In the Name of God

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Faculty of Mathematical Sciences and Computer

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Title

Homotopy Analysis Method in Solving Nonlinear Partial Differential Equations

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy (PhD)

By

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To

my family, specially my loving mother

who

dedicated her life to her kids

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Declaration

This thesis consists of four chapters. In chapter one the principles and basic concepts of homotopy analysis method are described. As well, other investigators experiences along with this thesis findings were used to submit suitable guidelines for HAM deployment. These guidelines can be useful to select proper parameters, auxiliary function and operator of HAM.

- The results obtained in chapter 2 are original and is accepted as
  S. Abbasbany, E. Babolian, M. Ashtiani, *Numerical solution of the generalized Zakharov equation by homotopy analysis method*, Commun. Nonlinear Sci. Numer. Simulat, **14** (2009) 4114–4121.

- The results of chapter 3 are original and is accepted as
  S. Abbasbandy, M. Ashtiani, E. Babolian, *Analytic solution of the Sharma-Tasso-Olver equation by homotopy analysis method*, Z. Naturforsch **65 a** (2010) 285–290.

Abstract

As we know, most phenomena in our world are essentially nonlinear and are described by nonlinear equations, so nonlinear phenomena play a crucial role in applied mathematics and physics. It is too important to find exact solutions of nonlinear partial differential equations. These equations are mathematical models of complex physical occurrences that arise in engineering, chemistry, biology, mechanics and physics. There are many explicit exact methods such as Bäcklund transformation, Cole-Hopf transformation, tanh method, sine-cosine method and so on. Many scientists and engineers delight to solve these kind of equations by numerical or analytic methods. Any method has its own excellences and faults. As pointed out by Liao, in the introduction of his book [27], The numerical techniques generally can be applied to nonlinear problems in complicated computation domain and most equations that arise in real problems are quite intractable by analytical means, so the computer is the only hope; this is an advantage of numerical methods over analytic ones that often handle nonlinear problems in simple domains. However, numerical methods usually give discontinuous points of a curve and thus it is often costly and time consuming to get a complete curve of the approximation of the exact solution function. Besides, from numerical results, it is hard to have a whole and essential understanding of a nonlinear problem. Additional numerical difficulties appear if a nonlinear problem contains singularities or has multiple solutions. Furthermore nonlinear equations are difficult to solve, especially analytically. Perturbation techniques [36] are widely applied in science and engineering, and do great contribution to help us understanding many nonlinear phenomena. However, perturbation methods are dependent upon small/large physical parameters, and are valid for weakly nonlinear problems. The non-perturbation techniques, such as the Adomian's decomposition method (ADM), are formally independent of small/large parameters. But, all of these traditional non-perturbation methods can not provide us with a convenient way to adjust convergence region and rate of approximation series

solutions. The present thesis focus on a new semi analytic method for solving functional equations. This method is called homotopy analysis method which is based on homotopy, a fundamental concept in topology and differential geometry, which can be traced back to Poincaré [31].

The homotopy analysis method (HAM) is a general semi analytic approach that is valid for various types of nonlinear equations even if a given problem does not contain any small/large parameters, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations and differential-difference equations. More importantly, HAM provides us with a convenient way to adjust the convergence region and rate of approximation series. Besides, different from all perturbation and previous non-perturbation methods HAM provides us with freedom to use different base functions to approximate solution to a nonlinear problem. Our work in the present thesis can be classified into two different parts. The first part would devote to the theoretical aspects of HAM and the second part concerns the applications of this method by introducing two equations, the generalized Zakharov equation (GZE) and Sharma-Tasso-Olver equation (STOE) which are applicable in engineering and physics. During solving these problems, we explain difficulties in applying HAM and give guidelines to remove or release some of them.

Mathematics subject classification 2010: 65H20, 35A20, 74G10, 35C10.

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Introduction

The homotopy method or homotopy continuation method maps an initial approximation to the exact solution through a homotopy function involving an auxiliary operator and an embedding parameter. Recently, through using symbolical computer softwares such as Maple, Matlab and Mathematica, obtaining exact solutions of nonlinear partial differential equations has become more interesting.

In 1992, Liao [24] first used the concept of homotopy to obtain analytic approximations of the nonlinear equation $A[u(r)] = 0$ by means of constructing a one-parameter family of equations (called the zeroth-order deformation equation). At first we construct the homotopy of this equation as

$$\mathcal{H}[\phi; q] = (1 - q)L[\phi(r; q) - u_0(r)] - qA[\phi(r; q)], \tag{0.0.1}$$

where $q \in [0, 1]$ is an embedding parameter, A is a nonlinear operator, $u(r)$ is an unknown function, $u_0(r)$ is a guess approximation and r denotes independent variable(s), respectively. Here, $\phi(r; q)$ denotes that ϕ is not only a function of the original independent variable, but also a function of the embedding parameter q. Obviously, we have $\phi(r; 0) = u_0(r)$ when $q = 0$, and $\phi(r; 1) = u(r)$ when $q = 1$. The Taylor series of $\phi(r; q)$ with respect to the embedding parameter q reads

$$\phi(r; q) = u_0(r) + \sum_{m=1}^{+\infty} u_m(r) q^m, \tag{0.0.2}$$

where

$$u_m(r) = \frac{1}{m!} \frac{\partial^m \phi(r; q)}{\partial q^m}\Big|_{q=0}. \tag{0.0.3}$$

Provided that the Taylor series (0.0.2) is convergent at $q = 1$, we have the series solution

$$\phi(r;q) = u_0(r) + \sum_{m=1}^{+\infty} u_m(r). \tag{0.0.4}$$

However, the above approach breaks down if the Taylor series (0.0.2) diverges at $q = 1$. To overcome this disadvantage, Liao [25] introduced a nonzero auxiliary parameter $\hbar$, which is now called the convergence-control parameter [31], to construct such a two-parameter family of equations (i.e. the zeroth-order deformation equation)

$$\mathcal{H}[\phi; q, \hbar] = (1-q)L[\phi(r;q,\hbar) - u_0(r)] - q\hbar A[\phi(r;q,\hbar)]. \tag{0.0.5}$$

Now, by entering the parameter $\hbar$ the solution $\phi(r;q,\hbar)$ of the above equation is not only dependent upon r and the embedding parameter q but also the convergence-control parameter $\hbar$. So, the term $u_m(r)$ given by (0-0-3) is also dependent upon $\hbar$ and therefore the convergence region of the Taylor series (0-0-2) is influenced by $\hbar$. The introduction of the convergence-control parameter $\hbar$ greatly improves the homotopy analysis method in theory, as shown by Liao [25–27], Abbasbandy [1–3], Liao and Tan [23], Sajid and Hayat [46], and suchlike. As you will see later, by means of HAM, instead of direct action to solve the nonlinear differential equation we transfer it into an infinite number of linear sub-problems and then approximate it by the sum of the solutions of first few sub-problems. Chapter 1 is devoted to explain principles of the method by considering the nonlinear differential equation $A[u(r)] = 0$ and consists of two sections. First section deals with definitions and fundamental concepts of homotopy and in second section we present some guideline to performing HAM. At the end of section two, we give short review on comparison between HAM and homotopy perturbation method (HPM). Chapters 2 and 3 are devoted to solving the generalized Zakharov equation [6] and Sharma-Tasso-Olver equation [4], respectively. In chapter 4, we present some limitations of this method by considering 3 problems.

