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NUMERICAL AND ANALYTICAL SOLUTIONS OF FUZZY DIFFERENTIAL EQUATIONS OF FRACTIONAL AND NATURAL ORDER

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Numerical and Analytical Solutions of Fuzzy Differential Equations of Fractional and Natural Order**" by **Soheil Salahshour** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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Abstract

The aim of this work is to use the generalized differentiability for solving Fuzzy Differential Equations about fractional order (for short FFDEs) and natural order (for short FDEs). To do so, The fuzzy Laplace transforms method, a new operator method and 1-cut expansion method have been applied to solve FDEs. To solve FFDEs, we have used the fuzzy Laplace transforms method under Riemann-Liouville differentiability. In this regard, some basic properties of the fuzzy Laplace transforms are stated. Furthermore, some well-known results are derived whenever the Riemann-Liouville differentiability is appeared. Moreover, several examples have been solved in details to illustrate the ability of proposed approaches.

Publications

1. **Solving fuzzy fractional differential equations by fuzzy Laplace transforms**, *Communications in Nonlinear Science and Numerical Simulation* 17 (2012) 1372-1381, (ISI, IF: 2.7).

2. **Explicit solutions of fractional differential equations with uncertainty**, *Soft Computing - A Fusion of Foundations, Methodologies and Applications* 16 (2012) 297-302, (ISI, IF: 1.52).

3. **A new method for solving fuzzy linear differential equations**, *Computing* 92 (2011) 181-197, (ISI, IF: 0.96).

4. **Euler method for solving hybrid fuzzy differential equation**, *Soft Computing - A Fusion of Foundations, Methodologies and Applications* 15 (2011) 1247-1253, (ISI, IF: 1.52).

5. **Fuzzy symmetric solutions of fuzzy linear systems**, *Journal of Computational and Applied Mathematics* 16 (2011) 4545-4553, (ISI, IF: 1.029).

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15. **A novel approach for ranking triangular intuitionistic fuzzy numbers** , *Computer Procedia Science*, (ISI Proceeding), In Press.

16. **A numerical method for solving Volterra and Fredholm integral equations using homotopy analysis method**, *Computer Procedia Science*, (ISI Proceeding), In Press.

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18. **Application of fuzzy concept to determine the house price and development costs**, *Computer Procedia Science*, (ISI Proceeding), In Press.

Introduction

The mathematical modeling of various physical phenomena involves two inconveniences. The first is excessive complexity of the situation being modeled and consequent uncertainty of the systems. On the other hand, in many cases, information about the behavior of a dynamical system is uncertain. In order to obtain a more realistic (not more exact!) model, we have to take into account these uncertainties. Also, in several cases the uncertainties are not of statistical type. For example, having some linguistic information and when we cannot repeat a measurement are such cases. This leads to two further consequences:

1. *We are not able to formulate the model,* and
2. *The model constructed is too complicated with uncertainties to be used in practice.*

Despite such imperfect knowledge, attempts have been made in some selected mathematical models to devise controllers that will steer the system in a certain required fashion.

The second inconvenience consists of indeterminacy caused by our subjective inability to differentiate events exactly, and its main property is vagueness. Thus there is a need for a mathematical apparatus to describe the vague notions to overcome the obstacles in modeling and such an apparatus is provided by fuzzy set theory.

Recently, the theory of fuzzy differential equations has been initiated and the basic results have been systematically investigated. Differential equations in fuzzy setting are a natural way to model uncertainty of dynamical systems. First order linear fuzzy differential equations are one of the simplest fuzzy differential equations which may appear in many applications. However, the form of such an equation is very simple, it raises many problems since under different fuzzy differential equation concepts, the behaviour of the solutions is different (depending on the interpretation used) [22].

The Hukuhara derivative of a fuzzy-valued function was introduced in [30] and its starting point in the Hukuhara derivative of multivalued functions. The first approach to modeling the uncertainty of dynamical systems uses the Hukuhara derivative. Under this setting, mainly the existence and uniqueness of the solution of a fuzzy differential equation are studied [37, 75, 41, 47, 48, 78].

This approach has the disadvantage that it leads to solutions with increasing support, fact which is solved by interpreting a fuzzy differential equation as a system of differential inclusions. But this last mentioned approach has at its turn some shortcomings.

This shortcoming is solved by interpreting a fuzzy differential equation as a system of differential inclusions [45]. The main shortcoming is that one cannot talk about the derivative of a fuzzy-valued function, since a fuzzy differential equation is directly interpreted with the help of differential inclusions without having a derivative. Also solutions are not necessarily fuzzy-valued functions.

Strongly generalized differentiability (for short generalized differentiability) was introduced in [21] and studied in [22]. This concept allows us to solve FDEs without the above mentioned weakness. Actually, the strongly generalized derivative is defined for a larger class of fuzzy-valued functions than the H-derivative. So, an FDE can solve such that its solution has a decreasing length of support.

Fractional Calculus and fractional differential equations have undergone expanded study in recent years as a considerable interest both in mathematics and in applications. They were applied in modeling of many physical and chemical processes and in engineering [17, 19, 20, 26, 34].

Fractional differential and integral equations play increasingly important roles in the modeling of engineering and science problems, as shown in [2, 34, 35, 55, 76]. It has been established that, in many situations, these models provide more suitable results than analogous models with integer derivatives. The calculus of fractional order derivatives and the theory of fractional differential equations has been studied comprehensively in [3, 14, 17, 61, 62, 63].

One of the recently influential works on the subject of fractional calculus is the monograph of Podlubny [76] and the other is the monograph of Kilbas et. al [55].

Consequently, several research papers were done to investigate the theory and solutions of fractional differential equation. See [2, 61, 62, 63] and references therein.

Due to similar reasons, we should investigate the solutions of differential equations about fractional order under uncertainty, represented by fuzzy-valued functions which so-called in short FFDEs.

To do so, we define the basic concepts like fractional integral and derivative of fuzzy-valued functions. Also, we need to obtain a Derivative theorem about fractional order for applying the fuzzy Laplace transforms method which is our adopted method to solve FFDEs.

Clearly, one of the useful system in such fractional models is fractional differential equation with initial value under Riemann-Liouville differentiability defined by

$$\begin{cases} \left({}^{RL}D_{a^+}^{\beta} y\right)(x) = f[x, y(x)] \\ \left({}^{RL}D_{a^+}^{\beta-1} y\right)(x_0) = \ {}^{RL}y_0^{(\beta-1)} \in \mathbb{R} \end{cases}$$

where f is stated appropriately and $x_0 \in (a, b)$.

In this model, the initial value $y(a)$ is obtained by the decision maker in accordance with the conditions of original problem. In this direction, some essential questions are existed as following:

Problem A. *Does the initial value is determined correctly?*

Problem B. *What is the effect of changing the initial value $y(a)$ in the structure* *of solution?.*

In order to answer these basic questions in the real sense, we used the fuzzy concepts in the fractional differential equations so-called fuzzy fractional differential equations (FFDEs). Moreover, there exists another basic question as follows.

Problem C. *How can we solve FFDEs, analytically?.*

We now discuss on the mentioned questions under uncertainty, represented by fuzzy concepts as following:

Discussing on the Problem A, B and C.

It is common sense that the initial value $y(a)$ is determined by decision maker under some conditions. So, by changing the conditions the original problem may be changed to the another problem with the same formula for f, and some variations for the initial value $y(a)$. In order to consider the original problem in a new sense, we intend to use fuzzy initial value instead of crisp initial value. In this direction, we should reconstruct the original problem in the fuzzy field. Therefore, each elements of fractional differential equation should be converted to the fuzzy frame, especially, fractional differentiability.

On the other hand, investigating the effects of changing the initial value on the solution can be interpreted as investigating the stability of solutions. After constructing the fuzzy fractional differential equations and their basic elements, it is essential to produce an analytical approach for solving such uncertain problems. One of the interesting analytical method in the deterministic case for the theory of fractional differential equations is the method of Laplace transforms.

So, for solving FFDEs, we adopted such operator method. To this end, we need to an essential theorem in order to connect between the Laplace transform of fuzzy fractional derivative of f and the Laplace transform of f which so-called Derivative Theorem. Based on the mentioned questions and related discussions we decided that for solving a fractional differential equation in the real case (not necessary exact case) we should use the fuzzy version of this problem which is named as fuzzy fractional differential equations.

Moreover, it is essential that the obtained solution and the deterministic solution (solution of 1-cut problem) should have the same behaviour. It means that, if the crisp solution goes to zero when the independent variable x goes to infinity, the corresponding problem in the fuzzy frame should have the same behaviour, i.e., the solution of FFDE goes to zero when x goes to infinity (asymptotically certain) and similar conditions should be valid for the other cases. To this end, we must use at least two type of differentiability. It is easy to verify that the theory of fuzzy fractional differential equations is richer than the theory in the deterministic case.

In the fractional literature, Mittag-Leffler function plays a major role in the theory of fractional calculus and fractional differential equations (see [55, 76]):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta > 0. \tag{0.0.1}$$

However, one can find some recently monographs and research papers in the field of fractional differential equations and their solutions in [25, 46, 57, 61, 62, **?**, 64, 67, 68, 80].

The thesis is organized as follows:

Chapter 2. Recently, Allahviranloo and Barkhordari in [8] have proposed the fuzzy Laplace transforms for solving first order fuzzy differential equations under generalized differentiability. Unfortunately, they only have defined the fuzzy Laplace transforms and did not state that under what conditions the fuzzy-valued functions have the fuzzy Laplace transform? However, they did not consider the important properties and related theorems for solving FDEs. Here we are going to consider the above mentioned conditions about the existence of fuzzy Laplace theorem and its inverse.
In the next section, we suggest a new method to solve FDEs which is constructed based on the equivalent integral form of the original problems. By using the lower and upper functions of the obtained integral equations, we can determine the lower and upper functions of solutions.
So, we construct a closed form of solutions by solving corresponding fuzzy Volterra integral equations.

In the sequel, a new approach for solving FDEs is considered under generalized differentiability. The idea of the presented approach is constructed based on the extending the 1-cut solution of original FDEs. Obviously, 1-cut of FDE is interval differential equation or ordinary differential equation. If the 1-cut of FDE be an interval differential equation, we solve it using Stefanini et al.'s method [82], otherwise, solving ordinary differential equation will be done as usual. Consequently, we try to fuzzify obtained 1-cut solution in order to determine solutions of original FDE. To this end, some unknown spreads are allocated to the 1-cut solution. Then, by replacing such fuzzified value into original FDE and also based on the type of differentiability, we can get the spreads of solution of FDE.

Chapter 3. Recently, Agarwal et al. [14] have proposed a concept of solution for fractional differential equations with uncertainty. They have considered the Riemann-Liouville differentiability concept based on the Hukuhara differentiability to solve FFDEs.

To do so, we propose Riemann-Liouville differentiability for a given fuzzy-valued function by using Hukuhara difference so-called *Riemann-Liouville differentiability*. To this end, a direct procedure is adopted to derive such concept which is constructed based on the combination of strongly generalized differentiability [23] and Riemann-Liouville derivative [55, 76].
Consequently, we intend an analytical (operator) method to solve FFDEs. Since, considering the solutions of FFDEs is a new subject, we should first implement the analytical method to solve it, then numerical methods can be applied.

However, such concepts are not explicitly implemented in the fractional case under uncertainty. So, we obtain the explicit solutions of FFDEs usnig the Mittag-Leffler presentation. In order to show the ability of the proposed methods, some illustrative examples have been solved in details.

Chapter 1

Preliminaries

Fuzziness is not a priori an obvious concept and demands some explanation. *Fuzziness* is *vagueness* i.e. to designate the kind of uncertainty which is both due to fuzziness and ambiguity. Generalizing refers to the application of a symbol to a multiplicity of objects in the field of reference, ambiguity to the association of a finite number of alternative meanings having the same phonetic form. But, the fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol applies.

1.1 Introduction

In this chapter, the basic definitions of fuzzy sets and algebraic operations are defined and extension principle are provided which is one of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets.

1.2 Fuzzy sets and some basic definitions

Definition 1.2.1. If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, A(x))|\ x \in X\}$$

$A(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in A that maps X to the membership space M (when M contains only the two points 0 and 1, A is nonfuzzy and $A(x)$ is identical to the characteristic function of nonfuzzy set).

The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite. Elements with a zero degree of membership are normally not listed. A fuzzy set is obviously a generalization of a classical set and the membership function a generalization of the characteristic function. Since we are generally referring to a universal (crisp) set X, some elements of fuzzy set may have the degree of membership zero. Often it is appropriate to consider those elements of the universe that have a nonzero degree of membership in a fuzzy set.

Definition 1.2.2. The *support* of a fuzzy set A is the ordinary subset of X:

$$supp(A) = \{x \in X|\ A(x) > 0\}.$$

Definition 1.2.3. The *height* of A is $hgt(A) = sup_{x\in X} A(x)$, i.e. the least upper bound of $A(x)$. A is said to be *normalized* iff $\exists x \in X, A(x) = 1$; this definition implies hgt(A)=1.

A more general and even more useful notion is that of an $r-$level set.

Definition 1.2.4. The set of elements that belong to the fuzzy set A at least to the degree r is called the $r-$level set or $r-$cut:

$$[A]_r = \{x \in X| \ A(x) \geq r\}$$

if nonequality is hold strictly then $[A]_r$ is called *strong* $r-$*level set.*

Definition 1.2.5. A triangular fuzzy number A, is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $A(x)$, the membership function of the fuzzy number A, is a triangle with base on the interval $[a_1, a_3]$ and vertex at $x = a_2$. We specify A as $(a_1/a_2/a_3)$. We will write: (1) $A > 0$ if $a_1 > 0$; (2) $A \geq 0$ if $a_1 \geq 0$; (3) $A < 0$ if $a_3 < 0$; and (4) $A \leq 0$ if $a_3 \leq 0$.

Convexity also plays an important role in fuzzy set theory. By contrast to classical set theory, however, convexity conditions are defined with reference to the membership function rather than the support of a fuzzy set.

Definition 1.2.6. A fuzzy set A is *convex* if

$$A(\lambda x_1 + (1-\lambda)x_2) \geq min\{A(x_1), A(x_2)\}, \forall x_1, x_2 \in X, \forall \lambda \in [0, 1].$$

Alternatively, a fuzzy set is convex if and only if all $r-$level sets are convex, [90].

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. In its elementary form, it was already implied in Zadeh's first contribution (1965).

Definition 1.2.7. Let X be a Cartesian product of universes $X = X_1 \times X_2 \times \ldots \times X_k$ and f be a mapping from X to a universe Y, $y = f(x_1, \ldots, x_k)$. Then the extension principle allows us to define a fuzzy set B in Y by

$$B = \{(y, B(y))|\ y = f(x_1, \ldots, x_k), (x_1, \ldots, x_k) \in X\}$$

where

$$B(y) = \begin{cases} sup_{(x_1,\ldots,x_k)\in f^{-1}(y)} min\{A_1(x_1), \ldots, A_k(x_k)\} & if \quad f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

.

A fuzzy function is a generalization of the concept of a classical function. A classical function f is a mapping from the domain D of definition of the function into a space S; $f(D) \subseteq S$ is called the range of f. Different features of the classical concept of a function can be considered to be fuzzy rather than crisp. Therefore different *degrees* of fuzzification of the classical notion of a function are conceivable.

1. There can be a crisp mapping from a fuzzy set that carries along the fuzziness of the domain and therefore generates a fuzzy set. The image of a crisp argument would again be crisp.

2. The mapping itself can be fuzzy, thus blurring the image of a crisp argument. This we shall call a *fuzzy function.* These are called *fuzzifying function* by Dubois and Prade.

3. Ordinary functions can have fuzzy properties or be constrained by fuzzy constrains, [90].

Definition 1.2.8. A classical function $f : X \to Y$ maps a fuzzy domain A in X into a fuzzy range B in Y if and only if

$$\forall x \in X, B(f(x)) \geq A(x).$$

Given a classical function $f : X \to Y$ and a fuzzy domain A in X, the extension principle yields the fuzzy range B with the membership function

$$B(y) = sup_{x \in f^{-1}(y)} A(x),$$

hence f is a function according to the above definition, [90].

Denote by κ^n the set of all nonempty compact subsets of R^n and by κ_c^n the subset of κ^n consisting of nonempty convex compact sets.

An open ϵ-neighborhood of $\mathrm{A} \in \kappa^n$ is the set

$$\mathrm{N}(\mathrm{A}, \epsilon) = \{x \in \mathrm{R}^n : \inf_{a \in A} \|x - a\| < \epsilon\} = \mathrm{A} + \epsilon \mathrm{B}^n,$$

where B^n is the open unit ball in R^n, [32].

Definition 1.2.9. A mapping $F : \mathrm{R}^n \to \kappa^n$ is *upper semicontinuous* (usc) at x_0 if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0)$ such that

$$F(x) \subset N(F(x_0), \epsilon) = F(x_0 + \epsilon B^n)$$

for all $x \in N(x_0, \delta)$.

1.3 Hausdorff metric

In this section we define a metric space by Hausdorff separation. Recall that

$$\rho(x, A) = \inf_{a \in A} \|x - a\|$$

is the distance of a point $x \in \mathrm{R}^n$ from $A \in \kappa^n$ and that the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \kappa^n$ is defined as

$$\rho(A, B) = \sup_{a \in A} \rho(a, B).$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$. The *Hausdorff metric* d_H on κ^n is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

and (κ^n, d_H) is a complete metric space. Let D^n denote the set of usc normal fuzzy sets on R^n with compact support. That is, $u \in D^n$, then $u : \mathrm{R}^n \to [0, 1]$ is usc, $supp(u)$ is compact and there exists at least one $\xi \in supp(u)$ for which $u(\xi) = 1$. The β-level set of u, $0 < \beta \leq 1$ is

$$[u]_\beta = \{x \in \mathrm{R}^n : u(x) \geq \beta\}.$$

Clearly, for $\alpha \leq \beta, [u]_\alpha \supseteq [u]_\beta$. The level sets are nonempty from normality and compact by usc and compact support. The metric d_H is defined on D^n as

$$d_\infty(u, v) = \sup\{d_H([u]_r, [v]_r) : 0 \leq r \leq 1\}, \quad u, v \in D^n$$

and (D^n, d_∞) is a complete metric space. Denote by E^n the subset of fuzzy convex elements of D^n. The metric space (E^n, d_∞) is also complete, [32].

The following properties are well-known [86]):

$(i) d_\infty(u + w, v + w) = d_\infty(u, v), \quad \forall u, v, w \in E,$

$(ii) d_\infty(k.u, k.v) = |k| d_\infty(u, v), \quad \forall k \in R, u, v \in E,$

$(iii) d_\infty(u + v, w + e) \leq d_\infty(u, w) + d_\infty(v, e), \quad \forall u, v, w, e \in E.$

Definition 1.3.1. Let I be a real interval. A mapping $x : I \to E$ is called a fuzzy process and its $r-$level set is denoted by

$$[x(t)]_r = [x_1(t;r), x_2(t;r)], \quad t \in I.$$

Now we have the following corollary for algebraic operations with $r-$level sets.

