha(t)$ is bounded variation on $[0, T]$, i.e., for arbitrary partition $0 = t_0 < t_1 <, \ldots, < t_n = T$ with step size $\Delta = t_{i+1} - t_i$, $\exists M > 0$ such that

$$\sum_{i=1}^{i=n} |x_\alpha(t_i) - x_\alpha(t_{i-1})| \leq M.$$

Let

$$\overline{m} = \max_{i=1,2,\ldots,n} |x_\alpha(t_i) - x_\alpha(t_{i-1})|,$$

then $\overline{m} \leq \frac{M}{n}$, therefore $\overline{m} \to 0$ as $\Delta t \to 0$. On the other hand, let $X_\alpha(t_i) = [x_{i,1}, x_{i,2}]$ then difference between $x_{i,1}$ and $x_{i-1,1}$ or $x_{i,2}$ and $x_{i-1,2}$ is small for large enough n. Refer to (3.6) by Euler scheme

$$\dot{x}_\alpha(t_i) \simeq \frac{x_\alpha(t_i + \Delta t) - x_\alpha(t_i)}{\Delta t},$$

hence we have

$$\frac{x_\alpha(t_i + \Delta t) - x_\alpha(t_i)}{\Delta t} \in F_\alpha(t_i + \Delta t, x_\alpha(t_i + \Delta t)).$$

3.5 Tuning of reachable set(TRS) Algorithm

Here, we introduce an easy algorithm, for solving FDIs, that estimates α-reachable set $X_\alpha(t_i)$.

TRS Algorithm:

1. Fix α in $[0, 1]$ and choose small positive ϵ.

2. Put $X_\alpha(0) = [x_{01\alpha}, x_{02\alpha}]$.

3. Repeat steps 4-7 for $i = 1, \dots, n$.

4. $x_{i2} := x_{i-1,2}$.

5. If $\left[\frac{x_{i,2}-x_{i-1,2}}{\Delta t}, \frac{x_{i,2}-x_{i-1,1}}{\Delta t}\right] \cap \left[\underline{F_\alpha}(t_i, x_{i,2}), \overline{F_\alpha}(t_i, x_{i,2})\right] = \phi$ then $x_{i,2}$ is greater than upper bound of $X_\alpha(t_i)$ and while this inequality is true take $x_{i,2} = x_{i,2} - \epsilon$, since there does not exist a trajectory $x_\alpha(t) \in \chi_\alpha$ such that $x_\alpha(t_i) = x_{i,2}$, and $x_\alpha(t_{i-1}) \in X_\alpha(t_{i-1})$;

else

$x_{i,2}$ is less than upper bound of $X_\alpha(t_i)$ and while this inequality is true take $x_{i,2} = x_{i,2} + \epsilon$, since there exists a trajectory $x_\alpha(t) \in \chi_\alpha$ such that $x_\alpha(t_i) = x_{i,2}$, and $x_\alpha(t_{i-1}) \in X_\alpha(t_{i-1})$.

6. $x_{i,1} := x_{i-1,1}$.

7. If $\left[\frac{x_{i,1}-x_{i-1,2}}{\Delta t}, \frac{x_{i,1}-x_{i-1,1}}{\Delta t}\right] \cap \left[\underline{F_\alpha}(t_i, x_{i,1}), \overline{F_\alpha}(t_i, x_{i,1})\right] = \phi$ then $x_{i,1}$ is less than lower bound of $X_\alpha(t_i)$ and while this inequality is true take $x_{i,1} = x_{i,1} + \epsilon$, since there does not exist a trajectory $x_\alpha(t) \in \chi_\alpha$ such that $x_\alpha(t_i) = x_{i,1}$, and $x_\alpha(t_{i-1}) \in X_\alpha(t_{i-1})$;

else

$x_{i,1}$ is greater than lower bound of $X_\alpha(t_i)$ and while this inequality is true take $x_{i,2} = x_{i,2} - \epsilon$, since there exists a trajectory $x_\alpha(t) \in \chi_\alpha$ such that $x_\alpha(t_i) = x_{i,1}$, and $x_\alpha(t_{i-1}) \in X_\alpha(t_{i-1})$.