Chapter 1

Preliminary

Throughout this chapter, we collect some basic definitions, theorems and facts needed in the chapters that follow and review the basic ideas of HAM. Our main references are the book of Shi-Jun Liao [27], Robert A. Van Gorder's paper [17] and Molabahrami's paper [34]. For the sake of simplicity consider a general kth order nonlinear ordinary differential equation and associated nonlinear differential operator $A : C^{(k)} \to \mathbb{R}$, where $C^{(k)}$ is the space of real-valued functions possessing continuous derivatives up to kth order. In order to solve such a nonlinear differential equation, we seek to understand the kernel of such a map A, which is simply the set of all $C^{(k)}$ functions $u(r)$ such that

$$A[u(r)] = 0, \tag{1.0.1}$$

for all r in the domain of interest. In practice, obtaining an exact solution $u(r)$ is not easy, and more likely impossible for an arbitrary A.

1.1 Basic ideas

In 1999, Liao [26] further generalized the homotopy analysis method by constructing such a deformation equation

$$\mathcal{H}[\phi; q, \hbar, H] = (1 - B(q))L[\phi(r; q) - u_0(r)] - E(q)\hbar H(r)A[\phi(r; q)], \tag{1.1.1}$$

where $u_0(r)$ is an initial approximation of $u(r)$, L denotes an auxiliary linear operator, $q \in [0, 1]$ is called homotopy-parameter, in topology q is called the embedding parameter, while $H(r)$ is the auxiliary function, $E(q)$ and $B(q)$ are two analytic functions in the region $|q| \leq 1$, called embedding functions which satisfy

$$E(0) = B(0) = 0, \quad E(1) = B(1) = 1. \tag{1.1.2}$$

Let

$$E(q) = \sum_{k=1}^{+\infty} \alpha_k q^k, \quad B(q) = \sum_{k=1}^{+\infty} \beta_k q^k \tag{1.1.3}$$

denote the Maclaurin series of $E(q)$ and $B(q)$, respectively, which are assumed to be convergent at $q = 1$, i.e.

$$\sum_{k=1}^{+\infty} \alpha_k = 1, \quad \sum_{k=1}^{+\infty} \beta_k = 1. \tag{1.1.4}$$

Define the two vectors

$$\vec{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, \cdots\}, \quad \vec{\beta} = \{\beta_1, \beta_2, \beta_3, \cdots\}. \tag{1.1.5}$$

Hence, the solution $\phi(r;q)$ of the equation (1.1.1) is not only depend upon the convergence-control parameter $\hbar$ but also the two vectors $\vec{\alpha}$ and $\vec{\beta}$. Thus, the generalized deformation equation (1.1.1) provides us greater freedom, or in other words, more possibility, to ensure the convergence of the Taylor series (0.0.2) at $q = 1$. This is the reason why $\vec{\alpha}$ and $\vec{\beta}$ are called the convergence-control vectors [31]. In Liao's book the auxiliary linear operator followed by the property

$$L[f] = 0 \qquad when \quad f = 0, \tag{1.1.6}$$

but the author of this dissertation believes that this property changes from one equation to another equation. Namely, relation (1.1.6) is revised as

$$L[f] = 0 \quad \Rightarrow f = w, \quad w \in Kernel\{L\}.$$

We will present it at next sections, clearly. We see that when $q = 0$, we have $\mathcal{H}[\phi; q, \hbar, H]\Big|_{q=0} = L[\phi(r;0) - u_0(r)]$, while when $q = 1$, we have $\mathcal{H}[\phi; q, \hbar, H]\Big|_{q=1} = A[\phi(r;1)]$. So, the solution of equation $\mathcal{H}[\phi; q, \hbar, H] = 0$, $\phi(r;q)$, agrees with the initial approximation at $q = 0$ and with a solution to the nonlinear differential equation of interest when $q = 1$. Briefly speaking, by means of the HAM, one constructs a continuous mapping of initial approximation to the exact solution of considered equations, in topology such a kind of continuous variation is called deformation. As we mentioned in the introduction, instead of direct action to solve the nonlinear differential equation we transfer it into an infinite number of linear sub-problems. But how we can gain these sub-problems?

Definition 1.1.1. Let ϕ be a function of the homotopy-parameter q, then

$$D_m(\phi) = \frac{1}{m!}\frac{\partial^m \phi}{\partial q^m}\Big|_{q=0}$$

is called the mth-order homotopy-derivative of ϕ, where $m \geq 0$ is an integer.

Due to $0 \leq q \leq 1$, Liao proposed a perturbation solution in which one regards the homotopy parameter q as the parameter about which we expand the solution. Expand $\phi(r;q)$ as a Taylor series, this is given by

$$\phi(r;q) = u_0(r) + \sum_{m=1}^{\infty} u_m(r) q^m. \tag{1.1.7}$$

Where $\phi(r;0) = u_0(r)$ is employed, and according to the definition (1.1.1)

$$u_m(r) = \frac{1}{m!}\frac{\partial^m \phi(r;q)}{\partial q^m}\Big|_{q=0} = D_m(\phi). \tag{1.1.8}$$

According to the theory of Taylor series, this power series is unique as one regards q as a small parameter. Here series (1.1.7) is called homotopy-series of ϕ. In this regard, $\mathcal{H}[\phi; q, \hbar, H] = 0$ serves as the zeroth-order deformation equation. Since we have freedom to select the initial approximation, auxiliary linear operator, auxiliary function, we must assume that they are properly chosen so that:

(i) The solution $\phi(r; q, \hbar, H)$ to the zeroth-order deformation (1.1.1) exists for all $q \in [0, 1]$; and

(ii) the series solution (1.1.7) converges at $q = 1$.

When these two assumptions hold, then using the relation $\phi(r, 1) = u(r)$, one has the so called homotopy-series solution

$$u(r) = u_0(r) + \sum_{m=1}^{\infty} u_m(r), \tag{1.1.9}$$

over the region of convergence for this representation. Thus, unlike all previous analytic techniques, the convergence region and rate of solution series given by the HAM might not be uniquely determined. This is indeed true, as shown later in this chapter.

To obtain $u_m(r)$, one recursively solves what are known as the mth-order deformation equations which can be deduced from the zeroth-order deformation equation. Before we derive the mth-order deformation equation it is necessary to introduce the following theorems.