Corollary 1.3.1. *Let $v, w \in E$ and s be a scalar, then for $r \in [0,1]$;*

$$[v + w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],$$

$$[v - w]_r = [v_1(r) - w_2(r), v_2(r) - w_1(r)],$$

$$[v.w]_r = [\min\{v_1(r).w_1(r), v_1(r).w_2(r), v_2(r).w_1(r), v_2(r).w_2(r)\},$$

$$\max\{v_1(r).w_1(r), v_1(r).w_2(r), v_2(r).w_1(r), v_2(r).w_2(r)\}],$$

$$[sv]_r = s[v]_r, \text{ [78].}$$

If $f : R^n \times R^n \to R^n$ is a function, then, according to Zadeh extension principle, we can extend f to $E^n \times E^n \to E^n$ by the equation

$$f(u,v)(z) = sup_{z=f(x,y)} min\{u(x), v(y)\}.$$

It is well known that

$$[f(u,v)]_r = f([u]_r, [v]_r)$$

for all $u, v \in E^n, r \in [0,1]$.

1.4 Fuzzy derivatives

Here, we state some previously reported fuzzy derivative of a given fuzzy function x.

1.4.1 Hukuhara derivative

The Hukuhara difference between two fuzzy numbers A and B defines as follows:

Definition 1.4.1. [75] If there exists a fuzzy number C so that $C + A = B$, then C is called the Hukuhara difference between B and A and we write this as $B \ominus A = C$.

Definition 1.4.2. A mapping $x : I \to E^n$ is Hukuhara differentiable at $t \in I$ if there exists $x'(t) \in E^n$ such that the limits

$$lim_{h\to 0^+}\frac{x(t+h)\ominus x(t)}{h}, \quad lim_{h\to 0^-}\frac{x(t)\ominus x(t+h)}{h}$$

exist and are equal to $x'(t)$. Here the limit is taken in the metric space (E^n, D). At the end points of I we consider only the one-side derivatives, [48].

Let $x : I \to E^n$ be a fuzzy process and differentiable on I, for each $r \in [0,1]$;.

$$[x(t)]_r = [x_1(t;r), x_2(t;r)],\ r \in [0,1].$$

The Hukuhara derivative $x'(t)$ of a fuzzy process x is defined by

$$[x'(t)]_r = [x_1'(t;r), x_2'(t;r)], \quad t \in I,$$

provided that this equation defines a fuzzy number for the partial of $x_i(t;r)$ with respect to $t, i = 1, 2$. [78].

Let $x(t)$ be a fuzzy number for each $t \in I$. Some kind of derivatives are defined namely Goetchel-Voxman derivative, the Seikkala derivative, the Dubois-Prade derivative, the Puri-Ralescu derivative, and the Kandel-Friedman-Ming derivative of $x(t)$. The other derivatives of fuzzy function, approaches are more abstract and therefore not directly applicable to solving the fuzzy initial value problem.

1.4.2 Goetchel-Voxman derivative

Definition 1.4.3. The Goetschel-Voxman derivative of $x(t)$, written $GVDx(t)$, was defined in [41]. The derivative of $x(t)$ at t_0 is defined as

$$GVDx(t_0) = lim_{h \to 0} \frac{x(t_0 + h) \ominus x(t_0)}{h},$$

provided that the limit exists with respect to the metric D. However the subtraction is not standard fuzzy subtraction because

$$[x(t_0 + h) \ominus x(t_0)]_r = [x_1(t_0 + h; r) - x_1(t_0; r), x_2(t_0 + h; r) - x_2(t_0; r)],$$

for all t, r. Standard fuzzy arithmetic would produce

$$[x_1(t_0 + h; r) - x_2(t_0; r), x_2(t_0 + h; r) - x_1(t_0; r)].$$

If $GVDx(t)$ exists then $[GVDx(t)]_r = [x_1'(t; r), x_2'(t; r)]$, for all $t \in I, r \in [0, 1]$. However $GVDx(t)$ may not be a fuzzy number for some t in I.

1.4.3 Seikkala derivative

Definition 1.4.4. The Seikkala derivative of $x(t)$, written $SDx(t)$, was defined in [78]. This definition is as follows: if $[x_1'(t; r), x_2'(t; r)]$ are the $r-$cuts of a fuzzy number for each $t \in I$, then $SDx(t)$ exists and $[SDx(t)]_r = [x_1'(t; r), x_2'(t; r)]$. Notice that this is the definition of derivative of a fuzzy function that we use in this work. That is, if $\frac{dy(t)}{dt}$ exists, then $SDx(t) = \frac{dy(t)}{dt}$. Also, $SDx(t)$ is a fuzzy number for all $t \in I$.

1.4.4 Dubois-Prade derivative

Definition 1.4.5. The Dubois-Prade derivative of $x(t)$, written $DPDx(t)$, was defined in [37]. $DPDx(t)$ always exists and its membership function is given by

$$DPDx(t)(x) = sup\{r|\ x = x_1'(t;r), x = x_2'(t;r)\}.$$

However, $DPDx(t)$ may not be a fuzzy number. We may have to add something to the definition of $DPDx(t)$ to obtain a fuzzy number.

1.4.5 Generalized Hukuhara differentiability

Definition 1.4.6. [9].

Let $f : (t_0, T) \times E \times E \to E$ and $x_0 \in (t_0, T)$. We say that f is generalized differentiable of the second-order at x_0, If there exists an element $f''(x_0) \in E$, such that

(i) for all $h > 0$ sufficiently small, $\exists f'(x_0 + h) \ominus f'(x_0)$, $\exists f'(x_0) \ominus f'(x_0 - h)$ and the limits (in the metric d)

$$\lim_{h \searrow 0} \frac{f'(x_0 + h) \ominus f'(x_0)}{h} = \lim_{h \searrow 0} \frac{f'(x_0) \ominus f'(x_0 - h)}{h} = f''(x_0),$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f'(x_0) \ominus f'(x_0 + h)$, $\exists f'(x_0 - h) \ominus f'(x_0)$ and the limits (in the metric d)

$$\lim_{h \searrow 0} \frac{f'(x_0) \ominus f'(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f'(x_0 - h) \ominus f'(x_0)}{-h} = f''(x_0),$$

or

(iii) for all $h > 0$ sufficiently small, $\exists f'(x_0 + h) \ominus f'(x_0)$, $\exists f'(x_0 - h) \ominus f'(x_0)$ and the limits (in the metric d)

$$\lim_{h\searrow 0}\frac{f'(x_0+h)\ominus f'(x_0)}{h}=\lim_{h\searrow 0}\frac{f'(x_0-h)\ominus f'(x_0)}{-h}=f''(x_0),$$

or

(iv) for all $h>0$ sufficiently small, $\exists f'(x_0)\ominus f'(x_0+h)$, $\exists f'(x_0)\ominus f'(x_0-h)$ and the limits (in the metric d)

$$\lim_{h\searrow 0}\frac{f'(x_0)\ominus f'(x_0+h)}{-h}=\lim_{h\searrow 0}\frac{f'(x_0)\ominus f'(x_0-h)}{h}=f''(x_0).$$

Definition 1.4.7. [8]. Let $f:(t_0,T)\times E\times\ldots\times E\to E$ and $x_0\in(t_0,T)$. We Define the $Nth-$order differential of f as follow: Let $f:(t_0,T)\to E$ and $x_0\in(t_0,T)$. We say that f is strongly generalized differentiable of the $nth-$order at x_0. If there exists an element $f^{(s)}(x_0)\in E,\qquad \forall s=1,\ldots,n$, such that

(i) for all $h>0$ sufficiently small, $\exists f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)$, $\exists f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)$ and the limits (in the metric d_∞)

$$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)}{h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)}{h}=f^{(s)}(x_0)$$

or

(ii) for all $h>0$ sufficiently small, $\exists f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h)$, $\exists f^{(s-1)}(x_0-h)\ominus f^{(s-1)}(x_0)$ and the limits (in the metric d_∞)

$$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h)}{-h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0-h)\ominus f(x_0)}{-h}=f^{(s)}(x_0)$$

or

(iii) for all $h>0$ sufficiently small, $\exists f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)$, $\exists f^{(s-1)}(x_0-h)\ominus$

$f^{(s-1)}(x_0)$ and the limits (in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f^{(s-1)}(x_0+h) \ominus f^{(s-1)}(x_0)}{h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0-h) \ominus f^{(s-1)}(x_0)}{-h} = f^{(s)}(x_0)$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0+h)$, $\exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0-h)$ and the limits (in the metric d_∞)

$$\lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0+h)}{-h} = \lim_{h \searrow 0} \frac{f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0-h)}{h} = f^{(s)}(x_0),$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively $\forall s = 1 \ldots n$)

1.5 Fuzzy differential equation

In this section we define *Fuzzy Initial Value Problem* (FIVP) or *Fuzzy Cauchy Problem.* Differential equation in a fuzzy environment has been suggested as a way of modeling uncertain and incompletely specified systems. Formulation of the concept usually interprets the solution as flow on some appropriate space of fuzzy sets and has been largely concerned with existence and uniqueness problems.

Definition 1.5.1. Fuzzy initial value problem is defined as

$$\begin{cases} x'(t) &= f(t, x(t)); \quad a \le t \le b, \\ x(a) &= x_0. \end{cases} \tag{1.5.1}$$

Assume that $f : T \times E^n \to E^n$ is continuous and $x_0 \in E^n$. The following theorems and lemmas show that the system (1.5.1) has solution.

Lemma 1.5.1. *[47]. A mapping $x : T \to E^n$ is a solution to (1.5.1) if and only if it is continuous and satisfies the integral equation*

$$x(t) = x_0 + \int_a^t f(s, x(s))ds \tag{1.5.2}$$

for all $t \in T$.

Note that we cannot extend lemma (1.5.1) for $t < a$. If f is Lipschitz continuous then (1.5.1) has a unique solution on T.

Theorem 1.5.2. *Let $f : T \times E^n \to E^n$ be continuous and assume that there exists a $k > 0$ such that*

$$D(f(t, x), f(t, y)) \leq kD(x, y)$$

for all $t \in T, x, y \in E^n$. Then (1.5.1) has a unique solution on T, [47].

Furthermore, the solution depends continuously on the initial value. Seikkala in [78] have solved (1.5.1) as follows:

The extension principle of Zadeh leads to the following definition of $f(t, x)$ when $x = x(t)$ is a fuzzy number

$$f(t, x)(s) = \sup\{x(\tau)|\ s = f(t, \tau)\}, \quad s \in R.$$

It follows that

$$[f(t, x)]_r = [f_1(t, x; r), f_2(t, x; r)], \quad r \in (0, 1].$$

where

$$\begin{aligned} f_1(t, x; r) &= min\{f(t, u)|\ u \in [x_1(t; r), x_2(t; r)]\}, \\ f_2(t, x; r) &= max\{f(t, u)|\ u \in [x_1(t; r), x_2(t; r)], \}. \end{aligned} \tag{1.5.3}$$

The function $x : R_+ \to E$ is a fuzzy solution of (1.5.1) on I, if

$$\begin{aligned} x_1'(t; r) &= min\{f(t, u)|\ u \in [x_1(r), x_2(r)]\},\ x_1(0; r) = x_{01}(r), \\ x_2'(t; r) &= max\{f(t, u)|\ u \in [x_1(r), x_2(r)]\},\ x_1(0; r) = x_{01}(r), \end{aligned} \tag{1.5.4}$$

for any $t \in I$ and $r \in [0,1]$. Thus for fixed r, we have an initial value problem in R^2. If we can solve it (uniquely), we have only to verify that the intervals $[x_1(t;r), x_2(t;r)],\ r \in [0,1]$, define a fuzzy number $x(t)$ in E.

Theorem 1.5.3. *Let f satisfy*

$$|f(t,v) - f(t,\overline{v})| \leq g(t, |v - \overline{v}|), \quad t \geq 0, \quad v, \overline{v} \in R,$$

where g on $R_+ \times R_+$ is a continuous mapping such that $r \longrightarrow g(t,r)$ is nondecreasing, the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \tag{1.5.5}$$

has a solution on R_+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (1.5.5) for $u_0 = 0$. Then the fuzzy initial value problem (1.5.1) has a unique fuzzy solution.

The other solution is introduced by Buckley and Feuring in [?]. They solve the FIVP (1.5.1) by fuzzifying the crisp solution to obtain fuzzy solution using the extension principle.

Remark 1.5.1. [22]. In the case of strongly generalized differentiability, to the fuzzy differential equation $y' = f(x,y)$ we may attach two different integral equations, while in the case of differentiability in the sense of the Definition of H-differentiable, we may attach only one. The second integral equation can be written in the form $y(x) = y_0 \ominus (-1). \int_{x_0}^{x} f(t, y(t))dt$.

The following theorems concern the existence of solutions of a fuzzy initial-value problem under generalized differentiability.

Theorem 1.5.4. *[22]. Suppose that the following conditions hold: (a)Let* $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q),\ p, q > 0,\ y_0 \in E,$ *where* $\overline{B}(y_0, q) = \{y \in E : D(y, y_0) \leq q\}$ *denote a closed ball in* E *and let* $f : R_0 \longrightarrow E$ *be a continuous function such that* $d_\infty(\widetilde{0}, f(x, y)) = \|f(x, y)\| \leq M$ *for all* $(x, y) \in R_0$ *(b) Let* $g : [x_0, x_0+p]\times[0, q] \longrightarrow E,$ *such that* $g(x, 0) \equiv 0$ *and* $0 \leq g(x, u) \leq M_1,\ \forall x \in [x_0, x_0 + p],\ 0 \leq u \leq q,$ *such that* $g(x, u)$ *is non-decreasing in* u *and* g *is such that the initial-value problem* $u^{'}(x) = g(x, u(x)), u(x_0) = 0$ *has only the solution* $u(x) \equiv 0$ *on* $[x_0, x_0 + p]$*. (c) We have* $d_\infty(f(x, y), f(x, z)) \leq g(x, d_\infty(y, z)), \forall(x, y),(x, z),(y, z) \in R_0$ *and* $d_\infty(y, z) \leq q.$ *(d) There exists* $d > 0$ *such that for* $x \in [x_0, x_0+d]$ *the sequence* $\overline{y}_n : [x_0, x_0+d] \longrightarrow E$ *given by* $\overline{y}_0(x) = y_0,\ \overline{y}_{n+1}(x) = y_0 \ominus (-1).\int_{x_0}^{x} f(t, \overline{y}_n)dt$ *is defined for any* $n \in N$*.*

Then the fuzzy initial -value problem

$$\begin{cases} y''(x) = f(x, y), \\ y(x_0) = y_0 \end{cases}$$

has two solutions (one (1)-differentiable and the other one (2)-differentiable) $y, \hat{y} : [x_0, x_0 + r] \longrightarrow B(y_0, q)$ *where* $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ *and the successive iterations*

$$y_0(x) = y_0, y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$$

and

$$\hat{y}_0(x) = y_0, \hat{y}_{n+1}(x) = y_0 \ominus (-1).\int_{x_0}^{x} f(t, \hat{y}_n(t))dt$$

converge to these two solutions respectively.

According to Theorem 1.5.4, we restrict our attention to functions which are (1)-

or (2)-differentiable on their domain except for a finite number of points (see also [22]).

Lemma 1.5.5. *Let* $u, v \in E$ *be such that* $u(1) - \underline{u}(0) > 0$, $\overline{u}(0) - u(1) > 0$ *and* $len(v) = (\overline{v}(0) - \underline{v}(0)) \leq \min\{u(1) - \underline{u}(0), \overline{u}(0) - u(1)\}$. *Then the H-difference* $u \ominus v$ *exists.*

The following corollary gives simple sufficient condition for the existence of fuzzy differential equations under generalized differentiability.

Corollary 1.5.6. *Let* $f : R_0 \longrightarrow E$ *where* $R_0 = [x_0, x_0+p] \times (\overline{B}(y_0, q) \cap E)$, *and* $y_0 \in E$ *such that* $y(0,1) - \underline{y}(0,0)$ *and* $\overline{y}(0,0) - y(0,1)$. *Let* $m = \min\{y(0,1) - \underline{y}(0,0), \overline{y}(0,0) - y(0,1)\}$. *Under the assumptions (a)-(c) of Theorem (1.5.4), the fuzzy initial-value problem*

$$\begin{cases} y' = f(x,y), \\ y(x_0) = y_0 \end{cases}$$

has two solutions $y, \overline{y} : [x_0, x_0 + r] \longrightarrow B(\widetilde{y}_0, q)$ *where* $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, \frac{m}{2M}\}$ *and the successive iterations in theorem (1.5.4) converge to these two solutions.*

Chapter 2

Fuzzy Differential Equations of Natural Order

In this chapter, we firstly express the fuzzy Lalpace transform and then some its well-known properties are investigated. In addition, an existence theorem is given for fuzzy-valued function which possess the fuzzy Lalpace transform. Consequently, we investigate the solutions of FDEs. Then, an operator method has been applied to solve FDEs using the equivalent integral form of original FDE. At the end stagem a novel approach based on the 1-cut expansion approach has been proposed. In each section, several examples have been stated to demonstrate the approach, clearly.

2.1 Second Order FDEs

In this section, we are going to review the definition of fuzzy initial value problems, particularly, the second order fuzzy differential equation under generalized differentiability which has been proposed in [9].

We define a general second-order fuzzy differential equation as follows:

$$\begin{cases} y''(t) + g(t, y(t), y'(t)) = f(t, y(t), y'(t)) \\ \widetilde{y}(x_0) = \widetilde{y}_0, \\ \widetilde{y}'(x_0) = \widetilde{y}'_0 \end{cases} \tag{2.1.1}$$

$y(t) = ((\underline{y}(t;\alpha), \bar{y}(t;\alpha))$ is a fuzzy function of t and $f(t, y(t), y'(t))$ and $g(t, y(t), y'(t))$ are continuous fuzzy-valued functions.

Now, we state a useful result states the parametric forms of second order derivatives for a given fuzzy-valued function F.

Theorem 2.1.1. *Let F and F' are two differentiable fuzzy-valued functions. Moreover, we denote α-cut representation of fuzzy-valued function $F(t)$ with $[F(t)]^r = [f(t,r), g(t,r)]$, then:*

(a) Let F and F' are $(1)-$differentiable, or, let F and F' are $(2)-$differentiable, then: f, g have first-order and second-order derivatives and

$$[F''(t)]^r = [f''(t,r), g''(t,r)],$$

(b) Let F is $(1)-$differentiable and F' is $(2)-$differentiable, or, let F is $(2)-$differentiable and F' is $(1)-$differentiable, then: f, g have first-order and second-order derivatives and

$$[F''(t)]^r = [g''(t,r), f''(t,r)].$$

Proof (a). Let F and F' are $(1)-$differentiable, then:

Since F is $(1)-$differentiable, we have $[F'(t)]^r = [f'(t,r), g'(t,r)]$, and

$$\begin{cases} [F'(t+h) \ominus F'(t)]^r = [f'(t+h,r) - f'(t,r), g'(t+h,r) - g'(t,r)], \\ [F'(t) \ominus F'(t-h)]^r = [f'(t,r) - f'(t-h,r), g'(t,r) - g'(t-h,r)] \end{cases}$$

and, multiplying by $\frac{1}{h}, h > 0$, we get:

$$\frac{[F'(t+h) \ominus F'(t)]^r}{h} = \left[\frac{f'(t+h,r) - f'(t,r)}{h}, \frac{g'(t+h,r) - g'(t,r)}{h}\right],$$

and

$$\frac{[F'(t) \ominus F'(t-h)]^r}{h} = \left[\frac{f'(t,r) - f'(t-h,r)}{h}, \frac{g'(t,r) - g'(t-h,r)}{h}\right]$$

Finally, by using the fact that $h \longrightarrow 0$ on both sides of above relation, the proof is completed.