Example 3.5.1. *[33]: Consider the fuzzy differential inclusion on* $J = [0, T]$

$$\begin{cases} \dot{x}(t) \in -X(t) + W \cos t, \\ x(0) \in X_0, \end{cases}$$

where W *and* X_0 *are symmetric triangular fuzzy numbers* $(0, 1, 1)$*, where* $[W]_\alpha = [\alpha - 1, 1 + \alpha]$ *and* $[X_0]_\alpha = [\alpha - 1, 1 + \alpha]$*. The* α*-solution set is given for* $t \geq 0$ *by*

$$x(t) \in \frac{1}{2}(\sin t + \cos t)[W]_\alpha + (x(0) - \frac{1}{2}[W]_\alpha)e^{-t} = \chi_\alpha.$$

By useing TRS algorithm, let $Y(t)$ *be the approximation of* $X(t)$*. The least square error at* $t = 1, 2, 5, 11$ *and* $\alpha_i \in \{0, 0.1, ..., 1\}$*, i.e.,*

$$(\sum_{i=0}^{i=10}(\underline{X}_{\alpha_i}(t_j) - \underline{Y}_{\alpha_i}(t_j))^2 + \sum_{i=0}^{i=10}(\overline{X}_{\alpha_i}(t_j) - \overline{Y}_{\alpha_i}(t_j))^2)^{\frac{1}{2}}$$

for $\Delta t = 0.01$ *is equal to* $0.032, 0.4681, 0.2843, 0.3385$*, respectively.*

Figure 3.1: Estimation of upper bound and lower bound of $\chi_{0.5}$

Figure 3.2: Solid and dot lines represent, respectively, the exact and approximation of $X(0.5)$.

Chapter 4

Fuzzy Integral equations

4.1 Introduction

Fuzzy systems are now used to study a variety of problems ranging from fuzzy metric spaces [17], fuzzy topological spaces [22] to control chaotic systems [8, 48], fuzzy differential equations [40, 41, 42] and particle physics [28, 29, 30, 51].

The topics of fuzzy integral equations(FIE) which attracted growing interest for some time, in particular in relation to fuzzy control, have been developed in resent years.

Prior to discussing fuzzy integral equations and their associated numerical algorithms, it is necessary to present an appropriate brief introduction to preliminary topics such as fuzzy number and fuzzy calculus.

4.2 Preliminaries

Let $u(r) = (\underline{u}(r), \overline{u}(r)), \quad 0 \leq r \leq 1$ be a fuzzy number, we take

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2},$$

$$u^d(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}.$$

It is clear that $u^d(r) \geq 0$ and $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$, also a fuzzy number $u \in E^1$ is said symmetric if $u^c(r)$ is independent of r for all $0 \leq r \leq 1$.

Theorem 4.2.1. *Let $u(r) = (\underline{u}(r), \overline{u}(r))$, $v(r) = (\underline{v}(r), \overline{v}(r))$ and also k, s are arbitrary real numbers. If $w = ku + sv$ then*

$$w^c(r) = ku^c(r) + sv^c(r),$$

$$w^d(r) = |k|u^d(r) + |s|v^d(r).$$

Proof 4.2.1. *Prior to proof, it is obvious that if x is a real number we can represent it as*

$$x = x^+ - x^- \quad where \;\; x^+,\; x^- \geq 0,$$

and $|x| = x^+ + x^-$

Now, by referring to the theorem we have

$$w = (k^+ - k^-)u + (s^+ - s^-)v = (k^+u + s^+v) - (k^-u + s^-v)$$

Therefore

$$\overline{w} = (k^+\overline{u} + s^+\overline{v}) + (-k^-\underline{u} + (-s^-)\underline{v}),$$

$$\underline{w} = (k^+\underline{u} + s^+\underline{v}) + (-k^-\overline{u} + (-s^-)\overline{v}).$$