Theorem 1.1.2. (Molabahrami and Khani's Theorem 1) *Let $D_m(\phi^k)$ denote the homotopy derivative of ϕ^k where $\phi = \sum_{n=0}^{+\infty} u_n q^n$, then*

$$D_m(\phi^k) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}} u_{r_{k-1}},$$

and

$$D_m(\phi^{k+1}) = \sum_{j=0}^{m} u_{m-j} D_j(\phi^k) = \sum_{j=0}^{m} u_j D_{m-j}(\phi^k).$$

Proof. The proof is by induction on k. Clearly, for $k = 2$, we have

$$D_m(\phi^2) = \sum_{j=0}^{m} u_{m-j} u_j.$$

Put $\phi^{k+1} = \phi^k \phi$, by the *Leibnitz*'s rule for higher derivatives of products, we have

$$(\phi^{k+1})^{(m)} = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} (\phi^k)^{(j)} \phi^{(m-j)}.$$

Thus

$$\begin{aligned} D_m(\phi^{k+1}) &= \frac{1}{m!} \left(\sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} (\phi^k)^{(j)} \phi^{(m-j)} \right)_{q=0} \\ &= \frac{1}{m!} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} (j! D_j(\phi^k))(m-j)! u_{m-j} \\ &= \sum_{j=0}^{m} u_{m-j} D_j(\phi^k). \end{aligned}$$

It is clear that

$$\sum_{j=0}^{m} u_{m-j} D_j(\phi^k) = \sum_{j=0}^{m} u_j D_{m-j}(\phi^k).$$

This ends the proof. □

Corollary 1.1.3. *From theorem (1.1.2), we have*

$$D_m(\phi^{k-1}\phi_x) = \sum_{r_1=0}^{m} u_{m-r_1} \sum_{r_2=0}^{r_1} u_{r_1-r_2} \sum_{r_3=0}^{r_2} u_{r_2-r_3} \cdots \sum_{r_{k-2}=0}^{r_{k-3}} u_{r_{k-3}-r_{k-2}} \sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}} (u_x)_{r_{k-1}}.$$

Theorem 1.1.4. (Molabahrami and Khani's Theorem 2) *Let* $s = \sum_{n=0}^{+\infty} u_n$. *Then, recall the assumptions in theorem (1.1.2), we have*

$$\sum_{m=0}^{+\infty} D_m(\phi^k) = s^k.$$

Proof. The proof is by induction on k. From theorem (1.1.2), for $k = 2$, we have

$$\sum_{m=0}^{+\infty} D_m(\phi^2) = \sum_{m=0}^{+\infty} \left(\sum_{j=0}^{m} u_{m-j} u_j \right).$$

Thus

$$\begin{aligned} \textstyle\sum_{m=0}^{+\infty} D_m(\phi^2) &= \textstyle\sum_{m=0}^{+\infty} \sum_{j=0}^{m} u_{m-j} u_j \\ &= \textstyle\sum_{j=0}^{+\infty} \sum_{m=j}^{+\infty} u_{m-j} u_j \\ &= \textstyle\sum_{j=0}^{+\infty} u_j \sum_{m=j}^{+\infty} u_{m-j} \\ &= s^2. \end{aligned}$$

Put $\phi^{k+1} = \phi^k \phi$, from theorem (1.1.2), we have

$$\sum_{m=0}^{+\infty} D_m(\phi^{k+1}) = \sum_{m=0}^{+\infty} \left(\sum_{j=0}^{m} u_{m-j} D_j(\phi^k) \right).$$

Thus

$$\begin{aligned} \textstyle\sum_{m=0}^{+\infty} D_m(\phi^{k+1}) &= \textstyle\sum_{m=0}^{+\infty} \sum_{j=0}^{m} u_{m-j} D_j(\phi^k) \\ &= \textstyle\sum_{j=0}^{+\infty} \sum_{m=j}^{+\infty} u_{m-j} D_j(\phi^k) \\ &= \textstyle\sum_{j=0}^{+\infty} D_j(\phi^k) \sum_{m=j}^{+\infty} u_{m-j} \\ &= s^k s \\ &= s^{k+1}. \end{aligned}$$

This ends the proof. □

1.1.1 Some properties of the homotopy derivative

In this section, some properties of the homotopy derivative are represented by the aid of the following theorems.

Theorem 1.1.5. *Let f and g be functions independent of the homotopy-parameter q. For homotopy-series*

$$\phi = \sum_{i=0}^{+\infty} u_i q^i, \quad \psi = \sum_{j=0}^{+\infty} v_j q^j,$$

it holds

$$D_m(f\phi + g\psi) = f D_m(\phi) + g D_m(\psi).$$

Proof. Because f and g are independent of q, and besides D_m defined by (1.1.1) is a linear operator, it obviously holds

$$D_m(f\phi + g\psi) = D_m(f\phi) + D_m(g\psi) = f D_m(\phi) + g D_m(\psi).$$

□

Theorem 1.1.6. *For homotopy-series*

$$\phi = \sum_{i=0}^{+\infty} u_i q^i, \quad \psi = \sum_{j=0}^{+\infty} v_j q^j,$$

it holds

(a) $D_m(\phi) = u_m$,

(b) $D_m(q^k \phi) = D_{m-k}(\phi)$,

(c) $D_m(\phi\psi) = \sum_{i=0}^{m} D_i(\phi) D_{m-i}(\psi) = \sum_{i=0}^{m} D_i(\psi) D_{m-i}(\phi)$,

(d) $D_m(\phi^n \psi^l) = \sum_{i=0}^{m} D_i(\phi^n) D_{m-i}(\psi^l) = \sum_{i=0}^{m} D_i(\psi^l) D_{m-i}(\phi^n)$,

where $m \geq 0$, $n \geq 0$, $l \geq 0$ and $0 \leq k \leq m$ are integers.

Proof. (a) According to Taylor theorem, the unique coefficient u_m of Maclurin series of ϕ about the point $q = 0$ is given by

$$u_m = \frac{1}{m!} \frac{\partial^m \phi}{\partial q^m}\bigg|_{q=0},$$

which gives (a) by means of the definition of $D_m(\phi)$.

(b) It holds

$$q^k\phi = q^k \sum_{i=0}^{+\infty} u_i q^i = \sum_{i=0}^{+\infty} u_i q^{i+k} = \sum_{m=k}^{+\infty} u_{m-k} q^m,$$

which gives by means of (a) that

$$D_m(q^k\phi) = u_{m-k} = D_{m-k}(\phi).$$

(c) According to *Leibnitz*'s rule for derivatives of product, it holds

$$\frac{\partial^m(\phi\psi)}{\partial q^m} = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} \frac{\partial^i\phi}{\partial q^i} \frac{\partial^{m-i}\psi}{\partial q^{m-i}} = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} \frac{\partial^i\psi}{\partial q^i} \frac{\partial^{m-i}\phi}{\partial q^{m-i}},$$

which gives by the definition (1.1.1):

$$D_m(\phi\psi) = \frac{1}{m!}\frac{\partial^m(\phi\psi)}{\partial q^m}\Big|_{q=0} = \sum_{i=0}^{m} \left(\frac{1}{i!}\frac{\partial^i\phi}{\partial q^i}\Big|_{q=0}\right)\left(\frac{1}{(m-i)!}\frac{\partial^{m-i}\psi}{\partial q^{m-i}}\Big|_{q=0}\right) = \sum_{i=0}^{m} D_i(\phi)D_{m-i}(\psi).$$

Similarity, it holds

$$D_m(\phi\psi) = \sum_{i=0}^{m} D_i(\psi)D_{m-i}(\phi).$$

(d) Write $\Phi = \phi^n$ and $\Psi = \psi^l$. According to (c), it holds

$$D_m(\phi^n\psi^l) = D_m(\Phi\Psi) = \sum_{i=0}^{m} D_i(\Phi)D_{m-i}(\Psi) = \sum_{i=0}^{m} D_i(\psi^l)D_{m-i}(\phi^n),$$

which ends the proof. □

Theorem 1.1.7. *Let L be a linear operator independent of the homotopy-parameter q. For homotopy-series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

it holds

$$D_m(L\phi) = L[D_m(\phi)].$$

Proof. Since L is independent of q, it holds

$$L\phi = \sum_{k=0}^{+\infty} [L(u_k)]q^k.$$

Taking mth-order homotopy-derivative on both sides of the above expression and using Theorem (1.1.6)(a), one has $D_m(L\phi) = L(u_m)$. On the other hand, according to Theorem (1.1.6)(a), it holds obviously $L[D_m(\phi)] = L(u_m)$. Thus, $D_m(L\phi) = L[D_m(\phi)]$ holds. □

Some other properties of the homotopy derivative have presented by the following theorems, for proof see [31].