Additionally, consider F and F' are $(2)-$differentiable fuzzy-valued functions, then, $[F'(t)]^r = [g'(t,r), f'(t,r)]$, and

$$\begin{cases} [F'(t) \ominus F'(t+h)]^r = [g'(t,r) - g'(t+h,r), f'(t,r) - f'(t+h,r)], \\ [F'(t-h) \ominus F'(t)]^r = [g'(t-h,r) - g'(t,r), f'(t-h,r) - f'(t,r)] \end{cases}$$

and, multiplying by $\frac{1}{-h}, h > 0$, we get:

$$\frac{[F'(t) \ominus F'(t+h)]^r}{-h} = \left[\frac{f'(t+h,r) - f'(t,r)}{h}, \frac{g'(t+h,r) - g'(t,r)}{h}\right]$$

and

$$\frac{[F'(t-h) \ominus F'(t)]^r}{-h} = \left[\frac{f'(t,r) - f'(t-h,r)}{h}, \frac{g'(t,r) - g'(t-h,r)}{h}\right]$$

Then, by applying $h \longrightarrow 0$ on both sides of above relation, the proof is completed.

Proof (b). Let F is $(1)-$differentiable and F' is $(2)-$differentiable, then, $[F'(t)]^r = [f'(t,r), g'(t,r)]$, and

$$\begin{cases} [F'(t) \ominus F'(t+h)]^r = [f'(t,r) - f'(t+h,r), g'(t,r) - g'(t+h,r)], \\ [F'(t-h) \ominus F'(t)]^r = [f'(t-h,r) - f'(t,r), g'(t-h,r) - g'(t,r)] \end{cases}$$

and, multiplying by $\frac{1}{-h}, h > 0$, we get:

$$\frac{[F'(t) \ominus F'(t+h)]^r}{-h} = \left[\frac{g'(t+h,r) - g'_\alpha(t,r)}{h}, \frac{f'(t+h,r) - f'(t,r)}{h}\right],$$

and

$$\frac{[F'(t-h) \ominus F'(t)]^r}{-h} = \left[\frac{g'(t,r) - g'(t-h,r)}{h}, \frac{f'(t,r) - f'(t-h,r)}{h}\right],$$

Then, by applying $h \longrightarrow 0$ on both sides of above relation, the proof is completed. □

2.1.1 The fuzzy Laplace transforms

Suppose that f is fuzzy-valued function and s is a real parameter. We define the fuzzy Laplace transform of f as following:

Definition 2.1.1. The fuzzy Laplace transform of fuzzy-valued function f is defined as follows:

$$\widehat{F}(s) = \mathbf{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt \tag{2.1.2}$$

$$= \lim_{\tau \longrightarrow \infty} \int_0^\tau e^{-st} f(t)dt, \tag{2.1.3}$$

i.e., for $\alpha \in [0,1]$ we have:

$$\widehat{F}(s;\alpha) = \left[\lim_{\tau \longrightarrow \infty} \int_0^\tau e^{-st} \underline{f}(t;\alpha)dt, \ \lim_{\tau \longrightarrow \infty} \int_0^\tau e^{-st} \overline{f}(t;\alpha)dt\right] \tag{2.1.4}$$

whenever the limits exist.

Consider the fuzzy-valued function f, then the lower and upper fuzzy Lapalce transform of this function are denoted based on the lower and upper of fuzzy-valued function f as follow:

$$\widehat{F}(s;r) = \mathbf{L}(f(t;r)) = \left[l(\underline{f}(t;r)), l(\overline{f}(t;r))\right]$$

where,

$$l(\underline{f}(t;r)) = \int_0^\infty e^{-st}\underline{f}(t;r)dt = \lim_{\tau\longrightarrow\infty}\int_0^\tau e^{-st}\underline{f}(t;r)dt,$$
$$l(\overline{f}(t;r)) = \int_0^\infty e^{-st}\overline{f}(t;r)dt = \lim_{\tau\longrightarrow\infty}\int_0^\tau e^{-st}\overline{f}(t;r)dt.$$

2.1.2 Convergence

Although the fuzzy Laplace operator can be applied to a large number of fuzzy-valued functions, there exist some examples that integral (2.1.2) does not converge.

Example 2.1.1. *Let the fuzzy-valued function $f(t) = ce^{t^2}$, where $c \in \mathbf{E}$. Then, we get:*

$$\lim_{\tau\longrightarrow\infty}\int_0^\tau c\, e^{-st}e^{t^2}dt \to \infty,$$

for any choice of the variable s, because the integrand grows without bound as $\tau \longrightarrow \infty$.

Now, we define a special types of convergence of the fuzzy Laplace integral.

Definition 2.1.2. The integral (2.1.2) is said to be absolutely convergent if:

$$\lim_{\tau\longrightarrow\infty}\int_0^\tau d_\infty\left(e^{-st}f(t), \widetilde{0}\right)dt \tag{2.1.5}$$

exists.

Now, we consider another type of convergence as follows:

Definition 2.1.3. The integral (2.1.2) is said to be converge uniformly for arbitrary s in domain Ω, if for any $\epsilon > 0$, there exists some number τ_0 such that if $\tau \geq \tau_0$, then

$$d_\infty\left(\int_\tau^\infty e^{-st}f(t)dt, \widetilde{0}\right) \leq \epsilon$$

Now, we investigate the continuity requirement for some fuzzy-valued function which will be possesses the fuzzy Laplace transform

Definition 2.1.4. A fuzzy-valued function f has a jump discontinuity at the point t_0 if both limits

$$\begin{aligned}
\lim_{t\longrightarrow t_0^-} f(t) &= \left[\lim_{t\longrightarrow t_0^-} \underline{f}(t,r), \lim_{t\longrightarrow t_0^-} \overline{f}(t,r)\right] \\
&= \left[\underline{f}(t_0^-,r), \overline{f}(t_0^-,r)\right], \\
\lim_{t\longrightarrow t_0^+} f(t) &= \left[\lim_{t\longrightarrow t_0^+} \underline{f}(t,r), \lim_{t\longrightarrow t_0^+} \overline{f}(t,r)\right] \\
&= \left[\underline{f}(t_0^+,r), \overline{f}(t_0^+,r)\right],
\end{aligned}$$

exist and

$$f(t_0^-) \neq f(t_0^+).$$

Here, $t \longrightarrow t_0^-$ and $t \longrightarrow t_0^+$ mean that $t \longrightarrow t_0$ from the left and the right, respectively.

Definition 2.1.5. A fuzzy-valued function f is piecewise continuous on the interval $[0, \infty)$ if:

(i) $\lim_{t \longrightarrow 0^+} f(t) = f(0^+)$ *exists.*

(ii) f is continuous on every finite interval (0,b) expect possibly at a finite number of points $\tau_1, \tau_2, \ldots, \tau_n$ in $(0, b)$ at which f has jump discontinuity.

Also, considered fuzzy-valued function f is also bounded which means that:

$$d_\infty(f(t), \widetilde{0}) \leq M_i.\widetilde{1}, \quad \tau_i < t < \tau_{i+1}, \quad i = 1, 2, 3, \ldots, n-1$$

for finite positive constants M_i.

In order to integrate piecewise continuous fuzzy-valued functions form 0 to b, one simply integrates f over each of the subintervals and takes the sum of these integrals, that is,

$$\int_0^b f(t)dt = \int_0^{\tau_1} f(t)dt + \int_{\tau_1}^{\tau_2} f(t)dt + \ldots + \int_{\tau_n}^{b} f(t)dt$$

Now, we consider the definition of exponential order of fuzzy-valued functions which is necessary for future purpose.

Definition 2.1.6. A fuzzy-valued function f has exponential order p if there exist constants $M > 0$ and p such that for some $t_0 \geq 0$, $d_\infty(f(t), \widetilde{0}) \leq Me^{pt}$, $t \geq t_0$.

Consequently, we show that a large class of fuzzy-valued functions can possesses the fuzzy Laplace transform.

Theorem 2.1.2. *If fuzzy-valued function f be bounded piecewise continuous on $[0, \infty)$ and of exponential order p, then the fuzzy Laplace transform $\widehat{F}(s) = L(f(t))$ exists for $s > p$ and converges absolutely.*

Proof. Since f is assumed has exponential order property, we get:

$$d_\infty(f(t), \widetilde{0}) \leq M_1 e^{pt},$$

for some real p and $t \geq t_0$. Also, f is piecewise continuous on $[0, t_0]$ is bounded, we obtain:

$$d_\infty(f(t), \widetilde{0}) \leq M_2, \quad 0 < t < t_0.$$

Since e^{pt} has a positive minimum on $[0, t_0]$, a constant M can be chosen sufficiently large as $M = \max\{M_1, M_2\}$ so that $d_\infty(f(t), \widetilde{0}) \leq Me^{pt}$, $t > 0$. Therefore,

$$\begin{aligned} \int_0^\tau d_\infty \left(e^{-st} f(t)\right) dt &\leq M \int_0^\tau e^{-(x-p)t} dt \\ &= \left(\frac{M}{x-p} - \frac{Me^{-(x-p)\tau}}{x-p}\right). \end{aligned}$$

By passing $\tau \longrightarrow \infty$ and since $s = x > p$, we get:

$$\int_0^\infty d_\infty \left(e^{-st} f(t), \widetilde{0}\right) \leq \frac{M}{x-p}.$$

Thus, the fuzzy Laplace integral converges absolutely and hence converges for $s > p$. □

2.1.3 Basic properties

Here, we investigate the important properties of the fuzzy Laplace transforms.

Linearity. One of the most basic useful properties of the fuzzy Laplace operator $\mathbf{L}$ is that of linearity, namely if $f_1 \in L$ for $s > p$, $f_2 \in \mathbf{L}$ for $s > q$, then $f_1 + f_2 \in \mathbf{L}$ for $(s) > \max\{p, q\}$ and $L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2)$ for arbitrary constants c_1, c_2.

Uniformly convergence. We have already seen that for fuzzy-valued function f which is piecewise continuous on $[0, \infty)$ and of exponential order, the fuzzy Laplace integral (2.1.2) converges absolutely, that is,

$$\int_0^\infty d_\infty \left(e^{-st} f(t), \widetilde{0}\right) dt$$

converges. Moreover, for fuzzy-valued function f, the fuzzy Laplace integral (2.1.2) is uniformly converges. Since, if $d(f(t), \widetilde{0}) \leq Me^{pt}$, $t \geq t_0$, then we have

$$\begin{aligned} d_\infty \left(\int_0^\infty e^{-st} f(t), \widetilde{0}\right) &\leq \int_{t_0}^\infty e^{-xt} d(f(t), \widetilde{0}) dt \\ &\leq M \int_{t_0}^\infty e^{-(x-p)t} dt \\ &= \left(\frac{Me^{-(x-p)t_0}}{x-p}\right) \end{aligned}$$

provided $x = s > p$. Considering $x \geq x_0 > p$, gives an upper bound for the last expression:

$$\frac{Me^{-(x-p)t_0}}{x-p} \leq \frac{Me^{-(x_0-p)t_0}}{x_0-p}. \tag{2.1.6}$$

By choosing t_0 sufficiently large, we can make the term on the right-hand side of (2.1.6) arbitrarily small; that is, for a given $\epsilon > 0$, there exists a positive value $T > 0$ such that:

$$d_\infty \left(\int_{t_0}^\infty e^{-st} f(t) dt, \widetilde{0}\right) \leq \epsilon \quad \textit{whenever } t_0 \geq T$$

for all values of s with $s \geq x_0 > p$. This is precisely the condition required for the uniform convergence of the Laplace integral in the region $s \geq x_0 > p$.

Theorem 2.1.3. *If fuzzy-valued function f is piecewise continuous on $[0, \infty)$ and has exponential order p, then $\widehat{F}(s) = \mathbf{L}(f(t)) \longrightarrow 0$, as $s \longrightarrow \infty$*

Proof. In fact, by using previous results, we have

$$d_\infty \left(\int_0^\infty e^{-st} f(t), \widetilde{0} \right) \leq \frac{M}{x-p}, \quad s = x > p$$

and then, $x \longrightarrow \infty$ gives the result.

2.1.4 Inverse of fuzzy Laplace transform

In order to application of the fuzzy Laplace transform in physical problems, it is necessary to invoke the inverse transform.

If $\mathbf{L}(f(t)) = \widehat{F}(s)$, then the inverse of fuzzy Laplace transform is denoted by $\mathbf{L}^{-1}(\widehat{F}(s)) = f(t)$, $t \geq 0$ which maps the fuzzy Laplace transform of a fuzzy-valued function f back to original fuzzy-valued function f.

Note that $\mathbf{L}^{-1}$ is linear, that is,

$$L^{-1}(a\widehat{F}(s) + b\widehat{G}(s)) = af(t) + bg(t)$$

if $\mathbf{L}(f(t)) = \widehat{F}(s)$, $\mathbf{L}(g(t)) = \widehat{G}(s)$ and $a, b \in R$. This follows from the linearity of $\mathbf{L}$ and holds in the domain to $\widehat{F}$ and $\widehat{G}$.

2.1.5 Translation theorems

Now, we present two useful results for determining the fuzzy Laplace transforms and their inverses.

An important function occurring in electrical system is the (delayed) unit step function

$$u_a(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a. \end{cases} \tag{2.1.7}$$

Theorem 2.1.4. *(First translation theorem) If* $\widehat{F}(s) = \mathbf{L}(f(t))$ *for* $s > 0$*, then* $\widehat{F}(s-a) = \mathbf{L}(e^{at}f(t))$*, a real and* $s > a$

Proof. For $s > a$,

$$\begin{aligned} \widehat{F}(s-a) &= \int_0^\infty e^{-(s-a)t} f(t)dt \\ &= \int_0^\infty e^{-st}e^{at} f(t)dt \\ &= \mathbf{L}(e^{at}f(t)) \end{aligned}$$

Theorem 2.1.5. *(Second translation theorem) If* $\widehat{F}(s) = \mathbf{L}(f(t))$ *then*

$$\mathbf{L}(u_a(t)f(t-a)) = e^{-as}\widehat{F}(s), \quad a \geq 0$$

Proof. This follows from the fact that:

$$\int_0^\infty e^{-st}\left(u_a(t)f(t-a)\right)dt = \int_a^\infty e^{-st} f(t-a)dt$$

and setting $\tau = t - a$, the right-hand integral becomes

$$\begin{aligned} \int_0^\infty e^{-s(\tau+a)} f(\tau)d\tau &= e^{-sa}\int_0^\infty e^{-s\tau} f(\tau)d\tau \\ &= e^{-sa}\widehat{F}(s). \end{aligned}$$

2.1.6 Fuzzy Laplace transform of derivative

In order to solve fuzzy differential equations, it is necessary to know the fuzzy Laplace transform of the derivative of f, f'. The virtue of $\mathbf{L}(f'(t))$ is that, it can be written in terms of $\mathbf{L}(f(t))$.

Theorem 2.1.6. *([8]). (Derivative theorem) Suppose that f is continuous fuzzy-valued function on $[0, \infty)$ and exponential order α and that f' is piecewise continuous fuzzy-valued function on $[0, \infty)$, then*

$$\mathbf{L}(f'(t)) = s\mathbf{L}(f(t)) \ominus f(0),$$

if f is $(1)-$differentiable,

$$\mathbf{L}(f'(t)) = (-f(0)) \ominus (-s\mathbf{L}(f(t))),$$

if f is $(2)-$differentiable

Moreover, we need to the fuzzy Laplace transform of second-order derivative of fuzzy-valued function for solving second-order fuzzy differential equation. So, we do this under generalized H-differentiability as follows:

Theorem 2.1.7. *Suppose that f and f' are continuous fuzzy-valued function on $[0, \infty)$ and of exponential order and that f'' is piecewise continuous fuzzy-valued function on $[0, \infty)$, then:*

$$\mathbf{L}(f''(t)) = s^2\mathbf{L}(f(t)) \ominus sf(0) \ominus f'(0), \tag{2.1.8}$$

if f and f' are (1)−differentiable,

$$\mathbf{L}(f''(t)) = -f'(0) \ominus (-s^2)\mathbf{L}(f(t)) - sf(0), \tag{2.1.9}$$

if f is (1)−differentiable and f' is (2)−differentiable,

$$\mathbf{L}(f''(t)) = -sf(0) \ominus (-s^2)\mathbf{L}(f(t)) \ominus f'(0), \tag{2.1.10}$$

if f is (2)−differentiable and f' is (1)−differentiable,

$$\mathbf{L}(f''(t)) = s^2\mathbf{L}(f(t)) \ominus sf(0) - f'(0), \tag{2.1.11}$$

if f and f' are (2)−differentiable.