Then

$$w^c = \frac{(k^+ + (-k^-))\overline{u} + (k^+ + (-k^-))\underline{u}}{2} + \frac{(s^+ + (-s^-))\overline{v} + (s^+ + (-s^-))\underline{v}}{2}$$

$$= (k^+ - k^-)u^c + (s^+ - s^-)v^c = ku^c + sv^c.$$

Also,

$$w^d = \frac{(k^+ + k^-)\overline{u} - (k^+ + k^-)\underline{u}}{2} + \frac{(s^+ + s^-)\overline{v} - (s^+ + s^-)\underline{v}}{2}$$

$$= (k^+ + k^-)u^d + (s^+ + s^-)v^d = |k|u^d + |s|v^d$$

Definition 4.2.1. For arbitrary fuzzy numbers $u, v \in E^1$, we use the distance [38]

$$D(u,v) = \sup_{0\leq r\leq 1} \max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\},$$

and it is shown that (E^1, D) is a complete metric space [25].

Remark 4.2.1. We have

$$|\overline{u}(r) - \overline{v}(r)| \leq |u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|,$$

$$|\underline{u}(r) - \underline{v}(r)| \leq |u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|,$$

hence for all $r \in [0,1]$,

$$\max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\} \leq |u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|,$$

and then

$$D(u,v) \leq \sup_{0\leq r\leq 1} \{|u^c(r) - v^c(r)| + |u^d(r) - v^d(r)|\}.$$

Therefore if

$|u^c(r) - v^c(r)|$ and $|u^d(r) - v^d(r)|$ tend to zero then $D(u, v)$ tends to zero.

Definition 4.2.2. [23, 38] Let $f : [a, b] \to E^1$, for each partition $P = \{t_0, ..., t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i],\ 1 \leq i \leq n$ suppose

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}),$$

$$\Delta := \max\{|t_i - t_{i-1}|, i = 1, ..., n\}.$$

The definite integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t)dt = \lim_{\Delta \to 0} R_p,$$

provided that this limit exists in the metric D.

If the fuzzy function f is continuous in the metric D, its definite integral exists [38], and also,

$$\underline{(\int_a^b f(t;r)dt)} = \int_a^b \underline{f}(t;r)dt,$$

$$\overline{(\int_a^b f(t;r)dt)} = \int_a^b \overline{f}(t;r)dt.$$

4.3 Fuzzy integral equation

The Fredholm integral equation of the second kind is [10]

$$F(t) = f(t) + \lambda \int_a^b K(s, t)F(s)ds, \qquad (4.1)$$

where $\lambda > 0$, $K(s, t)$ is an arbitrary given kernel function over the square $a \leq t, s \leq b$ and $f(t)$ is a given function of $t \in [a, b]$. If $f(t)$ is a crisp function then the solution of above equation is crisp as well. However, if $f(t)$ is a fuzzy function this equation may only possess fuzzy solution. Sufficient conditions for the existence equation of the second kind , where $f(t)$ is a fuzzy function, are given in [49]. For solving Eq. (4.1) in parametric form we take

$$\underline{U}(t;r) = \lambda \int_a^b \underline{K(s,t)F(s;r)}ds, \tag{4.2}$$

$$\overline{U}(t;r) = \lambda \int_a^b \overline{K(s,t)F(s;r)}ds. \tag{4.3}$$

Suppose $K(s,t)$ be continuous in $a \leq t \leq b$ and for fix s, $K(s,t)$ changes its sign in finite points as t_i where $t_i \in [a,b]$. For example, let $K(s,t)$ be nonnegative over $[a,t_1]$ and negative over $[t_1,b]$, therefore from Eqs. (4.2) and (4.3), we have

$$\underline{U}(t;r) = \lambda \int_a^{t_1} K(s,t)\underline{F(s;r)}ds + \lambda \int_{t_1}^b K(s,t)\overline{F(s;r)}ds,$$

$$\overline{U}(t;r) = \lambda \int_a^{t_1} K(s,t)\overline{F(s;r)}ds + \lambda \int_{t_1}^b K(s,t)\underline{F(s;r)}ds.$$