Theorem 1.1.8. *For homotopy-series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

the following recurrence formulas hold

$$D_0(e^{\phi}) = e^{u_0},$$

$$D_m(e^{\phi}) = \sum_{k=0}^{m-1}(1 - \frac{k}{m})D_k(e^{\phi})D_{m-k}(\phi),$$

where $m \geq 1$ *is an integer.*

Theorem 1.1.9. *For homotopy-series*

$$\phi = \sum_{k=0}^{+\infty} u_k q^k,$$

the following recurrence formulas hold

$$D_0(sin\phi) = sin(u_0), \quad D_0(cos\phi) = cos(u_0),$$

$$D_m(sin\phi) = \sum_{k=0}^{m-1}(1 - \frac{k}{m})D_k(cos\phi)D_{m-k}(\phi),$$

$$D_m(cos\phi) = -\sum_{k=0}^{m-1}(1 - \frac{k}{m})D_k(sin\phi)D_{m-k}(\phi),$$

where $m \geq 1$ *is an integer.*

Moustafa El-Shahed in his paper [14] prepared a section entitled *Differential transform method* which is close to the homotopy derivative and its properties.

1.1.2 Deformation equation

By considering the theorems and definitions in the previous section we want to deduce the so-called mth-order deformation equation from the so-called zeroth-order deformation equation.

Theorem 1.1.10. *Write*

$$E(q) = \sum_{k=1}^{+\infty} \alpha_k q^k, \quad B(q) = \sum_{k=1}^{+\infty} \beta_k q^k,$$

for the zeroth-order deformation equation

$$(1 - B(q))L[\phi(r; q, \hbar, H) - u_0(r)] = E(q)\hbar H(r)A[\phi(r; q, h, H)], \tag{1.1.10}$$

the corresponding mth-order deformation equation ($m \geq 1$) reads

$$L\left[u_m(r) - \sum_{k=1}^{m-1} \beta_k u_{m-k}(r)\right] = \hbar H(r)D_m(E(q)A[\Phi(r; q)]), \tag{1.1.11}$$

where

$$D_m(E(q)A[\Phi(r; q)]) = \sum_{k=1}^{m} \alpha_k \delta_{m-k}(r),$$

under the definition

$$\delta_n(r) = \frac{1}{n!} \frac{\partial^n A[\Phi(r; q)]}{\partial q^n}\Big|_{q=0}.$$

Proof. Differentiating zeroth-order deformation equation (1.1.10) m times with respect to the homotopy parameter q and then setting $q = 0$ and finally dividing it by $m!$, for $m \geq 1$ we have

$$\frac{1}{m!} \frac{d^m\{(1 - B(q))L[\phi(r; q) - u_0(r)]\}}{dq^m} = \frac{1}{m!} \frac{d^m\{\hbar E(q)H(r)A[\phi(r; q)]\}}{dq^m} \tag{1.1.12}$$

using theorem (1.1.6)(c)

$$\begin{aligned}
\frac{1}{m!}\frac{d^m\{(1-B(q))L[\phi(r;q)-u_0(r)]\}}{dq^m} &= \frac{1}{m!}\frac{d^m}{dq^m}L[\phi(r; q) - u_0(r)] - \frac{1}{m!}\frac{d^m}{dq^m}B(q)(L[\phi(r; q) - u_0(r)]) \\
&= L\left[\frac{1}{m!}\frac{d^m\phi(r;q)}{dq^m}\right] - \frac{1}{m!}\frac{d^m B(q)L[\phi(r;q)]}{dq^m} \\
&+ \frac{1}{m!}\frac{d^m B(q)L[u_0(r)]}{dq^m}.
\end{aligned}$$

Now setting $q = 0$ in the above expression and using the relation (1.1.8), we have

$$\frac{1}{m!}\frac{d^m\{(1-B(q))L[\phi(r;q)-u_0(r)]\}}{dq^m}\Big|_{q=0}$$
$$= L[u_m(r)] - \sum_{k=1}^{m}\beta_k L(u_{m-k}) + L[u_0(r)]\beta_m$$
$$= L[u_m(r)] - \sum_{k=1}^{m-1}\beta_k L(u_{m-k}).$$

Thus

$$\frac{1}{m!}\frac{d^m\{(1-B(q))L[\phi(r;q)-u_0(r)]\}}{dq^m}\Big|_{q=0} = L[u_m(r) - \sum_{k=1}^{m-1}\beta_k u_{m-k}].$$

Similarly, it holds

$$\frac{1}{m!}\frac{d^m\{\hbar E(q)H(r)A[\phi(r;q)]\}}{dq^m}\Big|_{q=0} = \hbar H(r)\left\{\frac{1}{m!}\frac{d^m E(q)A[\phi(r;q)]}{dq^m}\right\}\Big|_{q=0}$$
$$= \hbar H(r)\sum_{k=1}^{m}\alpha_k\delta_{m-k}(r),$$

where

$$\delta_n(r) = \frac{1}{n!}\frac{\partial^n A[\phi(r;q)]}{\partial q^n}\Big|_{q=0}.$$

□

In fact, by solving ordinary differential equation (ODE)(1.1.11) step by step we achieve $u_1(r), u_2(r), u_3(r), \cdots$, respectively. Hence in kth iteration $\delta_k(r)$ depends upon $u_0(r), u_1(r), \cdots, u_{k-1}(r)$ then we introduce the vector

$$\vec{u}_{k-1} = \{u_0(r), u_1(r), \cdots, u_{k-1}(r)\}$$

and revise the mth-order deformation equation such

$$L\left[u_m(r) - \sum_{k=1}^{m-1}\beta_k u_{m-k}(r)\right] = \hbar H(x)R_m(\vec{u}_{m-1}, r), \tag{1.1.13}$$

where

$$R_m(\vec{u}_{m-1}, r) = \sum_{k=1}^{m}\alpha_k\delta_{m-k}(r).$$

Although the general zeroth-order deformation Eq. (1.1.10) was published 11 years ago and even more generalized form was reported [31], most users of the HAM applied special case of Eq. (1.1.10) by considering $E(q) = B(q) = q$, mainly due to its simplicity. So, we have the following theorem.