Proof (2.1.8). For arbitrary fixed $r \in [0,1]$ we have:

$$s^2\mathbf{L}(f(t)) \ominus sf(0) \ominus f'(0) =$$
$$\left(s^2 l(\underline{f}(t,r)) - s\underline{f}(0,r)) - \underline{f'}(0,r), s^2 l(\overline{f}(t,r)) - s\overline{f}(0,r)) - \overline{f'}(0,r)\right).$$

Since,

$$\begin{cases} l(\overline{f''}(t,r)) = l(\overline{f}''(t,r)) = s^2 l(\overline{f}(t,r)) - s\overline{f}(0,r)) - \overline{f}'(0,r), \\ l(\underline{f''}(t,r)) = l(\underline{f}''(t,r)) = s^2 l(\underline{f}(t,r)) - s\underline{f}(0,r)) - \underline{f}'(0,r), \end{cases}$$

and, also since,

$$\begin{aligned} \overline{f}'(0,r) &= \overline{f'}(0,r), \;\; 0 \le r \le 1 \\ \underline{f}'(0,r) &= \underline{f'}(0,r), \;\; 0 \le r \le 1 \end{aligned}$$

and by applying linearity of $\mathbf{L}$, we get:

$$\begin{aligned} s^2\mathbf{L}(f(t)) \ominus sf(0) \ominus f'(0) &= (l(\underline{f''}(t,r)),\; l(\overline{f''}(t,r))) \\ &= \mathbf{L}(\underline{f''}(t,r), \overline{f''}(t,r)) \\ &= \mathbf{L}(f''(t)). \end{aligned}$$

Proof (2.1.9). For arbitrary fixed $r \in [0,1]$ we have:

$$-f'(0) \ominus (-s^2)\mathbf{L}(f(t)) - sf(0) =$$
$$(-\overline{f'}(0,r)) + s^2 l(\overline{f}(t,r)) - s\overline{f}(0,r), -\underline{f'}(0,r)) + s^2 l(\underline{f}(t,r)) - s\underline{f}(0,r)),$$

since,

$$\begin{cases} l(\overline{f''}(t,r)) = l(\underline{f}''(t,r)) = s^2 l(\underline{f}(t,r)) - s\underline{f}(0,r)) - \underline{f}'(0,r), \\ l(\underline{f''}(t,r)) = l(\overline{f}''(t,r)) = s^2 l(\overline{f}(t,r)) - s\overline{f}(0,r)) - \overline{f}'(0,r), \end{cases}$$

and, since, $\overline{f}'(0,r) = \overline{f'}(0,r)$, $\underline{f}'(0,r) = \underline{f'}(0,r)$ and by applying linearity of $\mathbf{L}$, we get:

$$\begin{aligned} -f'(0) \ominus (-s^2)\mathbf{L}(f(t)) - sf(0) &= (l(\underline{f''}(t,r)),\ l(\overline{f''}(t,r))) \\ &= \mathbf{L}(\underline{f''}(t,r), \overline{f''}(t,r)) \\ &= \mathbf{L}(f''(t)). \end{aligned}$$

Proof (2.1.10). For arbitrary fixed $r \in [0,1]$ we have:

$$-sf(0) \ominus (-s^2)\mathbf{L}(f(t)) \ominus f'(0) =$$
$$(-s\overline{f}(0,r) + s^2 l(\overline{f}(t,r)) - \underline{f}'(0,r), -s\underline{f}(0,r) + s^2 l(\underline{f}(t,r)) - \overline{f'}(0,r)),$$

since,

$$\begin{cases} l(\overline{f''}(t,r)) = l(\underline{f}''(t,r)) = s^2 l(\underline{f}(t,r)) - s\underline{f}(0,r)) - \underline{f}'(0,r), \\ l(\underline{f''}(t,r)) = l(\overline{f}''(t,r)) = s^2 l(\overline{f}(t,r)) - s\overline{f}(0,r)) - \overline{f}'(0,r), \end{cases}$$

and, since, $\overline{f}'(0,r) = \underline{f'}(0,r)$, $\underline{f}'(0,r) = \overline{f'}(0,r)$ and by applying linearity of $\mathbf{L}$, we get:

$$
\begin{aligned}
-sf(0) \ominus (-s^2)\mathbf{L}(f(t)) \ominus f'(0) &= (l(\underline{f''}(t,r)),\ l(\overline{f''}(t,r))) \\
&= \mathbf{L}(\underline{f''}(t,r), \overline{f''}(t,r)) \\
&= \mathbf{L}(f''(t)).
\end{aligned}
$$

Proof (2.1.11). For arbitrary fixed $r \in [0,1]$ we have:

$$
\begin{aligned}
&s^2\mathbf{L}(f(t)) \ominus sf(0) - f'(0) \\
&= (s^2 l(\underline{f}(t,r)) - s\underline{f}(0,r) - \overline{f'}(0,r), s^2 l(\overline{f}(t,r)) - s\overline{f}(0,r) - \underline{f'}(0,r)).
\end{aligned}
$$

Since,

$$
\begin{cases}
l(\overline{f''}(t,r)) = l(\overline{f}''(t,r)) = s^2 l(\overline{f}(t,r)) - s\overline{f}(0,r)) - \overline{f}'(0,r), \\
l(\underline{f''}(t,r)) = l(\underline{f}''(t,r)) = s^2 l(\underline{f}(t,r)) - s\underline{f}(0,r)) - \underline{f}'(0,r),
\end{cases}
$$

and, since, $\overline{f}'(0,r) = \underline{f'}(0,r)$, $\underline{f}'(0,r) = \overline{f'}(0,r)$ and by applying linearity of $\mathbf{L}$, we get:

$$
\begin{aligned}
s^2\mathbf{L}(f(t)) \ominus sf(0) - f'(0) &= (l(\underline{f''}(t,r)),\ l(\overline{f''}(t,r))) \\
&= \mathbf{L}(\underline{f''}(t,r)), \overline{f''}(t,r))) \\
&= \mathbf{L}(f''(t)),
\end{aligned}
$$

which completes the proof. □

2.1.7 Examples

Now, some second order FDEs are solved under generalized differentiability. Notice that in the first example, we found that all four solutions are valid, but in the next example we found that only two of them are valid.

Example 2.1.2. *([51]) Consider the following second- order FDE:*

$$\begin{cases} y''(t) = \sigma_0, \ \ \sigma_0 = (r-1, 1-r), \\ y(0,r) = (r-1, 1-r), \\ y'(0,r) = (r-1, 1-r) \end{cases}$$

Now, we consider this example in four cases as following:

Case(I). Let us consider y and y' are $(1)-$differentiable, then by applying (2.1.8), we get:

$$s^2\mathbf{L}(y(t)) \ominus sy(0) \ominus y'(0) = \frac{\sigma_0}{s},$$

Then, we get the $r-$cut representation of solution as following for all $t \in D_y = [0, \infty]$:

$$\begin{cases} \underline{y}(t,r) = (r-1)\left(\frac{t^2}{2} + t + 1\right), \\ \overline{y}(t,r) = (1-r)\left(\frac{t^2}{2} + t + 1\right). \end{cases}$$

Case(II). Let us consider y is $(1)-$differentiable and y' is $(2)-$differentiable, then by taking Laplace transform for both sides of the original problem and using (2.1.9), we get:

$$-y'(0) \ominus (-s^2)\mathbf{L}(y(t)) - sy(0) = \frac{\sigma_0}{s},$$

Then, we get the $r-$cut representation of solution as following for all $t \in D_y = (0, 1)$:

$$\begin{cases} \underline{y}(t,r) = (r-1)\left(-\frac{t^2}{2} + t + 1\right), \\ \overline{y}(t,r) = (1-r)\left(-\frac{t^2}{2} + t + 1\right). \end{cases}$$

Case(III). Let us consider y is $(2)-$differentiable y' are $(1)-$differentiable, then by taking Laplace transform for both sides of the original problem and using (2.1.10), we get:

$$-sy(0) \ominus (-s^2)\mathbf{L}(y(t)) \ominus y'(0) = \frac{\sigma_0}{s}.$$

Then, we get the $r-$cut representation of solution as following for all $t \in D_y = \left(0, \sqrt{3}-1\right)$:

$$\begin{cases} \underline{y}(t,r) = (r-1)\left(-\frac{t^2}{2} - t + 1\right), \\ \overline{y}(t,r) = (1-r)\left(-\frac{t^2}{2} - t + 1\right). \end{cases}$$

Case(IV). Let us consider y and y' are $(2)-$differentiable, then by taking Laplace transform for both sides of the original problem and using (2.1.11), we get:

$$-y'(0) \ominus sy(0) + s^2\mathbf{L}(y(t)) = \frac{\sigma_0}{s},$$

Then, we get the $r-$cut representation of solution as following for all $t \in D_y = (0,1)$:

$$\begin{cases} \underline{y}(t,r) = (r-1)\left(\frac{t^2}{2} - t + 1\right), \\ \overline{y}(t,r) = (1-r)\left(\frac{t^2}{2} - t + 1\right). \end{cases}$$

Example 2.1.3. *([51]). Consider second- order FDE:*

$$\begin{cases} y''(t) + y(t) = \sigma_0, \;\; \sigma_0 = (r, 2-r), \\ y(0,r) = (r-1, 1-r), \\ y'(0,r) = (r-1, 1-r) \end{cases}$$

Similarly, we consider the example in four cases.

Case(I). Let us consider y and y' are $(1)-$differentiable, then by applying (2.1.8), we get:

$$s^2\mathbf{L}(y(t)) \ominus sy(0) \ominus y'(0) + \mathbf{L}(y(t)) = \frac{\sigma_0}{s},$$

Then, we get the $r-$cut representation of solution as following:

$$\begin{cases} \underline{y}(t,r) = r\,(1+\sin(t)) - \sin(t) - \cos(t), \\ \overline{y}(t,r) = (2-r)\,(1+\sin(t)) - \sin(t) - \cos(t), \end{cases}$$

Clearly, y' is not (1)-differentiable. Hence, there is no solution in this case. For more details see [51].

Case(II). Let us consider y is $(1)-$differentiable and y' is $(2)-$differentiable, then by taking Laplace transform for both sides of the original problem and using (2.1.9), we get:

$$-y'(0) \ominus (-s^2)\mathbf{L}(y(t)) - sy(0) + \mathbf{L}(y(t)) = \frac{\sigma_0}{s}.$$

Then, we get the $r-$cut representation of solution as following:

$$\begin{cases} \underline{y}(t,r) = r\,(1+\sinh(t)) - \sinh(t) - \cos(t), \\ \overline{y}(t,r) = (2-r)\,(1+\sinh(t)) - \sinh(t) - \cos(t), \end{cases}$$

Since, y' is not (2)-differentiable, there is no solution in this case (see [51]).

Case(III). Let us consider y is $(2)-$differentiable y' are $(1)-$differentiable, then by taking Laplace transform for both sides of the original problem and using (2.1.10), we get:

$$-sy(0) \ominus (-s^2)\mathbf{L}(y(t)) \ominus y'(0) + \mathbf{L}(y(t)) = \frac{\sigma_0}{s}.$$

Then, we get the $r-$cut representation of solution as following for all $t \in D_y = \big(0, \ln(1+\sqrt{2})\big)$:

$$\begin{cases} \underline{y}(t,r) = r\,(1-\sinh(t)) + \sinh(t) - \cos(t), \\ \overline{y}(t,r) = (2-r)\,(1-\sinh(t)) + \sinh(t) - \cos(t), \end{cases}$$

Notice that, in this case, since y is $(2)-$differentiable and y' is $(1)-$differentiable, such solution is acceptable.

Case(IV). Let us consider y and y' are $(2)-$differentiable, then by taking Laplace transform for both sides of the original problem and using (2.1.11), we get:

$$-y'(0) \ominus sy(0) + s^2\mathbf{L}(y(t)) + \mathbf{L}(y(t)) = \frac{\sigma_0}{s}.$$

Then, we get the $r-$cut representation of solution as following for all $t \in D_y = \left(0, \frac{\pi}{2}\right)$:

$$\begin{cases} \underline{y}(t,r) = r\,(1-\sin(t)) + \sin(t) - \cos(t), \\ \overline{y}(t,r) = (2-r)\,(1-\sin(t)) + \sin(t) - \cos(t), \end{cases}$$

Also, obtained solution is valid, since, both y and y' are $(2)-$differentiable.

Remark 2.1.1. We point out that for an FDE, there exist at most 4 solutions, but may be all the obtained solutions did not acceptable. It is important result in the theory of FDEs for higher-order which was not discussed previously, only in [51]. Notice that by applying Khastan et al.'s approach [51], the original second-order fuzzy differential equation is transformed to four ordinary differential systems based on its r-cut representations. However, by applying fuzzy Laplace transform method, each solution can be obtained directly. Clearly, if the order of fuzzy differential equations increases, the assumed time will be increased in the Khastan et al.'s method, on the other hand, the highest order the most time.

2.2 Operator Method

In this section, we suggest a new method to solve fuzzy linear differential equations which is constructed based on the equivalent integral form of original problems. By using the lower and upper functions of obtained integral equations, we can determine the lower and upper functions of solutions. So, we construct a closed form of solutions by solving corresponding fuzzy Volterra integral equations, analytically.

The condition for determining the integral form of Eq. (2.1.1) in each case can now be stated as following (which is a direct generalization of Theorem 3.1 in [10]):

Theorem 2.2.1. *Let $x_0 \in [a,b]$ and assumed that $f : [a,b] \times E \times E \longrightarrow E$ is continuous. Also, suppose that $y(x)$ and $y'(x)$ are continuous. Then, Eq. (2.1.1) is equivalent to one of the integral equations:*

(1) if y and y' are (1)-differentiable, then:

$$y(x) + \int_{x_0}^{x}\int_{x_0}^{z} g(s,y(s),y'(s))dsdz \;=\; k_1 + k_2(x-x_0) + \int_{x_0}^{x}\int_{x_0}^{z} f(s,y(s),y'(s))dsdz,$$

(2) if y is (1)-differentiable and y' is (2)-differentiable, then

$$y(x) + \int_{x_0}^{x}\int_{x_0}^{z} g(s,y(s),y'(s))dsdz \;=\; k_1 + k_2(x-x_0) \ominus (-1)\int_{x_0}^{x}\int_{x_0}^{z} f(s,y(s),y'(s))dsdz,$$

(3) if y is (2)- differentiable and y' is (1)-differentiable, then

$$y(x) + \int_{x_0}^{x}\int_{x_0}^{z} g(s,y(s),y'(s))dsdz \;=\; k_1 \ominus (-1)\left(k_2(x-x_0) + \int_{x_0}^{x}\int_{x_0}^{z} f(s,y(s),y'(s))dsdz\right),$$

(4) if y and y' are (2)-differentiable, then:

$$y(x) + \int_{x_0}^{x}\int_{x_0}^{z} g(s, y(s), y'(s))dsdz = k_1 \ominus (-1)\left(k_2(x - x_0) \ominus (-1)\int_{x_0}^{x}\int_{x_0}^{z} f(s, y(s), y'(s))dsdz\right).$$

where $k_1 = y(x_0)$ and $k_2 = y'(x_0)$.

Here the equivalence between two equations means that any solution of an equation is a solution too for the other one.

Proof. Since f is a continuous fuzzy-valued function, then it is integrable (see [22]). Now, we determine the equivalent integral forms of fuzzy linear differential equation (2.1.1) under each type of strongly generalized differentiability as following:

(1) let us consider y and y' are (1)-differentiable fuzzy-valued functions, then the equivalent integral form of Eq. (2.1.1) can be written under (1)-differentiability of $y'(x)$ as follows:

$$y'(x) + \int_{x_0}^{z} g(s, y(s), y'(s))ds = k_2 + \int_{x_0}^{z} f(s, y(s), y'(s))ds, \tag{2.2.1}$$

and then based on (1)-differentiability of y, we get the following:

$$y(x) + \int_{x_0}^{x}\int_{x_0}^{z} g(s, y(s), y'(s))dsdz = k_1 + k_2(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{z} f(s, y(s), y'(s))dsdz.$$

(2) let us consider y is (1)-differentiable and y' is (2)-differentiable, then the equivalent integral form of Eq. (2.1.1) can be written under (2)-differentiability of y' as follows:

$$y'(x) + \int_{x_0}^{z} g(s, y(s), y'(s))ds = k_2 \ominus (-1)\int_{x_0}^{z} f(s, y(s), y'(s))ds, \tag{2.2.2}$$

and then based on (1)-differentiability of y, Eq. (2.2.2) is converted to:

$$y(x) + \int_{x_0}^{x} \int_{x_0}^{z} g(s, y(s), y'(s))dsdz = k_1 + k_2(x - x_0) \ominus (-1) \int_{x_0}^{x} \int_{x_0}^{z} f(s, y(s), y'(s))dsdz.$$

(3) let us consider y is (2)-differentiable and y' is (1)-differentiable, then the equivalent integral form of Eq. (2.1.1) can be written under (1)-differentiability of y' as follows:

$$y'(x) + \int_{x_0}^{z} g(s, y(s), y'(s))ds = k_2 + \int_{x_0}^{z} f(s, y(s), y'(s))ds, \tag{2.2.3}$$

and then based on (2)-differentiability of y, Eq. (2.2.3) is converted to:

$$y(x) + \int_{x_0}^{x} \int_{x_0}^{z} g(s, y(s), y'(s))dsdz = k_1 \ominus (-1) \left(k_2(x - x_0) + \int_{x_0}^{x} \int_{x_0}^{z} f(s, y(s), y'(s))dsdz \right).$$

(4) let us consider y and y' are (2)-differentiable fuzzy-valued functions, then the equivalent integral form of Eq. (2.1.1) can be written under (2)-differentiability of y' as follows:

$$y'(x) + \int_{x_0}^{z} g(s, y(s), y'(s))ds = k_2 \ominus (-1) \int_{x_0}^{z} f(s, y(s), y'(s))ds, \tag{2.2.4}$$

and then based on (2)-differentiability of y, we get:

$$\begin{aligned} y(x) + \int_{x_0}^{x} \int_{x_0}^{z} g(s, y(s), y'(s))dsdz &= k_1 \ominus \\ &\quad (-1) \left(k_2(x - x_0) \ominus (-1) \int_{x_0}^{x} \int_{x_0}^{z} f(s, y(s), y'(s))dsdz \right), \end{aligned}$$

which completes the proof. $\square$

Here, we describe the proposed method in order to solve fuzzy linear differential equations through some examples both first and second order fuzzy linear differential equations.

2.2.1 First order FDEs

Here, we explain our new method through solving some well-known examples.

Consider the following simplest first order fuzzy linear differential equation with fuzzy initial value:

$$\begin{cases} y'(x) = y(x), \\ y(0) = y_0 \in E \end{cases} \tag{2.2.5}$$

Taking an integral on both sides of Eq. (2.2.5) leads to obtain its equivalent integral form under case (1)-differentiability of y as follows:

$$y(x) = y(0) + \int_0^x y(t)dt. \tag{2.2.6}$$

Eq. (2.2.6) can be written as follows:

$$(1 - \boldsymbol{J})\underline{y}(x;r) = \underline{y}(0;r), \quad (1 - \boldsymbol{J})\overline{y}(x;r) = \overline{y}(0;r) \tag{2.2.7}$$

where $\boldsymbol{J}\underline{y}(t;r) = \int_0^x \underline{y}(t;r)dt$ and $\boldsymbol{J}\overline{y}(t;r) = \int_0^x \overline{y}(t;r)dt$.