Hence

$$U^c(t;r) = \lambda \int_a^b K(s,t)F^c(s;r)ds$$

$$U^d(t;r) = \lambda \int_a^b |K(s,t)|F^d(s;r)ds.$$

By referring to theorem 4.2.1 we have,

$$F^c(t;r) = f^c(t;r) + \lambda \int_a^b K(s,t)F^c(s;r)ds, \tag{4.4}$$

$$F^d(t;r) = f^d(t;r) + \lambda \int_a^b |K(s,t)|F^d(s;r)ds. \tag{4.5}$$

It is clear that we must solve two crisp Fredholm integral equations of the second kind provided that each of Eqs. (4.4) and (4.5) have solution.

4.4 Numerical method

Many numerical techniques have been used successfully for such equations and in this section we discuss in detail a straightforward yet generally applicable technique: *Nystrom* or *quadrature* method. In the operator form we can rewrite Eqs. (4.4) and (4.5) as

$$F^c(.;r) = f^c(.;r) + \lambda K F^c(.;r), \tag{4.6}$$

$$F^d(.;r) = f^d(.;r) + \lambda |K| F^d(.;r), \tag{4.7}$$

where

$$KF^c(t;r) = \int_a^b K(s,t) F^c(s;r) ds,$$
$$|K|F^d(t;r) = \int_a^b |K(s,t)| F^d(s;r)) ds.$$

It is convenient to begin by considering techniques based on using of the iterative sequences

$$F^c_{n+1}(t;r) = f^c(t;r) + \lambda K F^c_n(t;r),$$

$$F^d_{n+1}(t;r) = f^d(t;r) + \lambda |K| F^d_n(t;r),$$

with initial values $F^c_0(t;r) = f^c(t;r)$ and $F^d_0(t;r) = f^d(t;r)$.

In general we can not be able to carry out analytically the integrations, involved. In this case we naturally turn to numerical quadrature. We introduce a quadrature rule R for the interval $[a,b]$ with weights w_j and N nodes s_j, i.e.,

$$Rf = \sum_{j=1}^{N} w_j f(s_j) = If - Ef = \int_a^b f(s) ds - Ef,$$

where Ef is the error.

If we first ignore the error of this quadrature rule then the integral equations (4.6) and (4.7) are replaced by the approximate equations

$$F_R^c(t;r) = f^c(t;r) + \lambda \sum_{j=1}^{N} w_j K(s_j,t) F_R^c(s_j;r),$$

$$F_R^d(t;r) = f^d(t;r) + \lambda \sum_{j=1}^{N} w_j |K(s_j,t)| F_R^d(s_j;r),$$

and by Nystrom method we have

$$F_{R,n+1}^c(t;r) = f^c(t;r) + \lambda \sum_{j=1}^{N} w_j K(s_j,t) F_{R,n}^c(s_j;r),$$

$$F_{R,n+1}^d(t;r) = f^d(t;r) + \lambda \sum_{j=1}^{N} w_j |K(s_j,t)| F_{R,n}^d(s_j;r).$$

According to integral equation theory, sufficient condition for convergence of two last iterative sequences to unique solution is that in any matrix norm $\| \lambda \mathcal{K} \| < 1$, where $\langle \mathcal{K} \rangle_{ij} = w_j K(s_i, s_j)$ [21].