Theorem 1.1.11. *For the zeroth-order deformation equation defined by*

$$(1-q)L[\phi(r;q)-u_0] = q\hbar H(r)A[\phi(r;q)], \tag{1.1.14}$$

the corresponding mth-order deformation equation ($m \geq 1$) reads

$$L[u_m(r) - \chi_m u_{m-1}] = \hbar H(r) R_m(\vec{u}_{m-1}, r), \tag{1.1.15}$$

where χ_m is defined by

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2, \end{cases}$$

and

$$R_m(\vec{u}_{m-1}, r) = \frac{1}{(m-1)!} \frac{\partial^{m-1} A[\phi(r;q)]}{\partial q^{m-1}}\Big|_{q=0}.$$

There are two different approaches for deducing mth-order deformation equation (1.1.15). The first approach is introduced by S. J. Liao and Y. Tan, see [23].
The second approach is deduced by T. Hayat et al. [46] and M. Sajid et al. [41] by directly substituting homotopy-series (1.1.7) into zeroth-order equation (1.1.14) and equating the coefficients of the like power of q, one can get exactly the same equation as (1.1.15). Since L is a linear operator, we revise (1.1.15) as

$$L[u_m(r)] = \chi_m L[u_{m-1}(r)] + \hbar H(r) R_m(\vec{u}_{m-1}). \tag{1.1.16}$$

Hence, we have an expression for $u_m(r)$ in terms of all lower order terms $u_j(r)$, for $j = 0, 1, \cdots, m-1$. In principle, we have to solve an infinite number of inhomogeneous linear differential equations,

$$L[u_m(r)] = f_m(u_{m-1}(r), \cdots, u_0(r), r) = RHS_m(r),$$

that is much simpler than solving a nonlinear differential equation, where f_m is the right hand side (RHS) of Eq.(1.1.16). By solving this linear differential equation, subject to its initial and/or boundary conditions, the expression

$$u_m(r) = L^{-1}(RHS_m(r)) + J(x), \tag{1.1.17}$$

where L^{-1}, the inverse operator of L, is the inhomogeneous contribution due to $RHS_m(r)$, and J is the homogeneous contribution from the linear operator L, it means that $J(x)$ belongs to the kernel of operator L and satisfies the condition $L(J(x)) = 0$. The solutions of the first M linear subproblems are used to give Mth-order approximation:

Definition 1.1.12. The Mth-order approximation of $u(r)$ is given by

$$u(r) \approx \sum_{n=0}^{M} u_n(r).$$

Furthermore, it is called partial sum of order M and is shown by $S_M(r)$.

Theorem 1.1.13. (Liao's convergence Theorem)*As long as the series*

$$u_0(r) + \sum_{m=1}^{+\infty} u_m(r)$$

is convergent, where $u_m(r)$ is governed by the high-order deformation equation (1.1.11), it must be a solution of equation (1.0.1).

For proof see chapter 3 of [27]. In practice, a series may not converge over the whole domain of the problem. In such cases, the following result may be useful. Thus, in order to show that a series solution obtained via HAM converges to a solution of the desired nonlinear differential equation for some subset of the domain prescribed in the original problem, it suffices to show that the series converges over this domain.

Corollary 1.1.14. *Assume that the series $u_0(r) + \sum_{m=1}^{+\infty} u_m(r)$ is convergent over some subset Ω of the domain of the problem $A[u(r)] = 0$, and that $u_m(r)$ is governed by the mth-order deformation equation (1.1.11). Then, the series is a solution to the nonlinear differential equation $A[u(r)] = 0$ over the same subset Ω.*

Theorem 1.1.15. *As long as the series*

$$u_0(r) + \sum_{m=1}^{+\infty} u_m(r)$$

is convergent, where $u_m(r)$ *is governed by the* m*th-order deformation equation (1.1.11), it holds that*

$$\sum_{m=1}^{+\infty} R_m(\vec{u}_{m-1}, r) = \sum_{m=0}^{+\infty} \delta_m(r) = 0,$$

where

$$R_m(\vec{u}_{m-1}, r) = D_m(E(q)A[\Phi(r;q)]) = \sum_{k=1}^{m} \alpha_k \delta_{m-k}(r).$$

For proof refer to [27]. In addition according to Theorem (1.1.13) and Theorem (1.1.15), we need only focus on choosing the initial approximation $u_0(r)$, the auxiliary linear operator L, the embedding functions $E(q)$, $B(q)$, the convergence control parameter $\hbar$, and the auxiliary function $H(r)$ to ensure that the solution series (1.1.9) converges.

Theorem 1.1.16. (Necessary conditions for convergence [17])*For a specific nonlinear differential equation* $A[u(r)] = 0$*, let* $u(r)$ *and* $u_m(r)$ *be as defined above, and let* Ω *be the domain of interest. Then, in order for* $u(r)$ *to converge, it is necessary to have* $\lim_{m\to+\infty} |u_m(r)| = 0$ *for all* $r \in \Omega$*, and there must exist a positive integer* k *such that* $|u_m(r)| \leq |u_{m-1}(r)|$ *for all* $m > k$*, and all* $r \in \Omega$*.*

Theorem 1.1.17. (Sufficient conditions for convergence [17]) *For a specific nonlinear differential equation* $A[u(r)] = 0$*, let* $u(r)$ *and* $S_M(r)$ *be as defined above, and let* Ω *be the domain of interest. If for any real* $\delta > 0$ *there exists a positive integer* k *such that* $|u(r) - S_M(r)| < \delta$ *for all* $M > k$ *and all* $r \in \Omega$*, then the series solution* $u(r)$ *converges.*

Definition 1.1.18. The error of partial sum (or residual error) is defined as following

$$\varepsilon_M(r) = |A[S_M(r)]|.$$

Theorem 1.1.19. (Necessary conditions for convergence to the solution [17]) *For a specific nonlinear differential equation* $A[u(r)] = 0$*, let* $u(r)$ *and* $\varepsilon_M(r)$ *be as defined above, and let* Ω *be the domain of interest. Then, in order for* $u(r)$ *to converge to a solution of* $A[u(r)] = 0$*, it is necessary to have* $\lim_{M\to\infty} \varepsilon_M(r) = 0$ *for all* $r \in \Omega$*, and there must exist a positive integer* k *such that* $\epsilon_M(r) \leq \epsilon_{M-1}(r)$ *for all* $M > k$ *and all* $r \in \Omega$*.*

Theorem 1.1.20. (Sufficient condition for convergence to the solution [17]) *For a specific nonlinear differential equation* $A[u] = 0$*, let* $u(r)$ *and* $S_M(x)$ *be as defined above, and let* Ω *be the domain of interest. If for any real* $\delta > 0$ *there exists a positive integer* k *such that* $|A[u(r)] - A[S_M(r)]| < \delta$ *for all* $M > k$ *and all* $r \in \Omega$*, then the series solution* $u(r)$ *converges to a solution of* $A[u] = 0$.

It may be the case that a solution series requires only a finite number of terms to be calculated. Not only would such a series converge, but it would also converge to the solution of nonlinear differential equation. In the following chapter we show the application of the under consideration theorems (1.1.16) - (1.1.20) in practice.

1.2 Fundamental rules to direct us

In previous section, we mentioned that we have great freedom to choose the primal tools of HAM, such as the auxiliary linear operator, the initial approximation and the auxiliary function to construct zeroth-order deformation equation. But it seems that such a great freedom needs some fundamental rules to specify the bound of such freedom.