Then, by taking inverse of $(1 - \mathbf{J})$, $(1 - \mathbf{J})^{-1}$, on both sides of Eq. (2.2.7) we get:

$$\underline{y}(x;r) = (1 - \boldsymbol{J})^{-1}\left(\underline{y}(0;r)\right), \quad \overline{y}(x;r) = (1 - \boldsymbol{J})^{-1}\left(\overline{y}(0;r)\right). \tag{2.2.8}$$

Using

$$(1 - \boldsymbol{J})^{-1} = 1 + \boldsymbol{J} + \boldsymbol{J}^2 + \boldsymbol{J}^3 + \ldots \tag{2.2.9}$$

under assumption $||\mathbf{J}|| < 1$, we have for $r \in (0, 1]$

$$\underline{y}(x;r) = \left(1 + \boldsymbol{J} + \boldsymbol{J}^2 + \boldsymbol{J}^3 + \ldots\right)\underline{y}(0;r) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\underline{y}(0;r), \tag{2.2.10}$$

$$\overline{y}(x;r) = \left(1 + \boldsymbol{J} + \boldsymbol{J}^2 + \boldsymbol{J}^3 + \ldots\right)\overline{y}(0;r) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\overline{y}(0;r). \tag{2.2.11}$$

Finally, we get the solution of Eq. (2.2.5) under (1)-differentiability as follows:

$$\underline{y}(x;r) = \underline{y}(0;r)e^x, \quad \overline{y}(x;r) = \overline{y}(0;r)e^x, \quad \forall r \in (0,1].$$

Also, consider another well-known example in sense of (2)-differentiability as following:

$$\begin{cases} y'(x) = -y(x), \\ y(0) = y_0 \in E \end{cases} \tag{2.2.12}$$

Hence, the equivalent integral form of Eq. (2.2.12) is expressed as:

$$y(x) = y(0) \ominus \int_0^x y(t)dt.$$

Similarly, we can express the above equation in accordance with its lower and upper functions as following:

$$(1 + \boldsymbol{J})\underline{y}(x;r) = \underline{y}(0;r), \quad (1 + \boldsymbol{J})\overline{y}(x;r) = \overline{y}(0;r)$$

Then, by assumption $\|\boldsymbol{J}\| < 1$ we obtain for $r \in (0,1]$:

$$\underline{y}(x;r) = \left(1 - \boldsymbol{J} + \boldsymbol{J}^2 - \boldsymbol{J}^3 + \ldots\right)\underline{y}(0;r) = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\underline{y}(0;r),$$

and

$$\overline{y}(x;r) = \left(1 - \boldsymbol{J} + \boldsymbol{J}^2 - \boldsymbol{J}^3 + \ldots\right)\overline{y}(0;r) = \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\overline{y}(0;r).$$

Finally, we derive the solution of Eq. (2.2.12) in the closed form as follows:

$$\underline{y}(x,r) = \underline{y}(0;r)e^{-x}, \quad \overline{y}(x;r) = \overline{y}(0;r)e^{-x}.$$

Consider another example [8] as following:

$$\begin{cases} y'(x) = -y(x) + x + 1, \\ y(0) = y_0 \in E \end{cases} \tag{2.2.13}$$

Taking an integral on both sides of Eq. (2.2.13) and also suppose that $y(x)$ is (2)-differentiable, then we get:

$$\underline{y}(x;r) = (1+\boldsymbol{J})^{-1}\underline{y}(0;r) + (1+\boldsymbol{J})^{-1}\left(x + \frac{x^2}{2}\right),$$

and

$$\overline{y}(x;r) = (1+\boldsymbol{J})^{-1}\overline{y}(0;r) + (1+\boldsymbol{J})^{-1}\left(x + \frac{x^2}{2}\right).$$

Hence, we obtain solution in the closed form as follows:

$$\underline{y}(x;r) = \underline{y}(0;r)e^{-x} + x, \quad \overline{y}(x;r) = \overline{y}(0;r)e^{-x} + x.$$

Remark 2.2.1. Please notice that we solved Eqs. (2.2.12)-(2.2.13) in sense of (2)-differentiability, since we seek the asymptotic stable solution of fuzzy linear differential equations (for more detail see [23]). To this end, the deterministic solutions of Eqs. (2.2.12)-(2.2.13) tend to zero when $x \longrightarrow \infty$, i.e., $diam\{y(x)\} \longrightarrow 0$. So, we should chose an appropriate kind of differentiability in order to solve considered fuzzy linear equations both first and second order fuzzy linear differential equations, whose the behaviour of fuzzy solutions are as same as deterministic solutions.

2.2.2 Second order FDEs

Now, we extend the proposed method for solving second order fuzzy linear differential equations. Analogously, we describe the method through solving some examples in details.

Let us consider the following second order fuzzy linear differential equations:

$$\begin{cases} y''(x) + y(x) = \sigma_0, \ \sigma_0 = (r, 2-r), \\ y(0,r) = (r-1, 1-r), \\ y'(0,r) = (r-1, 1-r) \end{cases} \tag{2.2.14}$$

Based on various types of differentiability, we have the following 4 cases:

Case I. Let us consider $y(x)$ and $y'(x)$ are (1)-differentiable, then using Theorem 2.2.1, we get:

$$y(x) + \int_0^x \int_0^z y(s)dsdz = y(0) + y'(0)x + \sigma_0 \frac{x^2}{2},$$

or equivalently, we get for all $r \in (0, 1]$:

$$(1 + \boldsymbol{J}^2)\underline{y}(x; r) = \underline{y}(0; r) + \underline{y}'(0; r)x + \underline{\sigma_0}\frac{x^2}{2}$$

and

$$(1 + \boldsymbol{J}^2)\overline{y}(x; r) = \overline{y}(0; r) + \overline{y}'(0; r)x + \overline{\sigma_0}\frac{x^2}{2}$$

where $\boldsymbol{J}^2\underline{y}(t; r) = \int_0^x \int_0^z \underline{y}(t; r)dtdz$ and $\boldsymbol{J}^2\overline{y}(t; r) = \int_0^x \int_0^z \overline{y}(t; r)dtdz$.

Then,by taking inverse of $(1 + \mathbf{J}^2)$, $\left(1 + \mathbf{J}^2\right)^{-1}$, on both sides of above equations, we get:

$$\underline{y}(x; r) = \left(1 + \boldsymbol{J}^2\right)^{-1} \left[\underline{y}(0; r) + \underline{y}'(0; r)x + \underline{\sigma_0}\frac{x^2}{2}\right] \tag{2.2.15}$$

and

$$\overline{y}(x; r) = \left(1 + \boldsymbol{J}^2\right)^{-1} \left[\overline{y}(0; r) + \overline{y}'(0; r)x + \overline{\sigma_0}\frac{x^2}{2}\right]. \tag{2.2.16}$$

Also, by using:

$$\left(1+\boldsymbol{J}^2\right)^{-1} = \frac{1}{1+\boldsymbol{J}^2} = 1 - \mathbf{J}^2 + \mathbf{J}^4 - \mathbf{J}^6 + \mathbf{J}^8 - \ldots, \tag{2.2.17}$$

we get:

$$\begin{aligned}
\left(1+\boldsymbol{J}^2\right)^{-1} \underline{y}(0;r) &= \left(1 - \mathbf{J}^2 + \mathbf{J}^4 - \mathbf{J}^6 + \mathbf{J}^8 - \ldots\right) \underline{y}(0;r) \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \ldots\right)(r-1),
\end{aligned}$$

$$\begin{aligned}
\left(1+\boldsymbol{J}^2\right)^{-1} \left(\underline{y}'(0;r)x\right) &= \left(1 - \mathbf{J}^2 + \mathbf{J}^4 - \mathbf{J}^6 + \mathbf{J}^8 - \ldots\right) \left(\underline{y}'(0;r)x\right) \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \ldots\right)(r-1),
\end{aligned}$$

$$\begin{aligned}
\left(1+\boldsymbol{J}^2\right)^{-1} \left(\underline{\sigma_0}\frac{x^2}{2}\right) &= \left(1 - \mathbf{J}^2 + \mathbf{J}^4 - \mathbf{J}^6 + \mathbf{J}^8 - \ldots\right) \left(\underline{\sigma_0}\frac{x^2}{2}\right) \\
&= \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \frac{x^{10}}{10!} \ldots\right) r.
\end{aligned}$$

Then, by substituting obtained results into the Eq. (2.2.15) we get:

$$\underline{y}(x;r) = (r-1)\left(1 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots\right) \tag{2.2.18}$$

So, we get the lower function of solution of Eq. (2.2.14) in closed form by using Eq. (2.2.18) as follows:

$$\underline{y}(x;r) = r(1+\sin(x)) - \sin(x) - \cos(x).$$

Furthermore, one can easily obtained the upper function of solution of Eq. (2.2.14) by similar procedure as follows:

$$\overline{y}(x;r) = (2-r)(1+\sin(x)) - \sin(x) - \cos(x).$$

In this case, no solution exist, since y' is not (1)-differentiable fuzzy-valued function. Clearly, this is agree with the Khastan et al.'s result [51].

Please notice that the same procedure is done for determining the solutions of Eq. (2.2.14) under three following cases:

Case II. Let us consider y is (1)-differentiable and y' is (2)-differentiable, then using Theorem 2.2.1, we get:

$$y(x) + \int_0^x \int_0^z y(s)dsdz = y(0) + y'(0)x \ominus (-1)\sigma_0 \frac{x^2}{2}.$$

Then, similar to previous case we get the lower and upper functions of solution of Eq. (2.2.14) in the closed form for $r \in (0, 1]$ as follows:

$$\underline{y}(x; r) = r(1 + \sinh(x)) - \sinh(x) - \cos(x),$$

and

$$\overline{y}(x; r) = (2 - r)(1 + \sinh(x)) - \sinh(x) - \cos(x).$$

Similar **Case I**, in this case no solution exists.

Case III. Let us consider y is (2)-differentiable and y' is (1)-differentiable, then by using Theorem 2.2.1 we get:

$$y(x) + \int_0^x \int_0^z y(s)dsdz = y(0) \ominus (-1) \left(y'(0)x + \sigma_0 \frac{x^2}{2} \right).$$

Then, the lower and upper functions of solution of Eq. (2.2.14) are derived for all $x \in \big(0, \ln(\sqrt{2} + 1)\big)$ as following:

$$\underline{y}(x; r) = r(1 - \sinh(x)) + \sinh(x) - \cos(x),$$

and

$$\overline{y}(x;r) = (2-r)(1-\sinh(x)) + \sinh(x) - \cos(x).$$

In this case, since y is (2)-differentiable and y' is (1)-differentiable, the solution is acceptable.

Case IV. Let us consider y and y' are (2)-differentiable, then using Theorem 2.2.1, we get:

$$y(x) + \int_0^x \int_0^z y(s)dsdz = y(0) \ominus (-1)\left(y'(0)x \ominus (-1)\sigma_0 \frac{x^2}{2}\right).$$

Then, the lower and upper functions of solution of Eq. (2.2.14) in the closed form are obtained for $x \in (0, \frac{\pi}{2})$ as following:

$$\underline{y}(x;r) = r(1-\sin(x)) + \sin(x) - \cos(x),$$

and

$$\overline{y}(x;r) = (2-r)(1-\sin(x)) + \sin(x) - \cos(x).$$

Remark 2.2.2. It is easy to verify that the obtained results are agree with the previously reported solutions both first and second order [8, 51]. Also, by using Khastan et al.'s method, we should solve 4 related systems for determining the solutions of Eq. (2.2.14) for 4 cases. However, by using proposed method we used only the equivalent integral form of original problems for obtaining the solutions. It is clear that by increasing the order of differentiability to N, we should solve 2^N related systems when the Khatsan et al.'s method is applied.

Also, we solve a simple example in order to show the applicability of integral form of fuzzy linear equations which leads to obtain the solutions directly, instead of solving it by using 4 related systems [51].

Consider the following second order fuzzy linear differential equation:

$$\begin{cases} y''(x) = \sigma_0, \ \sigma_0 = (r-1, 1-r), \\ y(0,r) = (r-1, 1-r), \\ y'(0,r) = (r-1, 1-r), r \in (0,1] \end{cases} \tag{2.2.19}$$

Based on various types of differentiability, we have the following 4 cases:

Case I. Let us consider y and y' are (1)-differentiable, then by using case (1) in Theorem 2.2.1, we have:

$$y(x) = y(0) + y'(0)x + \int_0^x \int_0^z \sigma_0 ds dz.$$

Then, we can easily obtain the solution of Eq. (2.2.19) directly for all $x \geq 0$ and $r \in (0,1]$ as follows:

$$\underline{y}(x;r) = (r-1)\left(1 + x + \frac{x^2}{2}\right),$$

and

$$\overline{y}(x;r) = (1-r)\left(1 + x + \frac{x^2}{2}\right).$$

Case II. Let us consider y is (1)-differentiable and y' is (2)-differentiable, then using Theorem 2.2.1, we get:

$$y(x) = y(0) + y'(0)x \ominus (-1)\int_0^x \int_0^z \sigma_0 ds dz.$$

Then, the lower and upper functions of solution of Eq. (2.2.19) is obtained directly for all $x \in (0,1)$ as following:

$$\underline{y}(x;r) = (r-1)\left(1 + x - \frac{x^2}{2}\right),$$

and

$$\overline{y}(x;r) = (1-r)\left(1 + x - \frac{x^2}{2}\right).$$

Case III. Let us consider y is (2)-differentiable and y' is (1)-differentiable, then using Theorem 2.2.1, we get:

$$y(x) = y(0) \ominus (-1)\left(y'(0)x + \int_0^x \int_0^z \sigma_0 ds dz\right).$$

Then, we obtain the lower and upper functions of solution of Eq. (2.2.19) for all $x \in (0, \sqrt{3} - 1)$ as following:

$$\underline{y}(x;r) = (r-1)\left(1 - x - \frac{x^2}{2}\right),$$

and

$$\overline{y}(x;r) = (1-r)\left(1 - x - \frac{x^2}{2}\right).$$

Case IV. Let us consider y and y' are (2)-differentiable, then using Theorem 2.2.1, we get:

$$y(x) = y(0) \ominus (-1)\left(y'(0)x \ominus (-1)\int_0^x \int_0^z \sigma_0 ds dz\right).$$

Then, for all $x \in (0,1)$ we obtain the lower and upper functions of solution as following:

$$\underline{y}(x;r) = (r-1)\left(1 - x + \frac{x^2}{2}\right),$$

and

$$\overline{y}(x;r) = (1-r)\left(1 - x + \frac{x^2}{2}\right).$$

Obviously, these are agree with the Khastan et al.'s results [51].

Remark 2.2.3. One can easily seen that the usage of integral form of original problem in this special case leads to obtain the solution of Eq. (2.2.19) directly, but even in this special case we should solve 4 systems of equations for determining the solutions of Eq. (2.2.19) based on the Khastan et al.'s method [51]. It is easy to verify that since the right hand-side of Eq. (2.2.19) is independent with respect to y and y', we could derive the solution directly, instead of using proposed method.

2.3 1-Cut Expansion Method

Here, a new approach for solving first order fuzzy differential equations with fuzzy initial value is considered under generalized differentiability. In order to obtain solution of FDE, we extend the 1-cut solution of original problem. This extension is constructed based on the allocating some unknown spreads to 1-cut solution, then created value is replaced in the original FDE. However obtaining solutions of FDE is equivalent to determine the unknown spreads while 1-cut solution is derived via previous step (in general, 1-cut of FDE is interval differential equation).

Moreover, we extend the concepts of united solution set, tolerable solution set and controllable solution set for fuzzy differential equations. So, we will find solutions of first order fuzzy differential equation which are placed in mentioned solution sets. Moreover, we define three type of spreads while one of them is linear combination of the others. Such spread and related solution has pessimistic/optimistic attitude which is a new point of view to numerical solution of FDEs. Clearly, such property allows to the decision maker to inference or analyze the system in the real senses based on the pessimistic or optimistic desires. It seems that proposed method has flexible structure in order to obtain numerical solutions of FDEs in different attitude.

We consider the following first order fuzzy differential equation:

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0, \end{cases} \tag{2.3.1}$$

where $f : [a, b] \times E \longrightarrow E$ is fuzzy-valued function, $\widetilde{y}_0 \in E$.

Now, we describe our propose approach for solving FDE (2.3.1). In the beginning, we shall solve FDE (2.3.1) in sense of 1-cut as following:

$$\begin{cases} (y')^{[1]}(t) = f^{[1]}(t, y(t)), \\ y^{[1]}(t_0) = y_0^{[1]}, \ t_0 \in [0, T] \end{cases} \tag{2.3.2}$$

If Eq.(2.3.2) be a crisp differential equation we can solve it as usual, otherwise, if Eq.(2.3.2) be an interval differential equation we will solve it by Stefanini et al.'s method which is proposed and discussed in [82]. Notice that the solution of differential equation (2.3.2) is presented with notation $y^{[1]}$. Then, some unknown left spread $r_1(t; r)$ and right spread $r_2(t; r)$ are allocated to the 1-cut solution for all $0 \leq r \leq 1$. So, this approach leads to obtain

$$y(t) = \left[\underline{y}(t; r), \overline{y}(t; r)\right] = \left[\underline{y^{[1]}(t)} - r_1(t; r), \overline{y^{[1]}}(t) + r_2(t; r)\right] \tag{2.3.3}$$

as unknown solution of original FDE (2.3.1), then Eq.(2.3.3) is replaced into FDE (2.3.1). Hence, we have the following:

$$y'(t) = \left[\underline{y^{[1]}}(t) - r_1(t, r), \overline{y^{[1]}}(t) + r_2(t; r)\right]' =$$

$$\left[\underline{f}(t,\underline{y^{[1]}}(t)-r_1(t;r),\overline{y^{[1]}}(t)+r_2(t,r)),\overline{f}(t,\underline{y^{[1]}}(t)-r_1(t,r),\overline{y^{[1]}}(t)+r_2(t;r))\right].$$

Please notice that, we assumed the considered spreads and 1-cut solution are differentiable. Consequently, based on type of differentiability we have two following cases:

Case I. Suppose that y in Eq.(2.3.3) is (1)-differentiable, then we get:

$$y'(t)=\left[(\underline{y^{[1]}})'(t)-r_1'(t,r),(\overline{y^{[1]}})'(t)+r_2'(t,r)\right] \tag{2.3.4}$$

where, $r_1'(t,r)=\frac{\partial r_1(t,r)}{\partial t}$ and $r_2'(t,r)=\frac{\partial r_2(t,r)}{\partial t}$ for all $0\le r\le 1$. Consider Eq.(2.3.4) and original FDE (2.3.1), then we have the following for all $r\in[0,1]$:

$$\begin{cases} (\underline{y^{[1]}})'(t)-r_1'(t,r)=\underline{f}(t,\underline{y^{[1]}}(t)-r_1(t,r),\overline{y^{[1]}}(t)+r_2(t,r)),\ t_0\le t\le T \\ \\ (\overline{y^{[1]}})'(t)+r_2'(t,r)=\overline{f}(t,\underline{y^{[1]}}(t)-r_1(t,r),\overline{y^{[1]}}(t)+r_2(t,r)),\ t_0\le t\le T \end{cases} \tag{2.3.5}$$

Moreover, we modified fuzzy initial value y_0 in terms of unknown left and right spreads and 1-cut solution. Consider fuzzy initial value $y(t_0)=[\underline{y}(t_0,r),\overline{y}(t_0,r)]$ for all $0\le r\le 1$, then we can rewrite the lower and upper functions $\underline{y}(t_0,r)$ and $\overline{y}(t_0,r)$, respectively as following:

$$\begin{cases} \underline{y}(t_0,r)=\underline{y^{[1]}}(t_0)-r_1(t_0,r), \\ \\ \overline{y}(t_0,r)=\overline{y^{[1]}}(t_0)+r_2(t_0,r), \end{cases} \tag{2.3.6}$$

Thus, Eq.(2.3.5) and Eq.(2.3.6) simultaneously lead to obtain the following ODEs:

$$\begin{cases} (\underline{y^{[1]}})'(t)-r_1'(t,r)=\underline{f}(t,\underline{y^{[1]}}(t)-r_1(t,r),\overline{y^{[1]}}(t)+r_2(t,r)), \\ (\overline{y^{[1]}})'(t)+r_2'(t,r)=\overline{f}(t,\underline{y^{[1]}}(t)-r_1(t,r),\overline{y^{[1]}}(t)+r_2(t,r)), \\ \underline{y}(t_0,r)=\underline{y^{[1]}}(t_0)-r_1(t_0,r), \\ \overline{y}(t_0,r)=\overline{y^{[1]}}(t_0)+r_2(t_0,r), \end{cases} \tag{2.3.7}$$