Example 4.4.1. *[6] Consider the following fuzzy Fredholm integral equation*

$$\underline{f}(t;r) = rt + \frac{3}{26} - \frac{3}{26}r - \frac{1}{13}t^2 - \frac{1}{13}t^2 r,$$

$$\overline{f}(t;r) = 2t - rt + \frac{3}{26}r + \frac{1}{13}t^2 r - \frac{3}{26} - \frac{3}{13}t^2,$$

and kernel

$$K(s,t) = \frac{(s^2 + t^2 - 2)}{13}, \quad 0 \le s,t \le 2, \quad \lambda = 1,$$

and $a = 0,\ b = 2$. *The exact solution in this case is given by*

$$\underline{F}(t; r) = rt,$$

$$\overline{F}(t; r) = (2 - r)t.$$

We can see that

$$f^c(t; r) = t - \frac{2}{13}t^2,$$

$$f^d(t; r) = (1 - r)(t - \frac{1}{13}t^2 - \frac{3}{26}).$$

According to Eqs. (4.4) and (4.5) we have the following two crisp Fredholm integral equations

$$F^c(t; r) = f^c(t; r) + \int_0^2 \frac{(s^2 + t^2 - 2)}{13} F^c(s; r)ds, \tag{4.8}$$

$$F^d(t; r) = f^d(t; r) + \int_0^2 |\frac{(s^2 + t^2 - 2)}{13}| F^d(s; r)ds. \tag{4.9}$$

Now by applying Nystrom method for Eqs. (4.8) and (4.9), we get two sequences as follows

$$F^c_{n+1}(t; r) = f^c(t; r) + \int_0^2 \frac{(s^2 + t^2 - 2)}{13} F^c_n(s; r)ds,$$

$$F^d_{n+1}(t; r) = f^d(t; r) + \int_0^2 \frac{(s^2 + t^2 - 2)}{13} F^d_n(s; r)ds,$$

with initial value

$$F^c_0(t; r) = f^c(t; r),$$

$$F^d_0(t; r) = f^d(t; r).$$

According to Nystrom method by using the N-points Newton-Cotes method we have

$$F^c_{R,n+1}(t;r) = f^c(t;r) + \sum_{j=1}^{N} w_j K(s_j,t) F^c_{R,n}(s_j;r),$$

$$F^d_{R,n+1}(t;r) = f^d(t;r) + \sum_{j=1}^{N} w_j |K(s_j,t)| F^d_{R,n}(s_j;r).$$

Hence, if the above series converge then for large enough n

$$\underline{F}(t;r) \simeq \underline{F}_{R,n}(t;r) = F^c_{R,n}(t;r) - F^d_{R,n}(t;r),$$

$$\overline{F}(t;r) \simeq \overline{F}_{R,n}(t;r) = F^c_{R,n}(t;r) + F^d_{R,n}(t;r).$$

The exact and obtained solution of fuzzy Fredholm integral equation in Example 4.4.1 at $t = 1$, are shown in Figure. 4.1 and Figure. 4.2

Figure 4.1: Compares the solutions with 2 iterations and 6-points Newton-Cotes method (.... obtained solution and —— exact solution)

Figure 4.2: Compares the solutions with 4 iterations and 6-points Newton-Cotes method (.... obtained solution and —— exact solution)

Example 4.4.2. *[6, 23] Consider the following fuzzy Fredholm integral equation with*

$$\underline{f}(t;r) = \sin(\frac{t}{2})(\frac{13}{15}(r^2 + r) + \frac{2}{15}(4 - r^3 - r)),$$
$$\overline{f}(t;r) = \sin(\frac{t}{2})(\frac{2}{15}(r^2 + r) + \frac{13}{15}(4 - r^3 - r)),$$

and kernel

$$K(s,t) = 0.1 \sin(s) \sin(\frac{t}{2}), \quad 0 \leq s, t \leq 2\pi, \quad \lambda = 1,$$

and $a = 0,\ b = 2\pi$*. The exact solution in this case is given by*

$$\underline{F}(t;r) = (r^2 + r) \sin(\frac{t}{2}),$$

$$\overline{F}(t;r) = (4 - r^3 - r) \sin(\frac{t}{2}).$$

We can see that

$$f^c(t;r) = 0.5 \sin(\frac{t}{2})(4 + r^2 - r^3),$$

$$f^d(t;r) = \frac{11}{30}\sin(\frac{t}{2})(4 - 2r - r^2 - r^3).$$

According to Eqs. (4.4) and (4.5) we have the following two crisp Fredholm integral equations