1.2.1 The rule of solution existence

HAM is based on some assumptions, in section (1.1) we pointed to :
The solution $\Phi(r; q)$ of the zeroth-order deformation equation exists for all $q \in [0, 1]$ if the initial approximation, auxiliary linear operator and auxiliary function were properly chosen.
So, if the original nonlinear problem has a solution, all of these linear subproblems (1.1.17) should have solutions too, this rule is called *The rule of solution existence.*

1.2.2 The rule of solution expression

The essence of the analytic approximation is to express its solution by a proper set of base functions. It is clear that one can get better approximation by means of better basis functions. In other words, solutions of nonlinear differential equations can be approximated more efficiently by means of suitable basis functions. For example, it is convenient for a periodic solution to express by periodic basis functions than by polynomials. Due to the integrations required to recover $u_m(r), m = 1, 2, \cdots$, in addition to the homogeneous terms due to the inversion of auxiliary linear operator, the inhomogeneous terms will involve many integrations of the initial approximation. Thus, if one were to select an initial approximation in terms of polynomials, one would expect a solution in terms of polynomials and polynomials multiplying any homogeneous term contributions from the auxiliary linear operator and their integrals. Convenient basis functions, such as polynomials, decaying exponential, sines, cosines, rational functions and other elementary functions or products of such functions, are quite commonly employed. Because such choices are fairly easy to integrate, which decrease the computational time required to obtain the terms u_m from equation (1.1.17) and hence the approximate solutions. Further, it is well known that basis functions have close relationship with linear operators. To introduce *the rule of solution expression* practically, we follow two approaches.

In the first approach, by means of analyzing the physical background and/or initial/boundary conditions and/or its type of nonlinearity, we choose a set of base functions that is proper to represent the solution, even without solving a given nonlinear problem [see next chapter]. For example consider the set,

$$\{e_k | k = 0, 1, \cdots\},$$

as base functions. Then we can represent the solution in a series

$$u(r) = \sum_{n=0}^{\infty} c_n e_n(r), \tag{1.2.1}$$

where c_n's are coefficients to be determined. After determining the base functions, the auxiliary linear operator L, the auxiliary function $H(r)$ then the initial approximation will be choosen so that the representation (1.2.1) be abiding. In what follows we explain this approach more clearly. In the second approach, the set of base functions is not introduced but at first we choose the auxiliary linear operator L, then one may select an initial approximation in terms of functions which agree with the homogeneous solutions to the auxiliary linear operator.

Consequently, we can easily avoid the appearance of the so-called secular terms in solution expression by using this rule, also it practically provides us with a starting point and therefore plays a very important role within the frame of the homotopy analysis method. As we will see later the auxiliary function $H(r)$ plays a crucial role for performing this rule.

1.2.3 Selection of the initial approximation

For the selection of an adequate initial approximation we consider two principles:

(i) The initial approximation should satisfy the initial and/or the boundary condition(s) of the original nonlinear differential equation; and

(ii) be expressed in terms of functions which are both convenient (it means be easy to integrate) and useful (it means that functions should be selected which allow for the convergence of the series solutions).

In the following examples we explain selection of initial approximation to solution of some problems.

Example 1.2.1. *Yann Bouremel [10] considered the Glauert-jet's equation*

$$f''' + ff'' + 2f'^2 = 0,$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 0,$$

$$f(\xi) \to 1, \quad f'(\xi) \to 0 \quad as \quad \xi \to \infty,$$

where f denotes a dimensionless stream function and the primes denote derivatives with respect to the variable ξ. Through HAM, he has obtained an explicit series solution for this problem. According to the boundary conditions, the solution can be expressed by the following set of base functions (it seems that the boundary condition at infinity had influence on selecting such base function)

$$\left\{ e^{-n\xi} | n \geq 0, n \in \mathbb{N} \right\},$$

in the form

$$f(\xi) = \sum_{n=0}^{+\infty} c_n e^{-n\xi},$$

where c_ns $(n = 0, 1, \cdots)$ are coefficients. According to the rule of solution expression, it is straightforward to choose an initial approximation

$$f_0(\xi) = a_0 + a_1 e^{-\xi} + a_2 e^{-2\xi},$$

where $a_i (i = 0, 1, 2)$, are determined by the boundary conditions. So, the initial approximation is

$$f_0(\xi) = 1 - 2e^{-\xi} + e^{-2\xi}.$$

Sometimes the nonlinear differential equation have the following form

$$L[u(r)] + g(r) + N[u(r)] = 0, \quad B(u, \frac{du}{dr}, \cdots) = 0,$$

where L is a linear operator, r denotes independent variable, $u(r)$ is an unknown function, $g(r)$ is a known function, N is a nonlinear operator and B is a boundary operator. In such cases, it may be convenient to construct such a zeroth-order deformation equation

$$(1 - q) \left[L[\phi(r; q)] + g(r) \right] = q\hbar H(r) \left[L[\phi(r; q)] + g(r) + N[\phi(r; q)] \right].$$

Here we can obtain $u_0(r)$ from the above zeroth-order deformation equation for $q = 0$:

$$L[u_0(r)] + g(r) = 0, \quad B(u_0, \frac{du_0}{dr}, \cdots) = 0.$$

See the following example.

Example 1.2.2. *Consider the second type of the Painlevé equation which is formulated in the following form [12]*

$$u'' = 2u^3 + xu + \alpha,$$

with the following initial conditions

$$u(0) = 1, \quad u'(0) = 0.$$

According to the above comment consider $L(\phi) = \phi''$ *as the linear operator,* $g(x) = \alpha$ *as a known function and* $N[\phi] = 2\phi^3 + x\phi$ *as a nonlinear operator. Hence the initial approximation can be obtain from the following differential equation*

$$u_0'' - \alpha = 0, \quad u_0(0) = 1, \quad u'(0) = 0.$$

So, we have $u_0(x) = \alpha\frac{x^2}{2} + 1$*. Ellahi et al. [15] use this initial approximation in their paper.*

By catching a glimpse of problems which are handled by HAM, we can see some other approaches of determining initial guess approximation. For example in problems which are time dependent we can consider the initial condition at $t = 0$ as an initial guess approximation, for instance see [8, 40].
To answer following questions that are propounded by S.J. Liao and Y. Tan [23] we arrange next section.

(1) Must the linear operator L in (1.1.16) for linear subproblems have a close relationship with the linear operator appeared in the original nonlinear equation $A[u] = 0$?; and

(2) Must the linear operator L in (1.1.16) for the linear subproblems have the same order as that of original nonlinear operator A?

1.2.4 Selection of the auxiliary linear operator

At first we introduce some theorems on the selection of an optimal auxiliary linear operator.

Theorem 1.2.3. (Van Gorder and Vajravelu selection theorem [17]) *For any two auxiliary linear operators L and $\tilde{L}$ and resulting solutions $u_m(r)$ and $\tilde{u}_m(r)$, if the approximations $S_m(r) = S_{m-1}(r) + u_m(r)$ and $\tilde{S}_m(r) = \tilde{S}_{m-1}(r) + \tilde{u}_m(r)$ satisfy the relation $|A[\tilde{S}_m(r)]| \leq |A[S_m(r)]|$ for all $r \in \Omega$, where Ω is some subset of the domain given in the original nonlinear problem, then $\tilde{L}$ is a better choice of auxiliary operator than L, for the mth iteration of the method.*

Definition 1.2.4. (Optimal linear operator) *An optimal auxiliary linear operator for the mth order deformation equation is an operator L^* admitting a solution $u_m^*(r)$ satisfying $|A[S_{m-1}(r) + u_m^*(r)]| \leq |A[S_{m-1}(r) + u_m(r)]|$ for all $r \in \Omega$, where Ω is some subset of the domain given in the original nonlinear problem and for all other solutions $u_m(r)$ corresponding to auxiliary linear operators L.*