Clearly, in above ODEs (2.3.7), only left and right spreads $r_1(t,r)$ and $r_2(t,r)$ are unknown parameters. So, ODEs (2.3.7) can be rewritten as following:

$$\begin{cases} r_1'(t,r) = H_1(t, r_1(t,r), r_2(t,r)), \\ r_2'(t,r) = H_2(t, r_1(t,r), r_2(t,r)), \\ r_1(t_0,r) = \underline{y^{[1]}}(t_0) - \underline{y}(t_0,r), \\ r_2(t_0,r) = \overline{y}(t_0,r) - \overline{y^{[1]}}(t_0),\ 0 \le r \le 1,\ t_0 \in [0,T] \end{cases} \tag{2.3.8}$$

Indeed, we will find spreads $r_1(t,r)$ and $r_2(t,r)$ by solving ODEs (2.3.8). Hence, solution of original FDE (2.3.1) is derived based on the obtained spreads and 1-cut solution as follows:

$$y(t,r) = [\underline{y}(t,r), \overline{y}(t,r)]$$

where for all $0 \le r \le 1$ and $t \in [0,T]$ such that

$$\underline{y}(t,r) = \underline{y^{[1]}}(t) - r_1(t,r), \quad \overline{y}(t,r) = \overline{y^{[1]}}(t) + r_2(t,r)$$

Case II. Suppose that y in Eq.(2.3.3) is (2)-differentiable, then we get:

$$y'(t,r) = \left[(\overline{y^{[1]}})'(t) + r_2'(t,r), (\underline{y^{[1]}})'(t) - r_1'(t,r)\right] \tag{2.3.9}$$

Similarly, ODEs (2.3.7) can be rewritten in sense of II-differentiability as following:

$$\begin{cases} (\overline{y^{[1]}})'(t) + r_2'(t,r) = \underline{f}(t, y^{[1]}(t) - r_1(t;r), y^{[1]}(t) + r_2(t,r)), \\ (\underline{y^{[1]}})'(t) - r_1'(t,r) = \overline{f}(t, y^{[1]}(t) - r_1(t;r), y^{[1]}(t) + r_2(t;r)), \\ \underline{y}(t_0,r) = \underline{y^{[1]}}(t_0) - r_1(t_0,r), \\ \overline{y}(t_0,r) = \overline{y^{[1]}}(t_0) + r_2(t_0,r), \end{cases} \tag{2.3.10}$$

Since, only unknown parameters in ODEs (2.3.10) are $r_1(t,r)$ and $r_2(t,r)$, we can rewrite (2.3.10) in terms of $r_1(t,r)$ and $r_2(t,r)$ and their derivatives. So, we have the following:

$$\begin{cases} r_2'(t,r) = H_1(t, r_1(t,r), r_2(t,r)), \\ r_1'(t,r) = H_2(t, r_1(t,r), r_2(t,r)), \\ r_1(t_0,r) = \underline{y^{[1]}}(t_0) - \underline{y}(t_0,r), \\ r_2(t_0,r) = \overline{y}(t_0;r) - \overline{y^{[1]}}(t_0),\ 0 \le r \le 1,\ t_0 \in [0,T] \end{cases} \tag{2.3.11}$$

Finally, by solving above ODEs (2.3.11) unknown spreads are determined and follows we can derive solution of original FDE (2.3.1) in sense of II-differentiability by using

$$\underline{y}(t,r) = \underline{y^{[1]}}(t) - r_1(t,r), \quad \overline{y}(t,r) = \overline{y^{[1]}}(t) + r_2(t,r) \tag{2.3.12}$$

for all $0 \le r \le 1$.

Notice that solution of original FDE (2.3.1) is assumed fuzzy-valued function and under such assumption we determined the unknown left and right spreads $r_1(t,r)$ and $r_2(t,r)$. However, we will check that obtained spreads lead to derive fuzzy-valued function as solution of original FDE (2.3.1).

Theorem 2.3.1. *Suppose that left spread $r_1(t,r)$ and right spread $r_2(t,r)$ are obtained from ODEs (2.3.8) or (2.3.11). Then the following affirmations are equivalent:*

(1) *$r_1(t,r)$ and $r_2(t,r)$ are nonincreasing positive functions for all $0 \le r \le 1$, $t_0 \le t \le T$,*

(2) *y is fuzzy-valued function.*

2.3.1 Three new solution sets

In this section, we try to extend concepts of united solution set (`USS`), tolerable solution set(`TSS`) and controllable solution set (`CSS`) to the theory of fuzzy differential equations.

Definition 2.3.1. Let us consider FDE (2.3.1), then united solution set, tolerable solution set and controllable solution set are defined, respectively, as following:

$$\mathtt{Y}_{\exists\exists} = \{\widehat{y}|\ \widehat{y}'(t) \cap f(t, y(t)) \neq \emptyset\}\,, \tag{2.3.13}$$

$$\mathtt{Y}_{\forall\exists} = \{\widehat{y}|\ \widehat{y}'(t) \subseteq f(t, y(t))\}\,, \tag{2.3.14}$$

$$\mathtt{Y}_{\exists\forall} = \{\widehat{y}|\ \widehat{y}'(t) \supseteq f(t, y(t)), \quad t \in [a, b]\}\,. \tag{2.3.15}$$

Subsequently, we are going to obtain solution of FDE which are placed in TSS or CSS. To this end, some discussions are given to construct a solution of FDE such that it has pessimistic/optimistic attitude where pessimistic attitude is happened in TSS and optimistic attitude is placed in CSS. Clearly, in this sense, we obtain a connected solution between TSS and CSS. However, this approach is coincide with real application while decision maker can obtain interested solution and can inference the systems in general cases.

Let us consider left and right spreads $r_1(t, r), r_2(t, r)$ are derived similar to previous section. Then, we define some spreads as following:

$$r^-(t, r) = \min\{r_1(t, r), r_2(t, r)\}, \quad t_0 \leq t \leq T,\ 0 \leq r \leq 1, \tag{2.3.16}$$

$$r^+(t, r) = \max\{r_1(t, r), r_2(t, r)\}, \quad t_0 \leq t \leq T,\ 0 \leq r \leq 1, \tag{2.3.17}$$

$$r^\lambda(t; r) = \lambda\ r^+(t, r) + (1 - \lambda)\ r^-(t; r), \quad t_0 \leq t \leq T,\ \lambda \in [0, 1],\ 0 \leq r \leq 1. \tag{2.3.18}$$

Also, we define new solutions corresponding to each spreads (2.3.16)-(2.3.18), respectively, as following:

$$y^-(t) = \left[\underline{y^{[1]}}(t) - r^-(t; r), \overline{y^{[1]}}(t) + r^-(t; r)\right], \tag{2.3.19}$$

$$y^{+}(t) = \left[\underline{y^{[1]}}(t) - r^{+}(t;r), \overline{y^{[1]}}(t) + r^{+}(t;r)\right], \tag{2.3.20}$$

$$y^{\lambda}(t) = \left[\underline{y^{[1]}(t)} - r^{\lambda}(t;r), \overline{y^{[1]}}(t) + r^{\lambda}(t;r)\right], \tag{2.3.21}$$

Proposition 2.3.2. *Let us consider the spreads (2.3.16)-(2.3.17) and corresponding solutions (2.3.19)-(2.3.20), then we have the following:*

(1) $y^{-}(t) \in \mathit{TSS}$

(2) $y^{+}(t) \in \mathit{CSS}$

Proposition 2.3.3. *Let us consider $y^{\lambda}(t)$ which is defined by (2.3.21). Also, suppose that $\{\lambda_k\}_{k=0}^{\infty}$ is a nondecreasing consequence with initial value $\lambda_0 = 0$ such that $\lambda_k \longrightarrow 1$ when $k \longrightarrow \infty$. Then*

$$y^{\lambda_k}(t) = y^{-}(t) \in \mathit{TSS} \longrightarrow y^{+}(t) \in \mathit{CSS}, \quad \lambda_0 = 0, \ k \longrightarrow \infty$$

Proposition 2.3.4. *Let us consider $y^{\lambda}(t)$ which is defined by (2.3.21). Also, suppose that $\{\lambda_k\}_{k=0}^{\infty}$ is a nonincreasing consequence with initial value $\lambda_0 = 1$ such that $\lambda_k \longrightarrow 0$ when $k \longrightarrow \infty$. Then*

$$y^{\lambda_k}(t) = y^{+}(t) \in \mathit{CSS} \longrightarrow y^{-}(t) \in \mathit{TSS}, \quad \lambda_0 = 1, \ k \longrightarrow \infty$$

2.3.2 Examples

In this part, some examples are given to illustrate the technique. Notice that Example 2.3.1 is solved under I-differentiability and Example 2.3.2 is considered under II-differentiability.

Example 2.3.1. *Let us consider the following FDE*

$$\begin{cases} y'(t) = y(t), \\ \\ y(0;r) = [1+r, 5-2r], \;\; 0 \le r \le 1 \end{cases} \tag{2.3.22}$$

Based on the proposed approach, 1-cut system is derived as follows:

$$\begin{cases} (y^{[1]})'(t) = y^{[1]}(t), \\ y^{[1]}(0) = [2,3], \end{cases} \tag{2.3.23}$$

Above interval differential equation is solved by Stefanini et. al's method [**?**] as follows:

$$\begin{cases} (\underline{y^{[1]}})'(t) = \underline{y^{[1]}(t)}, \\ (\overline{y^{[1]}})'(t) = \overline{y^{[1]}(t)}, \\ \underline{y^{[1]}(0)} = 2, \\ \overline{y^{[1]}(0)} = 3, \end{cases} \tag{2.3.24}$$

Then, solution of Eq.(2.3.24) is obtained $y(t) = [2e^t, 3e^t]$. Based on ODEs (2.3.8), we get:

$$\begin{cases} (y^{[1]})'(t) - r_1'(t;r) = y^{[1]}(t) - r_1(t;r), \\ (y^{[1]})'(t) + r_2'(t;r) = y^{[1]}(t) - r_2(t;r), \\ r_1(0;r) = \underline{y^{[1]}}(0) - \underline{y}(0;r) = 1-r, \\ r_2(0;r) = \overline{y}(0;r) - \overline{y^{[1]}}(0) = 2(1-r), \end{cases} \tag{2.3.25}$$

Hence, ODEs (2.3.25) is rewritten based on the left and right unknown spreads as follows:

$$\begin{cases} r_1'(t;r) = r_1(t;r), \\ r_2'(t;r) = r_2(t;r), \\ r_1(0;r) = 1-r, \\ r_2(0;r) = 2(1-r), \end{cases} \tag{2.3.26}$$

By solving ODEs (2.3.26), we get the spreads as following:

$$r_1(t;r) = r_1(0;r)\; e^t = (1-r)e^t$$

$$r_2(t;r) = r_2(0;r)\ e^t = (2-2r)e^t$$

Finally, solution of original FDE (2.3.22)

$$y(t) = [(1+r)\ e^t, (5-2r)e^t].$$

Clearly, our approach is coincide with the results of Bede et. al [?], Chalco-Cano et.al [29] and similar papers. It seems that proposed method has new point of view to solve FDE based on extending the 1-cut solution.

Now, we derive new solutions which are placed in `CSS` or `TSS`. Additionally, some pessimistic/optimistic solution is obtained, that is connected solution between `TSS` and `CSS`. So, by applying Eqs.(2.3.16)-(2.3.18) we have the following:

$$r^-(t;r) = \min\{(1-r)\ e^t, (2-2r)e^t\} = (1-r)\ e^t, \quad \forall t \in [t_0, T],,\ 0 \le r \le 1,$$

$$r^+(t;r) = \max\{(1-r)\ e^t, (2-2r)e^t\} = (2-2r)\ e^t, \quad \forall t \in [t_0, T],,\ 0 \le r \le 1,$$

$$r^\lambda(t;r) = \lambda r^+(t;r) + (1-\lambda)r^-(t;r) = (1-\lambda)(1-r)e^t, \quad \forall t \in [t_0, T]$$

Therefore, corresponding solutions for above spreads are achieved as:

$$y^-(t;r) = \left[(1+r)e^t, (4-r)e^t\right],$$
$$y^+(t;r) = \left[2re^t, (5-2r)e^t\right],$$
$$y^\lambda(t;r) = \left[(1+\lambda+r(1-\lambda))e^t, (4-\lambda+r(\lambda-1))e^t\right].$$

It easy to see that $y^-(t) \in$ `TSS`, $y^+(t) \in$ `CSS` and $y^\lambda(t)$ is pessimistic/optimistic solution for each $\lambda \in [0,1]$.

Example 2.3.2. *Let us consider the following FDE*

$$\begin{cases} y'(t) = -y(t), \\ \\ y(0;r) = [1+r, 5-r], \ \ 0 \le r \le 1 \end{cases} \tag{2.3.27}$$

1-cut solution of above FDE is derived via Stefanini et al's method as $y^{[1]}(t) = [2e^{-t}, 4e^{-t}]$. Similar to Example 2.3.1, the original FDE is transformed to the following ODEs

$$\begin{cases} r_1'(t;r) = -r_1(t;r), \\ r_2'(t;r) = -r_2(t;r), \\ r_1(0;r) = 1-r, \\ r_2(0;r) = 1-r, \end{cases} \tag{2.3.28}$$

By solving ODEs (2.3.28), we get spreads

$$r_1(t;r) = r_1(0;r) \ e^{-t} = (1-r)e^{-t}$$

$$r_2(t;r) = r_2(0;r) \ e^{-t} = (1-r)e^{-t}$$

Finally, solution of original FDE (2.3.27) is derived

$$y(t) = [(1+r) \ e^{-t}, (5-r)e^{-t}].$$

Analogously, we determine new spreads based on Eqs.(2.3.16)-(2.3.18) as following:

$$r^-(t;r) = r^+(t;r) = (1-r) \ e^{-t}, \quad \forall t \in [t_0, T], \ 0 \le r \le 1,$$

$$r^\lambda(t;r) = \lambda r^+(t;r) + (1-\lambda) r^-(t;r) = (1-r)e^{-t}, \quad \forall t \in [t_0, T], \quad \forall \lambda \in [0,1]$$

Then, we obtained relation solutions:

$$y^-(t;r) = y^+(t;r) = y^\lambda(t;r) = \left[(1+r)e^{-t}, (5-r)e^{-t}\right], \forall t \in [t_0, T], \ \forall \lambda \in [0,1], \ 0 \le r \le 1.$$

Chapter 3

Fuzzy Differential Equations of Fractional Order

3.1 Introduction

This chapter deals with the solutions of fuzzy fractional differential equations (FFDEs) under Riemann-Liouville differentiability by fuzzy Laplace transforms. In order to solve FFDEs, it is necessary to know the fuzzy Laplace transform of the Riemann-Liouville H-derivative of f, $\left({}^{RL}D^{\beta}_{a^+}f\right)(x)$. The virtue of $\mathbf{L}\left[\left({}^{RL}D^{\beta}_{a^+}f\right)(x)\right]$ is that can be written in terms of $\mathbf{L}[f(x)]$. Moreover, some illustrative examples are solved to show the efficiency and utility of Laplace transforms method.

We are going to solve FFDEs under Riemann-Liouville differentiability using fuzzy Laplace transforms. To do so, we investigate the Laplace transform of fractional derivatives which is important tool to solve FFDEs with Laplace transforms. To the best of our knowledge this is the first time in the literature that FFDEs are investigated under Riemann-Liouville differentiability using fuzzy Laplace transforms.

3.2 Riemann-Liouville differentiability

In this section, we state definition of fuzzy Riemann-Liouville integrals and derivatives under Hukuhara difference. We are giong to produce such definitions and statements similar to the non-fractional one in fuzzy context [22].
We denote $C^{\mathbb{F}}[a,b]$ as the space of all continuous fuzzy-valued functions on $[a,b]$. Also, we denote the space of all Lebesque integrable fuzzy-valued functions on the bounded interval $[a,b] \subset \mathbb{R}$ by $L^{\mathbb{F}}[a,b]$.

Now, we define a fuzzy Riemann-Liouville integral of fuzzy-valued function as follows:

Definition 3.2.1. Let $f \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$. The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:

$$\left(I_{a+}^{\beta} f\right)(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\beta}}, \quad x > a, \ 0 < \beta \leq 1. \tag{3.2.1}$$

Let us consider the r-cut representation of fuzzy-valued function f as $f(x,r) = \left[\underline{f}(x,r), \overline{f}(x,r)\right]$, for $0 \leq r \leq 1$, then we can indicate the fuzzy Riemann-Liouville integral of fuzzy-valued function f based on the lower and upper functions as following:

Theorem 3.2.1. *[13, 18]. Let $f \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$ is a fuzzy-valued function. The fuzzy Riemann-Liouville integral of a fuzzy-valued function f can be expressed as follows:*

$$\left(I^{\beta}_{a+}f\right)(x;r) = \left[\left(I^{\beta}_{a+}\underline{f}\right)(x;r), \left(I^{\beta}_{a+}\overline{f}\right)(x;r)\right], \quad 0 \leq r \leq 1, \tag{3.2.2}$$

where

$$\left(I^{\beta}_{a+}\underline{f}\right)(x;r) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\underline{f}(t;r)dt}{(x-t)^{1-\beta}}, \tag{3.2.3}$$

$$\left(I^{\beta}_{a+}\overline{f}\right)(x;r) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{\overline{f}(t;r)dt}{(x-t)^{1-\beta}}. \tag{3.2.4}$$

Now, we define fuzzy Riemann-Liouville fractional derivatives about order $0 < \beta < 1$ for fuzzy-valued function f.

Definition 3.2.2. Let $f \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$, x_0 in (a,b) and $\Phi(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f(t)dt}{(x-t)^{\beta}}$. We say that f is Riemann-Liouville H-differentiable about order $0 < \beta < 1$ at x_0, if there exists an element $\left({}^{RL}D^{\beta}_{a^+}f\right)(x_0) \in \mathbb{E}$, such that for $h > 0$ sufficiently small

$$(i) \qquad \left({}^{RL}D^{\beta}_{a^+}f\right)(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0+h) \ominus \Phi(x_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0-h)}{h},$$

or

$$(ii) \qquad \left({}^{RL}D^{\beta}_{a^+}f\right)(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0+h)}{-h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0-h) \ominus \Phi(x_0)}{-h},$$

or

$$(iii) \qquad \left({}^{RL}D^{\beta}_{a^+}f\right)(x_0) = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0+h) \ominus \Phi(x_0)}{h} = \lim_{h \longrightarrow 0^+} \frac{\Phi(x_0-h) \ominus \Phi(x_0)}{-h},$$

or

$$(iv) \qquad \left({}^{RL}D^{\beta}_{a^+}f\right)(x_0) = \lim_{h\longrightarrow 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0+h)}{-h} = \lim_{h\longrightarrow 0^+} \frac{\Phi(x_0)\ominus\Phi(x_0-h)}{h},$$

For sake of simplicity, we say that the fuzzy-valued function f is ${}^{RL}[(1)-\beta]-differentiable$ if it is differentiable as in the Definition 3.2.2 case (i), and f is ${}^{RL}[(2)-\beta]-differentiable$ if it is differentiable as in the Definition 3.2.2 case (ii) and so on for the other cases.