$$F^c(t;r) = f^c(t;r) + \int_0^{2\pi} 0.1\sin(s)\sin(\frac{t}{2})F^c(s;r)ds,$$
$$F^d(t;r) = f^d(t;r) + \int_0^{2\pi} |0.1\sin(s)\sin(\frac{t}{2})|F^d(s;r)ds.$$

Now by applying the Nystrom method we get two sequences as follows

$$F^c_{n+1}(t;r) = f^c(t;r) + \int_0^{2\pi} 0.1\sin(s)\sin(\frac{t}{2})F^c(s;r)ds,$$

$$F^d_{n+1}(t;r) = f^d(t;r) + \int_0^{2\pi} |0.1\sin(s)\sin(\frac{t}{2})|F^d(s;r)ds,$$

with initial value

$$F^c_0(t;r) = f^c(t;r),$$

$$F^d_0(t;r) = f^d(t;r).$$

The exact and obtained solution of fuzzy Fredholm integral equation in Example 4.4.2 at $t = \pi$*, are shown in Figure. 4.3 and Figure. 4.4*

The exact and obtained solution after 5 *iterations at* $t = \pi$ *for ten* $r - level$ *as* $r = 0, 0.1, 0.2, ..., 1$ *are given in table 1 and also* $D(F, F_{R,5}) \approx 0.00168$.

Table 1

The exact and obtained solution of Example 4.4.2

$r-level$	$\underline{F}(\pi;r)$	$\underline{F}_{R,5}(\pi;r)$	$\overline{F}(\pi;r)$	$\overline{F}_{R,5}(\pi;r)$
0	0.00000	0.00016	3.99999	4.00167
0.1	0.10999	0.10841	3.89899	3.90058
0.2	0.23999	0.23851	3.79199	3.79348
0.3	0.38999	0.38862	3.67299	3.67437
0.4	0.55999	0.55875	3.53599	3.53724
0.5	0.74999	0.74889	3.37499	3.37610
0.6	0.95999	0.95906	3.18399	3.18493
0.7	1.18999	1.18925	2.95699	2.95774
0.8	1.43999	1.43947	2.68799	2.68852
0.9	1.70999	1.70972	2.37099	2.37127
1.0	2.00000	1.99999	2.00000	1.99999

Figure 4.3: Compares the solutions with 8 iterations and 4-point Newton-Cotes method (.... obtained solution and —— exact solution)

Figure 4.4: Compares the solutions with 8 iterations and 5-point Newton-Cotes method (.... obtained solution and —— exact solution)

Conclusions

In this work, we have outlined a new numerical method for solving one-dimensional fuzzy differential equation based on inclusion, TRS algorithm and fuzzy Fredholm integral equation based on parametric form of fuzzy numbers. In TRS algorithm we tune fuzzy α-reachable set, since $x_\alpha(t)$ is of bounded variation .

Numerical example shows the efficiency of implemented algorithm. According to our experiment, TRS algorithm can be modified for solving FDIs with higher dimension. Also the method described for fuzzy Fredholm integral equation can be extended for arbitrary integral equation. The implementation of such methods will be a relevant aspect of future researches.

xBib.bib Thesis.bbl

Bibliography

[1] A. I. Panasyuk and V. I. Panasyuk. An equation generated by a differential inclusion. Mathematicheskii Zametki, 27(3):429-437, 1980.

[2] B. Heshmaty and A. Kandel, Fuzzy linear regression and its applications to forecasting in uncertain environments, Fuzzy Sets and Systems, 15 (1985)159-191.

[3] C.V. Negoita and D.A. Ralescu, Applications of fuzzy sets to systems analysis, Wiley, Now York, 1975.