Theorem 1.2.5. (Van Gorder and Vajravelu global optimal auxiliary linear operator theorem [17]) *Assume that an optimal auxiliary linear operator L_m^* exists for each mth order deformation equation. If there exists an optimal auxiliary linear operator over all mth order deformation equations, say $\tilde{L}$, then it must be the case that $\tilde{L} = L_1^* = L_2^* = \cdots$. That is, such an operator must be optimal for each $m = 1, 2, \cdots$.*

Proof. Assume that for some iteration m, a proposed optimal operator $\tilde{L}$ fails to be optimal. That is, on the mth iteration, there exists another operator L_m^* such that the error in the approximate solution $S_m^*(r)$ obtained via L_m^* is strictly less than the error in the approximate solution $\tilde{S}_m(r)$ obtained via $\tilde{L}$. By the definition of an optimal auxiliary linear operator given above, $\tilde{L}$ cannot be an optimal auxiliary linear operator over all mth order deformation equations. Yet, we assumed that $\tilde{L}$ was an optimal auxiliary linear operator over all mth order deformation equations, so we have reached a contradiction. □

Corollary 1.2.6. *As such, it must be the case that, if $\tilde{L}$ is an optimal auxiliary operator over all mth order deformation equations, it must be optimal for each iteration (i.e., there is no other linear operator which is more optimal, on any iteration). By a uniformly optimal auxiliary linear operator, we mean any optimal auxiliary linear operator which is optimal over all $m = 1, 2, \cdots$ order deformation equations, and hence satisfies Theorem (1-2-5).*

Question 1.2.7. How we can choose optimal auxiliary linear operator?

In practice it is not easy to find optimal linear operator but we give some useful guidelines via examples that help us to select good auxiliary linear operators.

1.2.5 Method of linear partition matching

This section is entitled by Van Gorder and Vajravelu. If the nonlinear operator A be separable into a linear part L_0 and nonlinear part N_0, such that L_0 has constant coefficients then we can set $L = L_0$.

Example 1.2.8. *Consider the nonlinear differential equation of the form [3]*

$$\gamma\mu^4 w^{(4)} + \beta\mu^2 w'' + \frac{\alpha}{2} a w^2 - cw = 0$$

subject to the boundary conditions

$$w(0) = 1, \quad w'(0) = 0, \quad w(\infty) = 0, \quad w'(\infty) = 0,$$

where prime denotes the derivative with respect to η and α, β, γ and $\mu > 0$ are some arbitrary constants. Furthermore, we have the term

$$\gamma\mu^4 + \beta\mu^2 = c,$$

where c is the wave speed. Under linear partition matching, Abbasbandy [3] takes the auxiliary linear operator to be the linear part of the original equation

$$L[\phi(\eta;q)] = (\gamma\mu^4\frac{\partial^4}{\partial\eta^4} + \beta\mu^2\frac{\partial^2}{\partial\eta^2} - c)\phi(\eta;q).$$

As we see, equation $L[\phi(\eta;q)] = 0$ *is a linear equation with constant coefficients, indeed it has solutions of the form* $\phi(\eta) = e^{\alpha\eta}$, *where the values of* α *are the zeros of the so-called characteristic polynomial. So, we have four values for* α

$$-1, \quad 1, \quad -x, \quad x,$$

where

$$x = \pm\sqrt{-1 - \frac{\beta}{\gamma\mu^2}}.$$

So, L has the property,

$$L[C_1e^{-\eta} + C_2e^{\eta} + C_3e^{-x\eta} + C_4e^{x\eta}] = 0.$$

Now we construct the relation (1.1.17) for this example,

$$w_m(\eta) = L^{-1}(RHS_m(\eta)) + C_1e^{-\eta} + C_2e^{\eta} + C_3e^{-x\eta} + C_4e^{x\eta}.$$

Abbasbandy has considered the traveling-wave solution expression by

$$w(\eta) = \sum_{m=1}^{+\infty} d_m e^{-m\eta}.$$

So, according to the rule of solution expression the terms $e^{-x\eta}$, $e^{x\eta}$ *and* e^{η} *are out of considered set of base functions. Therefore, in this example they are called secular terms and should be omitted from* $w_m(\eta)$ *at each iteration. So, we set* $C_2 = C_3 = C_4 = 0$.

Example 1.2.9. *In this example, we consider the thin film flow problem [5] which is modeled by the equation*

$$\frac{d^2u}{dy^2} + 6\beta(\frac{du}{dy})^2\frac{d^2u}{dy^2} + 1 = 0,$$

subject to the boundary conditions

$$u(0) = 0, \quad u'(1) + 2\beta(u'(1))^3 = 0.$$

Again, under linear partition matching, Abbasbandy et al. [5] take the auxiliary linear operator to be the linear part of the original equation

$$L[\phi(y;q)] = (\frac{\partial^2}{\partial y^2})\phi(y;q).$$

As we see, equation $L[\phi(y;q)] = 0$ *is a linear equation with constant coefficient, and its characteristic equation has* 0 *as a root with multiplicity* 2*. So, its solutions are of the form* $\phi(y) = y^k e^{\alpha y}$*, where* $\alpha = 0$ *and* $k = 0, 1$*. Hence* L *has the property,*

$$L[C_1 + C_2 y] = 0,$$

in which C_1 *and* C_2 *are constants.*

Example 1.2.10. *Yongyan Wu et al. [52] solved the Vakhnenko equation by means of HAM. This equation, after transformation, is of the form*

$$g'''(\theta) + \gamma[g'(\theta)]^2 - g'(\theta) = 0,$$

subject to the boundary conditions

$$g(0) = -1, \quad g''(0) = 0, \quad g(+\infty) = 0.$$

They take

$$L[\phi(\theta;q)] = (\frac{\partial^3}{\partial \theta^3} - \frac{\partial}{\partial \theta})\phi(\theta;q).$$

The zeros of equation $L[\phi] = 0$ *consists of*

$$1, \quad e^{\theta}, \quad e^{-\theta},$$

so, L *has the property*

$$L[C_1 + C_2 e^{\theta} + C_3 e^{-\theta}] = 0.$$

Despite fabulous results at previous examples, Liao and Tan [23], in the following example, showed that the method of linear partition matching might completely mislead us.

Example 1.2.11. *Consider a nonlinear oscillation, governed by*

$$u''(t) + \lambda u(t) + \epsilon u^3(t) = 0, \quad u(0) = 1, \quad u'(0) = 0,$$

where $\lambda \in (-\infty, +\infty)$ *and* $\epsilon \geq 0$ *are physical parameters. Mathematically, the above equation contains the linear and nonlinear parts:*

$$L_0[u] = u'' + \lambda u, \quad N_0[u] = \epsilon u^3.$$

Physically, the oscillation motion is periodic, no matter λ *is positive or negative. Thus, from physical points of view, it is easy to know that* $u(t)$ *is periodic, even if we do not directly solve the equation.*
Such as previous examples, we seek the solutions of the ordinary differential equation $L(\phi(t; q)) = 0$, *when* $\lambda \leq 0$, *so*

$$e^{\sqrt{|\lambda|}t}, \quad e^{-\sqrt{|\lambda|}t},$$

are these solutions. Therefore when $\lambda \leq 0$, $L = L_0$ *can not give us good approximation as is shown in [23]. Henceforth, it seems unnecessary that the linear operator* L *in the linear subproblems (1-1-16) should have a close relationship with* L_0 *in the original nonlinear equation* $A[u] = L_0 u + N_0 u = 0$.