Theorem 3.2.2. *[13]. Let $f \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$, x_0 in (a,b) and $0<\beta<1$. Then:*

(i) Let us consider f is ${}^{RL}[(1)-\beta]$-differentiable fuzzy-valued function, then:

$$\left({}^{RL}D^{\beta}_{a^+}f\right)(x_0;r) = \left[\left({}^{RL}D^{\beta}_{a^+}\underline{f}\right)(x_0,r), \left({}^{RL}D^{\beta}_{a^+}\overline{f}\right)(x_0,r)\right], \quad 0 \le r \le 1, \tag{3.2.5}$$

(ii) Let us consider f is ${}^{RL}[(2)-\beta]$-differentiable fuzzy-valued function, then:

$$\left({}^{RL}D^{\beta}_{a^+}f\right)(x_0;r) = \left[\left({}^{RL}D^{\beta}_{a^+}\overline{f}\right)(x_0;r), \left({}^{RL}D^{\beta}_{a^+}\underline{f}\right)(x_0,r)\right], \quad 0 \le r \le 1, \tag{3.2.6}$$

where

$$\left({}^{RL}D^{\beta}_{a^+}\underline{f}\right)(x_0;r) = \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x \frac{\underline{f}(t;r)dt}{(x-t)^{\beta}}\right]_{x=x_0}, \tag{3.2.7}$$

and

$$\left({}^{RL}D^{\beta}_{a^+}\overline{f}\right)(x_0;r) = \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x \frac{\overline{f}(t,r)dt}{(x-t)^{\beta}}\right]_{x=x_0}. \tag{3.2.8}$$

Theorem 3.2.3. *Let $f \in C^{\mathbb{F}}[a,b] \cap L^{\mathbb{F}}[a,b]$ be a Riemann-Liouville differentiable of order $0<\beta<1$ on each point $x \in (a,b)$ in the sense of Definition 3.2.2 case (iii) or case (iv). Then $\left({}^{RL}D^{\beta}_{a^+}f\right)(x) \in \mathbb{R}$ for all $x \in (a,b)$.*

Proof. By substituting Φ in Theorem 7 [22] instead of f, the proof is obvious. $\square$

Remark 3.2.1. Please notice that the subjects of sum, multiplication and switching points of fuzzy-valued functions in order to determine types of Riemann-Liouville differentiability are out of the scope of this paper and will be studied carefully in the future.

3.3 The fuzzy Laplace transforms of Riemann-Liouville derivative

Theorem 3.3.1. *(Second translation theorem). Let us consider* $\mathbf{F}(p) = \mathbf{L}[f(x)]$. *Then:*

$$\mathbf{L}[u_a(x) \odot f(x-a)] = e^{-ap} \odot \mathbf{F}(p), \quad a \geq 0 \tag{3.3.1}$$

Proof. This follows from the basic fact that

$$\int_0^\infty e^{-st} \odot (u_a(x) \odot f(x-a))\, dx = \int_a^\infty e^{-px} \odot f(x-a) dt$$

and setting $\tau = t - a$, the right-hand integral becomes

$$\int_0^\infty e^{-p(\tau+a)} \odot f(\tau) d\tau = e^{-pa} \odot \int_0^\infty e^{-p\tau} \odot f(\tau) d\tau = e^{-ap} \odot \mathbf{F}(p). \quad \square$$

In order to solve FFDEs, it is necessary to know the fuzzy Laplace transform of the Riemann-Liouville H-derivative of f, $\left({}^{RL}D^{\beta}_{a^+}f\right)(x)$. The virtue of $\mathbf{L}\left[\left({}^{RL}D^{\beta}_{a^+}f\right)(x)\right]$ is that can be written in terms of $\mathbf{L}[f(x)]$.

Theorem 3.3.2. *(Derivative theorem). Suppose that $f \in C^{\mathbb{F}}[0,\infty) \cap L^{\mathbb{F}}[0,\infty)$. Then:*

$$\mathbf{L}\left[\left({}^{RL}D^{\beta}_{a^+}f\right)(x)\right] = p^{\beta}\mathbf{L}\left[f(t)\right] \ominus \ \left({}^{RL}D^{\beta-1}_{a^+}f\right)(0), \tag{3.3.2}$$

if f is ${}^{RL}[(1)-\beta]$-differentiable, and

$$\mathbf{L}\left[\left({}^{RL}D^{\beta}_{a^+}f\right)(x)\right] = - \ \left({}^{RL}D^{\beta-1}_{a^+}f\right)(0) \ominus \left(-p^{\beta}\mathbf{L}\left[f(t)\right]\right) \tag{3.3.3}$$

if f is ${}^{RL}[(2)-\beta]$-differentiable, provided the mentioned Hukuhara differences exist.

Proof. For arbitrary fixed $r \in [0,1]$ we have:

$$\begin{aligned}&p^{\beta}\mathbf{L}\left[f(x;r)\right] \ominus \left({}^{RL}D^{\beta-1}_{a^+}f\right)(0;r) = \\ &\left[p^{\beta}\ell[\underline{f}(x;r)] - \left({}^{RL}D^{\beta-1}_{a^+}\underline{f}\right)(0;r), p^{\beta}\ell[\overline{f}(x;r)] - \left({}^{RL}D^{\beta-1}_{a^+}\overline{f}\right)(0;r)\right].\end{aligned}$$

Since f is ${}^{RL}[(1)-\beta]$, we get:

$$\begin{aligned}\left({}^{RL}D^{\beta}_{a^+}f\right)(x;r) &= \left[\left(\underline{{}^{RL}D^{\beta}_{a^+}f}\right)(x;r), \left(\overline{{}^{RL}D^{\beta}_{a^+}f}\right)(x;r)\right] \qquad (3.3.4)\\ &= \left[\left({}^{RL}D^{\beta}_{a^+}\underline{f}\right)(x;r), \left({}^{RL}D^{\beta}_{a^+}\overline{f}\right)(x;r)\right]. \qquad (3.3.5)\end{aligned}$$

Hence, we have:

$$\ell\left[\left(\overline{{}^{RL}D^{\beta}_{a^+}f}\right)(x;r)\right] = \ell\left[\left({}^{RL}D^{\beta}_{a^+}\overline{f}\right)(x;r)\right] = p^{\beta}\ell\left[\overline{f}(x;r)\right] - \left({}^{RL}D^{\beta-1}_{a^+}\overline{f}\right)(0;r),$$

and

$$\ell\left[\left(\underline{{}^{RL}D^{\beta}_{a^+}f}\right)(x;r)\right] = \ell\left[\left({}^{RL}D^{\beta}_{a^+}\underline{f}\right)(x;r)\right] = p^{\beta}\ell\left[\underline{f}(x;r)\right] - \left({}^{RL}D^{\beta-1}_{a^+}\underline{f}\right)(0;r).$$

Then, we conclude that:

$$p^{\beta}\mathbf{L}[f(x;r)] \ominus \left({}^{RL}D_{a^+}^{\beta-1}f\right)(0;r) = \left[\ell\left[\left({}^{RL}D_{a^+}^{\beta}\underline{f}\right)(x;r)\right], \ell\left[\left({}^{RL}D_{a^+}^{\beta}\overline{f}\right)(x;r)\right]\right],$$

by linearity of $\mathbf{L}$,

$$p^{\beta}\mathbf{L}[f(x;r)] \ominus \left({}^{RL}D_{a^+}^{\beta-1}f\right)(0;r) = \mathbf{L}\left[\left({}^{RL}D_{a^+}^{\beta}\underline{f}\right)(x;r), \left({}^{RL}D_{a^+}^{\beta}\overline{f}\right)(x;r)\right].$$

Using Eq. (3.3.4), leads to obtain:

$$p^{\beta}\mathbf{L}[f(x;r)] \ominus \left({}^{RL}D_{a^+}^{\beta-1}f\right)(0;r) = \mathbf{L}\left[\left({}^{RL}D_{a^+}^{\beta}f\right)(x;r)\right].$$

Now we assume that f is ${}^{RL}[(2)-\beta]$, then for arbitrary fixed $r \in [0,1]$ we have:

$$\begin{aligned}&-\left({}^{RL}D_{a^+}^{\beta-1}f\right)(0;r) \ominus \left(-p^{\beta}\mathbf{L}\left[f(x;r)\right]\right) = \\ &\left[-\left(\overline{{}^{RL}D_{a^+}^{\beta-1}f}\right)(0;r) + p^{\beta}\ell[\overline{f}(x;r)], -\left(\underline{{}^{RL}D_{a^+}^{\beta-1}f}\right)(0,r) + p^{\beta}\ell[\underline{f}(x;r)]\right],\end{aligned}$$

since f is ${}^{RL}[(2)-\beta]$, we get:

$$\begin{aligned}\left({}^{RL}D_{a^+}^{\beta}f\right)(x;r) &= \left[\left(\underline{{}^{RL}D_{a^+}^{\beta}f}\right)(x;r), \left(\overline{{}^{RL}D_{a^+}^{\beta}f}\right)(x;r)\right] &(3.3.6)\\ &= \left[\left({}^{RL}D_{a^+}^{\beta}\overline{f}\right)(x;r), \left({}^{RL}D_{a^+}^{\beta}\underline{f}\right)(x;r)\right]. &(3.3.7)\end{aligned}$$

Thus, we have:

$$\begin{aligned}&-\left({}^{RL}D_{a^+}^{\beta-1}f\right)(0;r) \ominus \left(-p^{\beta}\mathbf{L}\left[f(x;r)\right]\right) = \\ &\left[p^{\beta}\ell\left[\overline{f}(x;r)\right] - \left({}^{RL}D_{a^+}^{\beta-1}\overline{f}\right)(0;r), p^{\beta}\ell\left[\underline{f}(x;r)\right] - \left({}^{RL}D_{a^+}^{\beta-1}\underline{f}\right)(0;r)\right].\end{aligned}$$

So, we have:

$$-\left({}^{RL}D_{a^+}^{\beta-1}f\right)(0;r)\ominus\left(-p^{\beta}\mathbf{L}\left[f(x;r)\right]\right)=\left[\ell\left[\left({}^{RL}D_{a^+}^{\beta}\overline{f}\right)(x;r)\right],\ell\left[\left({}^{RL}D_{a^+}^{\beta}\underline{f}\right)(x;r)\right]\right].$$

Then, we deduce:

$$-\left({}^{RL}D_{a^+}^{\beta-1}f\right)(0;r)\ominus\left(-p^{\beta}\mathbf{L}\left[f(x;r)\right]\right)=\mathbf{L}\left[\left({}^{RL}D_{a^+}^{\beta}f\right)(x;r)\right],$$

which completes the proof. □

3.4 Analytical solution

Consider the following fuzzy fractional differential equation of order $0<\beta<1$ with the initial condition

$$\begin{cases}\left({}^{RL}D_{a^+}^{\beta}y\right)(x)=f[x,y(x)]\\ \left({}^{RL}D_{a^+}^{\beta-1}y\right)(x_0)=\ {}^{RL}y_0^{(\beta-1)}\in\mathbb{E}\end{cases}\tag{3.4.1}$$

where $f\in C^{\mathbb{F}}(a,b)\cap L^{\mathbb{F}}(a,b)$ and $x_0\in(a,b)$.

3.4.1 Determining solutions

In this subsection, we provide the fuzzy Laplace transform and its inverse to derive solutions of FFDE (3.4.1). By taking Laplace transform on the both sides of Eq. (3.4.1), we get the following:

$$\mathbf{L}\left[\left({}^{RL}D_{a^+}^{\beta}y\right)(x)\right]=\mathbf{L}[f(x,y(x)].\tag{3.4.2}$$

Then, based on the types of Riemann-Liouville differentiability we have the following cases:

Case I. Let us consider y is a ${}^{RL}[(1)-\beta]$-differentiable function, then Eq.(3.4.2) is extended based on the its lower and upper functions as follows:

$$\begin{cases} \ell[\underline{f}(x,y(x);r)] = p^{\beta}\ell\left[\underline{y}(x;r)\right] - \left({}^{RL}D_{a^+}^{\beta-1}\underline{y}\right)(0;r),\ 0\le r\le 1, \\ \\ \ell[\overline{f}(x,y(x);r)] = p^{\beta}\ell\left[\overline{y}(x;r)\right] - \left({}^{RL}D_{a^+}^{\beta-1}\overline{y}\right)(0;r),\ 0\le r\le 1, \end{cases} \tag{3.4.3}$$

where

$$\underline{f}(x,y(x);r) = \min\left\{f(x,u)|u\in[\underline{y}(x;r),\overline{y}(x;r)]\right\} \tag{3.4.4}$$

and

$$\overline{f}(x,y(x);r) = \max\left\{f(x,u)|u\in[\underline{y}(x;r),\overline{y}(x;r)]\right\}. \tag{3.4.5}$$

To solve the linear system (3.4.3), for simplify we assume that:

$$\ell[\underline{y}(x;r)] = H_1(p;r)$$

$$\ell[\overline{y}(x;r)] = K_1(p;r)$$

where $H_1(p;r)$ and $K_1(p;r)$ are solutions of system (3.4.3). By using inverse Laplace transform, $\underline{y}(x;r)$ and $\overline{y}(x;r)$ are computed as following:

$$\underline{y}(x,r) = \ell^{-1}\left[H_1(p;r)\right], \tag{3.4.6}$$

$$\overline{y}(x;r) = \ell^{-1}\left[K_1(p;r)\right]. \tag{3.4.7}$$

Case II. Let us consider y is ${}^{RL}[(2)]-\beta]$-differentiable, then Eq. (3.4.2) can be written as follows:

$$\begin{cases} \ell[\underline{f}(x,y(x);r)] = p\ell[\underline{y}(x;r)] - \left({}^{RL}D_{a^+}^{\beta-1}\underline{y}\right)(0;r), \\ \\ \ell[\overline{f}(x,y(x);r)] = p\ell[\overline{y}(x;r)] - \left({}^{RL}D_{a^+}^{\beta-1}\overline{y}\right)(0;r). \end{cases} \tag{3.4.8}$$

where

$$\underline{f}(x, y(x); r) = \min\{f(x, u) | u \in [\underline{y}(x; r), \overline{y}(x; r)]\} \tag{3.4.9}$$

and

$$\overline{f}(x, y(x); r) = \max\{f(x, u) | u \in [\underline{y}(x; r), \overline{y}(x; r)].\} \tag{3.4.10}$$

to solve the linear system (3.4.8), we for simplify:

$$\ell[\underline{y}(x; r)] = H_2(p; r),$$

$$\ell[\overline{y}(x; r)] = K_2(p; r),$$

where $H_2(p; r)$ and $K_2(p; r)$ are solutions of system (3.4.8). By using inverse Laplace transform, $\underline{y}(x; r)$ and $\overline{y}(x; r)$ are computed as follows:

$$\underline{y}(x; r) = \ell^{-1}[H_2(p; r)], \tag{3.4.11}$$

$$\overline{y}(x; r) = \ell^{-1}[K_2(p; r)]. \tag{3.4.12}$$

3.4.2 Examples

In this section, we consider two examples in details to solve FFDEs under Riemann-Liouville H-differentiability.

Example 3.4.1. *(Fuzzy fractional nuclear decay equation). Let us consider the following FFDE:*

$$\begin{cases} \left({}^{RL}D^{\beta}_{0^+} y\right)(x) = \lambda \odot y(x), \ 0 < \beta, x < 1 \\ \left({}^{RL}D^{\beta-1}_{0^+} y\right)(0) = \ {}^{RL}y_0^{(\beta-1)} \in \mathbb{E} \end{cases}$$

where, y is the number of radionuclides present in a given radioactive, λ is a decay constant. We solve this example according two following cases for $\lambda \in \mathbb{R}$.

Case I. Suppose that $\lambda \in \mathbb{R}^{+} = (0, +\infty)$, then applying Laplace transform on the both sides of above equation, we have:

$$\mathbf{L}\left[\left({}^{RL}D_{0^+}^{\beta}y\right)(x)\right] = \mathbf{L}(\lambda \odot y(x)). \tag{3.4.13}$$

using ${}^{RL}[(i)-\beta]-$differentiability, we get:

$$\begin{cases} \lambda\ell[\underline{y}(x;r)] = p^{\beta}\ell\left[\underline{y}(x;r)\right] - \left({}^{RL}D_{0^+}^{\beta-1}\underline{y}\right)(0;r), \\ \\ \lambda\ell[\overline{y}(x;r)] = p^{\beta}\ell\left[\overline{y}(x;r)\right] - \left({}^{RL}D_{0^+}^{\beta-1}\overline{y}\right)(0;r), \end{cases} \tag{3.4.14}$$

Then, after some manipulations we get the following:

$$\begin{cases} (p^{\beta}-\lambda)\ell[\underline{y}(x;r)] = \left({}^{RL}D_{0^+}^{\beta-1}\underline{y}\right)(0;r), \\ \\ (p^{\beta}-\lambda)\ell[\overline{y}(x;r)] = \left({}^{RL}D_{0^+}^{\beta-1}\overline{y}\right)(0;r), \end{cases} \tag{3.4.15}$$

Consequently, applying inverse of Laplace on the both sides of Eq. (3.4.15) we have:

$$\begin{cases} \underline{y}(x;r) = \left({}^{RL}D_{a^+}^{\beta-1}\underline{y}\right)(0;r)\ell^{-1}\left[\frac{1}{p^{\beta}-\lambda}\right],\ 0 \le r \le 1, \\ \\ \overline{y}(x;r) = \left({}^{RL}D_{a^+}^{\beta-1}\overline{y}\right)(0;r)\ell^{-1}\left[\frac{1}{p^{\beta}-\lambda}\right],\ 0 \le r \le 1 \end{cases} \tag{3.4.16}$$

Finally, we determine the solution of FFDE as following:

$$\begin{cases} \underline{y}(x;r) = \ {}^{RL}\underline{y}_0^{(\beta-1)}(r)(x)^{\beta-1}E_{\beta,\beta}\left[\lambda x^{\beta}\right],\ 0 \le r \le 1, \\ \overline{y}(x;r) = \ {}^{RL}\overline{y}_0^{(\beta-1)}(r)(x)^{\beta-1}E_{\beta,\beta}\left[\lambda x^{\beta}\right],\ 0 \le r \le 1. \end{cases} \tag{3.4.17}$$

Case II. Suppose that $\lambda \in \mathbb{R}^{-} = (-\infty, 0)$, then using ${}^{RL}[(ii)-\beta]-$differentiability and Theorem 3.3.2 the solution will obtain similar to Eq. (3.5.11).