[4] D. Dubois and H. Prade, Fuzzy number: an overview, in: The Analysis of Fuzzy Information, Vol. 1, Mathematics, J. C. Bezdek (Ed.), CRC Press, Boca Raton, FL, 1987, 3-39.

[5] D. Dubois and H. Prade, Towards fuzzy differential calculus, Fuzzy Sets and System, 8 (1982) 1-7, 105-116, 225-233.

[6] E. Babolian, H. Sadeghi Goghary and S. Abbasbandy, Numerical solution of linear Fredholm fuzzy integral equations of the second kind by Adomian method, Appl. Math. Comput. 161 (2005) 733-744.

[7] E. Hüllermeier, An approach to modeling and simulation of uncertain dynamical systems, Int. J. Uncertainty, Fuzziness, Knowledge-Based Syst., 5 (1997) 117-137.

[8] G. Feng and G. Chen, Adaptive control of discrete-time chaotic systems: a fuzzy control approach, Chaos Solitons Fractals 23 (2005) 459-467.

[9] G. J. Klir, U.S. Clair and B. Yuan, Fuzzy Set Theory: Foundations and Applications, Prentice-Hall Inc., 1997.

[10] H. Hochstadt, Integral equations, Wiley, New York, 1973.

[11] H. J. Zimmerman, Fuzzy programming and linear programming with several objective functions, Fuzzy Sets and Systems 1 (1978) 45-55.

[12] H. T. Negoita, A note on the extension principle for fuzzy sets, Journa of Mathematical Analysis and Applications, 64(1978) 369-380.

[13] J. J. Buckley, T. Feuring, Fuzzy differential equations, Fuzzy sets and systems 110 (2000) 43-54.

[14] J. J. Buckley, Fuzzy eigenvalues and input-output analysis, Fuzzy Sets and systems, 34 (1990) 187-195.

[15] J. J. Buckley, Solving fuzzy equations, Fuzzy Sets and Systems, 50 (1992) 1-14.

[16] J. P. Aubin, Fuzzy differential inclusions, Problems of control and information theory, 19 (1990) 55-67.

[17] J. H. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons Fractals 22 (2004) 1039-1046.

[18] J. Vrba, A note on inverses in arithmetic with fuzzy numbers, Fuzzy Sets and Systems 50 (1992) 267-278.

[19] K. Deimling, Multivalued Differential Equations, Walter de Gruyter, New York, 1992.

[20] K. K. Majumdar, One dimensional fuzzy differential inclusion, J. Intell. Fuzzy Syst., 13 (2003) 1-5.

[21] L. M. Delves and J. L. Mohamed, Computational methods for integral equations, Cambridge University Press, 1985.

[22] M. Caldas and S. Jafari, θ-Compact fuzzy topological spaces, Chaos Solitons Fractals, 25 (2005) 229-232.

[23] M. Friedman, M. Ma and A. Kandel, Numerical solutions of fuzzy differential and integral equations, Fuzzy Sets and Systems 106 (1999) 35-48.

[24] M. L. Puri and D. Ralescu, Differential for fuzzy function, J. Math. Anal. Appl. 91 (1983) 552-558.

[25] M. L. Puri and D. Ralescu, Fuzzy random variables, J. Math Anal. Appl. 114 (1986) 409-422.

[26] M. Matloka, On fuzzy integrals, Proc. 2nd Polish Symp. on Interval and Fuzzy Mathematics, Politechnika Poznansk, 1987, 167-170.

[27] M. Ma, M. Friedman and A. Kandel, A new fuzzy arithmetic, Fuzzy Sets and Systems 108 (1999) 83-90.

[28] MS. Elnaschie, A review of E-infinity theory and the mass spectrum of high energy particle physics, Chaos, Solitons & Fractals, 19 (2004) 209-236.

[29] MS. Elnaschie, On a fuzzy Kãhler manifold which is consistent with the two slit experiment, Int. J. Nonlinear Science and Numerical Simulation 6 (2005) 95-98.

[30] MS. Elnaschie, The concepts of E infinity: An elementary introduction to the Cantorian-fractal theory of quantum physics, Chaos, Solitons & Fractals 22 (2004) 495-511.

[31] O. Kaleva, Fuzzy differential equations, Fuzzy sets and systems, 24 (1987) 301-317.

[32] P. Diamond and P. Kloden, Metric Spaces of Fuzzy sets, Singapore: World Scientific, 1994.

[33] P. Diamond, Stability and periodicity in fuzzy differential equations, IEEE Trans. Fuzzy Syst., 8 (2000) 583-590.

[34] P. Diamond, Time-dependent differential inclusion, cocycle atractors and fuzzy differential equations, IEEE Trans. Fuzzy Syst., 7 (1999) 734-740.

[35] R. Goetschel and W. Vaxman, A pseudometric for fuzzy sets and certain related result, J. Math. Anal. Appl. 81 (1981) 507-523.

[36] R. Goetschel and W. Vaxman, Topological properties of fuzzy numbers, Fuzzy Sets and Systems, 10 (1983) 87-99.

[37] R. Goetschel and W. Vaxman, Eigen fuzzy number sets, Fuzzy Sets and Systems, 16 (1985) 75-85.

[38] R. Goetschel and W. Voxman, Elementary calculus, Fuzzy Sets and Systems, 18 (1986) 31-43.

[39] R. R. Yager, Fuzzy prediction based upon regression models, Inform. Sci. 26 (1982) 45-63.

[40] S. Abbasbandy and T. Allah Viranloo, Numerical solution of fuzzy differential equation by Runge-Kutta method, Nonlinear Stud. 11 (2004) 117-129.

[41] S. Abbasbandy, T. Allah Viranloo, Óscar López-Pouso and J. J. Nieto, Numerical Methods for Fuzzy Differential Inclusions, Comput. Math. Appl. 48 (2004) 1633-1641.

[42] S. Abbasbandy, J. J. Nieto and M. Alavi, Tuning of reachable set in one dimensional fuzzy differential inclusions, Chaos, Solitons & Fractals, 26 (2005) 1337-1341.

[43] S. L. Chang and L. A. Zadeh, On fuzzy mapping and control, IEEE Trans. Syst. Man Cybernet. 2 (1972) 30-34.

[44] S. Seikkala, On the Fuzzy initial value problem, Fuzzy sets and systems, 24 (1987) 319-330.

[45] V. A. Baidosov, Differential inclusions with fuzzy right hand side, Soviet mathematics, 40 (1990) 567-569.

[46] V. A. Baidosov, Fuzzy differential inclusions, Prikl. Math. Mekh. USSR, 54 (1990) 8-13.

[47] V. N. Bobylev, A possibilistic argument for irreversibility, Fuzzy sets and systems, 34 (1990) 73-80.

[48] W. Jiang and Q. Guo-Dong and D. Bin, H_∞ Variable universe adaptive fuzzy control for chaotic system, Chaos Solitons Fractals, 24 (2005) 1075-1086.

[49] Wu Congxin and Ma Ming, On the integrals, series and integral equations of fuzzy set-valued functions, J. Harbin Inst. Technol. 21 (1990) 11-19.

[50] Wu Congxin and Ma Ming, On embedding problem of fuzzy number spaces, Fuzzy Sets and Systems 44 (1991) 33-38, 45 (1992) 189-202, 46 (1992) 281-286.

[51] Y. Tanaka, Y. Mizuno and T. Kado, Chaotic dynamics in the Friedman equation, Chaos, Solitons & Fractals 24 (2005) 407-422.

[52] Z. Ding and A. Kandel, Existence and stability of fuzzy differential equations, J. Fuzzy Math. 5 (1997) 681-697.

[53] Zhao Renhong and R. Govind, Solutions of algebraic equations involving generalized fuzzy number, Inform Sci. 56 (1991) 199-243.