For special case, let us consider $\beta = 0.5, \lambda = 1$ and $\left({}^{RL}D_{a^+}^{-0.5}\underline{y}\right)(0;r) = [1+r, 3-r]$, then we get the solution for **Case I** as following:

$$y(x;r) = [1+r, 3-r] \odot \left(\frac{1}{\sqrt{\pi x}} + e^x \mathsf{erfc}(-\sqrt{x})\right), \tag{3.4.18}$$

and for **Case II** with $\lambda = -1$ is as following:

$$y(x;r) = [1+r, 3-r] \odot \left(\frac{1}{\sqrt{\pi x}} - e^x \mathsf{erfc}(\sqrt{x})\right), \tag{3.4.19}$$

where $\mathsf{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$.

Now, let us consider another example which is solved in non-fractional case in [8].

Example 3.4.2. *Let us consider the following FFDE*

$$\begin{cases} \left({}^{RL}D_{0^+}^{\beta} y\right)(x) = (-1) \odot y(x) + x + 1, \;\; 0 < \beta, x < 1 \\ \left({}^{RL}D_{0^+}^{\beta-1} y\right)(0) = \;\; {}^{RL}y_0^{(\beta-1)} \in \mathbb{E} \end{cases}$$

Applying Laplace transform on the both sides of above equation, we obtain:

$$\mathbf{L}\left[\left({}^{RL}D_{a^+}^{\beta} y\right)(x)\right] = \mathbf{L}[(-1) \odot y(x) + x + 1]. \tag{3.4.20}$$

Applying ${}^{RL}[(2)-\beta]-$differentiability and Theorem 3.3.2, we have the following:

$$\begin{cases} -\ell[\underline{y}(x;r)] + \ell[x] + \ell[1] = p^\beta \ell\left[\underline{y}(x;r)\right] - \left({}^{RL}D_{a^+}^{\beta-1}\underline{y}\right)(0;r), \\ \\ -\ell[\overline{y}(x;r)] + \ell[x] + \ell[1] = p^\beta \ell\left[\overline{y}(x;r)\right] - \left({}^{RL}D_{a^+}^{\beta-1}\overline{y}\right)(0;r), \end{cases} \tag{3.4.21}$$

After some manipulations, we get:

$$\begin{cases} (p^{\beta}+1)\ell[\underline{y}(x;r)] = \ell[x]+\ell[1]+\left({}^{RL}D_{a^+}^{\beta-1}\underline{y}\right)(0,r), \\ \\ (p^{\beta}+1)\ell[\overline{y}(x;r)] = \ell[x]+\ell[1]+\left({}^{RL}D_{a^+}^{\beta-1}\overline{y}\right)(0;r), \end{cases} \tag{3.4.22}$$

Applying inverse of Laplace transform on the both sides of Eq. (3.4.22), we get the following:

$$\begin{cases} \underline{y}(x;r) = \ell^{-1}\left[\frac{1}{p^2(p^{\beta}+1)}\right]+\ell^{-1}\left[\frac{1}{p(p^{\beta}+1)}\right]+\ell^{-1}\left[\frac{\left({}^{RL}D_{a^+}^{\beta-1}\underline{y}\right)(0;r)}{p^{\beta}+1}\right], \\ \\ \overline{y}(x;r) = \ell^{-1}\left[\frac{1}{p^2(p^{\beta}+1)}\right]+\ell^{-1}\left[\frac{1}{p(p^{\beta}+1)}\right]+\ell^{-1}\left[\frac{\left({}^{RL}D_{a^+}^{\beta-1}\overline{y}\right)(0;r)}{p^{\beta}+1}\right], \end{cases} \tag{3.4.23}$$

Solving Eq. (3.4.23), leads to determine the lower and upper functions of solution as following:

$$\begin{cases} \underline{y}(x;r) = & {}^{RL}\underline{y}_0^{(\beta-1)}(r)(x)^{\beta-1}E_{\beta,\beta}\left[\lambda x^{\beta}\right]+\int_0^x(x-t)^{\beta-1}E_{\beta,\beta}\left[\lambda(x-t)^{\beta}\right](t+1)dt, \\ \overline{y}(x;r) = & {}^{RL}\overline{y}_0^{(\beta-1)}(r)(x)^{\beta-1}E_{\beta,\beta}\left[\lambda x^{\beta}\right]+\int_0^x(x-t)^{\beta-1}E_{\beta,\beta}\left[\lambda(x-t)^{\beta}\right](t+1)dt. \end{cases}$$

For special case, let us consider $\beta = 0.5$ and $\left({}^{RL}D_{a^+}^{-0.5}\underline{y}\right)(0;r) = [1+r, 3-r]$, then we get the solution as following:

$$\begin{aligned} \underline{y}(x;r) = \quad & \frac{4}{3\sqrt{\pi}}x^{1.5}\mathsf{hypergeom}\left([1],[2.5],x\right) \\ & -0.5x^2\mathsf{hypergeom}([1],[3],x)+\frac{2x^{0.5}}{\sqrt{\pi}}\mathsf{hypergeom}([1],[1.5],x)-x\mathsf{hypergeom}([1],[2],x) \\ + \quad & (1+r)\left(\frac{1}{\sqrt{\pi t}}-e^x\mathsf{erfc}(\sqrt{x})\right),\ 0\le r\le 1, \end{aligned}$$

and

$$\begin{aligned} \overline{y}(x;r) = \quad & \frac{4}{3\sqrt{\pi}}x^{1.5}\mathsf{hypergeom}([1],[2.5],x) \\ & -0.5x^2\mathsf{hypergeom}([1],[3],x)+\frac{2x^{0.5}}{\sqrt{\pi}}\mathsf{hypergeom}([1],[1.5],x)-x\mathsf{hypergeom}([1],[2],x) \\ + \quad & (3-r)\left(\frac{1}{\sqrt{\pi t}}-e^x\mathsf{erfc}(\sqrt{x})\right),\ 0\le r\le 1, \end{aligned}$$

where hypergeom is defined as following:

hypergeom(n, d, z) is the generalized hypergeometric function $F(n, d, z)$, also known as the Barnes extended hypergeometric function. For scalar a, b, and c, hypergeom$([a, b], c, z)$ is the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$. The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined in the unit disc as the sum of the hypergeometric series

$$ {}_2F_1(a; b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{3.4.24} $$

where $|z| < 1$ and $(a)_k$ is the *Pochhammer* symbol, defined by

$$ \begin{aligned} (a)_0 &= 1, \\ (a)_n &= a(a+1)\cdots(a+n-1), \quad n \in \mathbb{N}. \end{aligned} $$

For more details see MATLAB Help and [55].

3.5 Explicit Solutions

We give the explicit solutions of fuzzy fractional differential equations under Riemann-Liouville differentiability by using Mittag-Leffler functions.

Lemma 3.5.1. *If $0 < \beta < 1$ and $f(x) \in L_p^{\mathbb{F}}(a, b)$, $1 \le p < \infty$, then the following equality is hold almost everywhere on $[a, b]$ for case of ${}^{RL}[(1) - \beta]$-differentiability:*

$$ \left({}^{RL}D_{a^+}^{\beta} \, I_{a^+}^{\beta} \, f \right)(x; r) = \left[\underline{f}(x; r), \overline{f}(x; r) \right], \quad 0 \le r \le 1, \tag{3.5.1} $$

and the following equality is hold almost everywhere on $[a,b]$ *for case of* ${}^{RL}[(2)-\beta]$*-differentiability:*

$$\left({}^{RL}D^{\beta}_{a^+}\ I^{\beta}_{a^+}\ f\right)(x;r)=\left[\overline{f}(x;r),\underline{f}(x;r)\right],\quad 0\leq r\leq 1. \tag{3.5.2}$$

Proof. Using Definition 3.2.2 and Theorem 3.2.2, the proofs for both cases are straightforward. □

The composition of the fuzzy fractional integration operator I^{β}_{a+} with the fuzzy fractional differentiation operator ${}^{RL}D^{\beta}_{a^+}$ is given by the following result.

Lemma 3.5.2. *Let* $f(x)\in L^{\mathbb{F}}_p(a,b)$*,* $f_{1-\beta}(x)\in AC^{\mathbb{F}}[a,b]$ *and* $0<\beta<1$*, then we have*

$$\left(I^{\beta}_{a^+}\ {}^{RL}D^{\beta}_{a^+}f\right)(x)=f(x)\ \ominus\ \frac{f_{1-\beta}(a)}{\Gamma(\beta)}(x-a)^{\beta-1}, \tag{3.5.3}$$

for case ${}^{RL}[(1)-\beta]$*-differentiability and we have*

$$\left(I^{\beta}_{a^+}\ {}^{RL}D^{\beta}_{a^+}f\right)(x)=-\left(\frac{f_{1-\beta}(a)}{\Gamma(\beta)}(x-a)^{\beta-1}\right)\ominus(-f(x)), \tag{3.5.4}$$

for case ${}^{RL}[(2)-\beta]$*-differentiability, where* $f_{1-\beta}(a)=\left(I^{1-\beta}_{a^+}f\right)(a)$ *and provided that the mentioned Hukuhara differences exist. Also,* $-f(x)=[-\overline{f}(x),-\underline{f}(x)]$*.*

Proof. Indeed, we have by direct computation for case of ${}^{RL}[(1)-\beta]$-differentiability:

$$\begin{aligned}\left(I^{\beta}_{a^+}\ {}^{RL}D^{\beta}_{a^+}f\right)(x;r) &= \left[\left(I^{\beta}_{a^+}\ {}^{RL}D^{\beta}_{a^+}\underline{f}\right)(x;r),\left(I^{\beta}_{a^+}\ {}^{RL}D^{\beta}_{a^+}\overline{f}\right)(x;r)\right]\\ &= \left[\underline{f}(x;r)-\frac{\underline{f}_{1-\beta}(a)}{\Gamma(\beta)}(x-a)^{\beta-1},\overline{f}(x;r)-\frac{\overline{f}_{1-\beta}(a)}{\Gamma(\beta)}(x-a)^{\beta-1}\right],\end{aligned}$$

and for ${}^{RL}[(2)-\beta]$-differentiability:

$$\begin{aligned}\left(I_{a^+}^{\beta}\ {}^{RL}D_{a^+}^{\beta}f\right)(x;r) &= \left[\left(I_{a^+}^{\beta}\ {}^{RL}D_{a^+}^{\beta}\overline{f}\right)(x;r),\left(I_{a^+}^{\beta}\ {}^{RL}D_{a^+}^{\beta}\underline{f}\right)(x;r)\right]\\ &= \left[\overline{f}(x;r)-\frac{\overline{f}_{1-\beta}(a)}{\Gamma(\beta)}(x-a)^{\beta-1},\underline{f}(x;r)-\frac{\underline{f}_{1-\beta}(a)}{\Gamma(\beta)}(x-a)^{\beta-1}\right],\end{aligned}$$

for all $0\leq r\leq 1$ which complete the proofs. □

Now, we derive the explicit solutions to the fuzzy linear fractional differential equations under Riemann-Liouville differentiability. To this end, consider the following FFDE:

$$\left({}^{RL}D_{a^+}^{\beta}y\right)(x)=\lambda y(x)+f(x),\quad \left({}^{RL}D_{a^+}^{\beta-1}y\right)(a)=\ {}^{RL}y_0^{\beta-1} \tag{3.5.5}$$

where $\lambda>0$ and we use ${}^{RL}[(1)-\beta]$-differentiability. So, Eq. (3.5.5) is equivalent to the following Volterra integral equation:

$$y(x)=\frac{{}^{RL}y_0^{\beta-1}(x-a)^{\beta-1}}{\Gamma(\beta)}+\frac{\lambda}{\Gamma(\beta)}\int_a^x\frac{y(t)dt}{(x-t)^{\beta-1}}+\frac{1}{\Gamma(\beta)}\int_a^x\frac{f(t)dt}{(x-t)^{\beta-1}} \tag{3.5.6}$$

In order to solve mentioned fuzzy Volterra integral equation, we adopt successive approximations method. So, based on this approach we set:

$$\begin{aligned}y_0(x)&=\frac{{}^{RL}y_0^{\beta-1}(x-a)^{\beta-1}}{\Gamma(\beta)},\\ y_{n+1}(x)&=y_0(x)+\frac{\lambda}{\Gamma(\beta)}\int_a^x\frac{y(t)dt}{(x-t)^{\beta-1}}+\frac{1}{\Gamma(\beta)}\int_a^x\frac{f(t)dt}{(x-t)^{\beta-1}}\end{aligned}$$

Then, we find for y_1 that

$$y_1(x)=y_0(x)+\lambda\left(I_{a^+}^{\beta}y_0\right)(x)+\left(I_{a^+}^{\beta}f\right)(x)$$

that is,

$$y_1(x)=\ {}^{RL}y_0^{\beta-1}\sum_{k=1}^{2}\frac{\lambda^{k-1}(x-a)^{\beta k-1}}{\Gamma(\beta k)}+\frac{1}{\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}f(t)dt$$

continuing this process, we find $y_n(x)$ as following:

$$y_n(x) = {}^{RL}y_0^{\beta-1} \sum_{k=1}^{n+1} \frac{\lambda^{k-1}(x-a)^{\beta k-1}}{\Gamma(\beta k)} + \int_a^x \left[\sum_{k=1}^{n} \frac{\lambda^{k-1}}{\Gamma(\beta k)} (x-t)^{\beta k-1} \right] f(t)dt \quad (3.5.7)$$

Then, taking the limit $n \longrightarrow \infty$ and by replacing the index of summation k by $k-1$, we obtain:

$$y(x) = {}^{RL}y_0^{\beta-1} \sum_{k=0}^{\infty} \frac{\lambda^{k}(x-a)^{\beta k-\beta-1}}{\Gamma(\beta k+\beta)} + \int_a^x \left[\sum_{k=0}^{\infty} \frac{\lambda^{k}}{\Gamma(\beta k+\beta)} (x-t)^{\beta k+\beta-1} \right] f(t)dt \quad (3.5.8)$$

Finally, by applying definition of Mittag-Leffler function $E_{\beta,\beta}(z)$ we get the following:

$$y(x) = {}^{RL}y_0^{\beta-1}(x-a)^{\beta-1} E_{\beta,\beta}\left[\lambda(x-a)^{\beta}\right] + \int_a^x (x-t)^{\beta-1} E_{\beta,\beta}\left[\lambda(x-t)^{\beta}\right] f(t)dt \quad (3.5.9)$$

So, this yields an explicit solution to the fuzzy Volterra integral equation and hence to the original FFDE (3.5.5).

Please notice that, one can easily extend this explicit solution under ${}^{RL}[(2)-\beta]$-differentiability and assumption $\lambda < 0$ such that:

$$y(x) = {}^{RL}y_0^{\beta-1}(x-a)^{\beta-1} E_{\beta,\beta}\left[\lambda(x-a)^{\beta}\right] \ominus \int_a^x (-1).(x-t)^{\beta-1} E_{\beta,\beta}\left[\lambda(x-t)^{\beta}\right] f(t)dt, \quad (3.5.10)$$

provided that the H-difference exists.

Remark 3.5.1. Please notice that, one can easily establish the convergence of mentioned successive approximation method by extending Theorem 2.1 in [64] to the fuzzy literature.

3.5.1 Examples

Now, we present two examples to solve FFDEs under Riemann-Liouville differentiability in order to show the application of obtained explicit solutions. The first and the second examples are the homogeneous and non-homogeneous nuclear decay equations with decay parameter λ, respectively.

Example 3.5.1. *Let us consider the following FFDE*

$$\begin{cases} \left({}^{RL}D_{0^+}^{\beta} y\right)(x) = \lambda y(x), \ 0 < \beta < 1 \\ \left({}^{RL}D_{0^+}^{\beta-1} y\right)(0) = \ {}^{RL}y_0^{(\beta-1)} \in \mathbb{E} \end{cases}$$

We solve this example according to two following cases for $\lambda \in \mathbb{R}$.

Case I. Suppose that $\lambda \in \mathbb{R}^+ = (0, +\infty)$, then using ${}^{RL}[(1) - \beta]-$differentiability and Eq. (3.5.9), we get the solution as following:

$$\begin{cases} \underline{y}(x;r) = \ {}^{RL}\underline{y}_0^{(\beta-1)}(r)(x)^{\beta-1} E_{\beta,\beta}\left[\lambda x^{\beta}\right], \ 0 \le r \le 1, \\ \overline{y}(x;r) = \ {}^{RL}\overline{y}_0^{(\beta-1)}(r)(x)^{\beta-1} E_{\beta,\beta}\left[\lambda x^{\beta}\right], \ 0 \le r \le 1. \end{cases} \tag{3.5.11}$$

Case II. Suppose that $\lambda \in \mathbb{R}^- = (-\infty, 0)$, then under ${}^{RL}[(2) - \beta]-$differentiability and applying Eq. (3.5.10), the solution will be obtained by Eq. (3.5.11).

For special case, let us consider $\beta = 0.5, \lambda = 1$ and $\left({}^{RL}D_{a^+}^{-0.5} \underline{y}\right)(0;r) = (1+r^2, 8-r^3)$, then we get the solution for **Case I** as following:

$$y(x) = (1 + r^2, 8 - r^3).\left(\frac{1}{\sqrt{\pi x}} + e^x \mathsf{erfc}(-\sqrt{x})\right), \tag{3.5.12}$$

and for **Case II** with $\lambda = -1$ is as following:

$$y(x) = (1 + r^2, 8 - r^3).\left(\frac{1}{\sqrt{\pi x}} - e^x \mathsf{erfc}(\sqrt{x})\right), \tag{3.5.13}$$

where $\mathsf{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$.

Example 3.5.2. *Let us consider the following FFDE*

$$\begin{cases} \left({}^{RL}D_{0^+}^{\beta} y\right)(x) = \lambda y(x) + f(x), \ 0 < \beta \leq 1 \\ \left({}^{RL}D_{0^+}^{\beta-1} y\right)(0) = \ {}^{RL}y_0^{(\beta-1)} \in \mathbb{E} \end{cases}$$

where $y(x)$ and $f(x)$ are fuzzy-valued functions. Then, similar to Ex. 3.5.1, we consider two cases for $\lambda \in \mathbb{R}$.

Case I. Suppose that $\lambda \in \mathbb{R}^+ = (0, +\infty)$, then using ${}^{RL}[(1) - \beta]-$differentiability and Eq. (3.5.9), we get the solution as following:

$$\begin{cases} \underline{y}(x;r) = \ {}^{RL}\underline{y}_0^{(\beta-1)}(r)(x)^{\beta-1} E_{\beta,\beta}\left[\lambda x^\beta\right] + \int_0^x (x-t)^{\beta-1} E_{\beta,\beta}\left[\lambda(x-t)^\beta\right] \underline{f}(t;r)dt, \\ \overline{y}(x;r) = \ {}^{RL}\overline{y}_0^{(\beta-1)}(r)(x)^{\beta-1} E_{\beta,\beta}\left[\lambda x^\beta\right] + \int_0^x (x-t)^{\beta-1} E_{\beta,\beta}\left[\lambda(x-t)^\beta\right] \overline{f}(t;r)dt. \end{cases} \tag{3.5.14}$$

Case II. Suppose that $\lambda \in \mathbb{R}^- = (-\infty, 0)$, then under ${}^{RL}[(2) - \beta]-$differentiability and applying Eq. (3.5.10), solution will be obtained by Eq. (3.5.14).

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