I think alluding bellow definition and theorem could be useful.

Definition 1.2.12. *A differential equation is said to be stable if for every set of initial data (at r=0) the solution of the differential equation remains bounded as* r *approaches infinity [13].*

A differential equation is called strongly stable if, for every set of initial data (at r=0) the solution not only remains bounded, but approaches zero as r approaches infinity.

Theorem 1.2.13. *A linear differential equation with constant coefficients is stable if and only if all of the roots of its characteristic equation lie in the left half plane, and those that lie on the imaginary axis, if any, are simple. Such a equation is strongly* *stable if and only if all of the roots of its characteristic equation lie in the left half plane, and none lie on the imaginary axis [13].*

Unfortunately, all of the current techniques for nonlinear problems, such as perturbation techniques [35–37], Adomian's decomposition method [7] and so on, have been based on the existence of the linear part operator hence, can not provide us such kind of freedom to choose the linear operator L as HAM can do. Another option for the selection of an auxiliary linear operator is:

1.2.6 Method of highest order differential matching

In this method, the highest order derivative of the equation appears in the linear part of operator A. So, we employ the highest derivative as the linear operator L.

Example 1.2.14. *Hang Xu et al. [53] used the rule of highest order differential matching to find solutions of the Navier-Stokes equations. This equation after transformation has the form*

$$s'''' + \alpha(\eta s''' + 3s'') + R(ss''' - s's'') = 0,$$

subject to the boundary conditions

$$s(0) = 0, \quad s''(0) = 0, \quad s(1) = 1, \quad s'(1) = 0,$$

where the prime denotes derivative with respect to η. They noted that, even for the nonlinear ODE, it is not convenient to employ its entire linear part $L = s'''' + \alpha(\eta s''' + 3s'')$ as the linear operator. Otherwise, the $L = 0$ would contain the error function $erfi[\sqrt{\frac{\alpha}{2}}\eta]$. For showing this error function we use the Mathematica command

$$L[\phi\_] := Expand[D[\phi, \{\eta, 4\}] + \alpha(\eta D[\phi, \{t, 3\}] + 3D[\phi, \{t, 2\}])];$$

now for instance, we see the influence of L^{-1} on f:

$$L^{-1}(f) = \frac{f\eta^2}{6\alpha} + \frac{C_1 e^{-\frac{\eta^2\alpha}{2}}}{\alpha} + (\frac{\sqrt{\frac{\pi}{2}} e^{-\frac{\eta^2\alpha}{2}} \sqrt{\eta^2\alpha} \quad erfi\left(\frac{\sqrt{\eta^2\alpha}}{\sqrt{2}}\right)}{2\eta\alpha^2} - \frac{\eta}{2\alpha})C_2 + C_3 + C_4\eta.$$

So, "it would then become exceedingly difficult to retrieve higher order approximations". This obstacle can be overcome by selecting an auxiliary operator of equal order, in this case, of fourth order, such as

$$L\Phi = \frac{d^4\Phi}{d\eta^4}.$$

Solutions of the equation $L[\phi] = 0$ are

$$1, \quad \eta, \quad \eta^2, \quad \eta^3.$$

Hence the auxiliary linear operator L has the property

$$L(C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3) = 0,$$

where C_i $(i = 0, \cdots, 3)$ are integral constants. Such operators afford us the most convenience when solving for the $s_m(\eta)$'s as they allow for simple integrations to be performed over $RHS_m(\eta)$.

It is obvious that, if the highest order is the only linear term, this method becomes a special case of the linear partition matching.

1.2.7 Method of complete differential with decreasing order

Let the highest order of derivative which is present in the nonlinear differential operator A be k. In this method we construct the auxiliary linear operator as follow

$$L(\phi) = \phi^{(k)} + a_{k-1}\phi^{(k-1)} + \cdots + a_0\phi,$$

where a_i, $(i = k-1, \cdots, 0)$, are unknowns to be determined by selecting suitable base functions and good kernel for auxiliary linear operator L.

Example 1.2.15. *To illustrate this method clearly we consider the Thomas- Fermi equation [28] which is used to calculate the electrostatic potential in atom model.*

$$x[u''(x)]^2 - u^3(x) = 0. \tag{1}$$

with boundary conditions

$$u(0) = 1, \quad u(+\infty) = 0, \tag{2}$$

in the common case. Note that Eq.(1) contains no linear terms. Liao, by considering the boundary conditions (2) and physical meaning of $u(x)$, chose the set of base functions [28]

$$\left\{(1+x)^{-m}|m \geq 1\right\}.$$

The highest order derivative which has appeared in Eq.(1) is 2*. So we construct the auxiliary linear operator as*

$$L\phi = \phi'' + a_1\phi' + a_0\phi.$$

For determining unknowns a_0 and a_1, Liao has considered 1 *and $\frac{1}{(1+x)}$ to be zeros of equation $L[\phi] = 0$. Consequently*

$$L[\phi(x;q)] = \frac{\partial^2\phi(x;q)}{\partial x^2} + \frac{2}{(1+x)}\frac{\partial\phi(x;q)}{\partial x},$$

such that

$$L[C_1(1+x)^{-1} + C_2] = 0.$$

In chapters 2 and 3 you will see other examples that are handled by this method. The choices proposed above are useful in some, but certainly not in all cases. In [23], Liao gave an example which uses an auxiliary linear operator with a higher order of derivative than the order of derivative of the original nonlinear operator. As a consequence of the previous subsections on choosing linear operator L, we can say that by means of HAM, a nonlinear ODE is often replaced by an infinite number of linear

ODEs, and a nonlinear PDE can be transferred into an infinite number of linear ODEs. Besides, a nonlinear differential equation with variable coefficients can be replaced by an infinite number of linear differential equations with constant coefficients.

1.2.8 The rule of coefficient ergodicity

For illustrating the concept of this rule consider

$$\{e_k(r)|k = 0, 1, 2, \cdots\}$$

as a set of base functions and then express the solution in the form of

$$u(r) = \sum_{n=0}^{+\infty} c_n e_n(r).$$

When we say *The rule of coefficient ergodicity* is abiding, one might rather say that all coefficients c_n $(n = 0, 1, \cdots)$ in the solution expression should be present to ensure the completeness of the set of base functions. Also there should not exist any secular terms (it means any term out of the base functions). One of the crucial factors to follow this rule, is the selection of the auxiliary function $H(r)$ which ensures us the mth-order deformation equations are closed. Liao, in chapter 2 of his book [27], investigates different kinds of base functions and gives the corresponding $H(r)$. Consider the homotopy equation

$$(1 - q)L(u - u_0) = q\hbar H(r)A[u]. \tag{1.2.2}$$

Often, $H(r) \neq 0$ for any r in the relevant domain of consideration. However, under such condition, we define

$$\widetilde{L}[u] = \frac{1}{H(r)} L[u]. \tag{1.2.3}$$

So, the above homotopy is modified into

$$(1 - q)\widetilde{L}[u - u_0] = q\hbar A[u]. \tag{1.2.4}$$

Yet, for such a homotopy, there exists an optimal linear operator, as discussed in the previous section. It is clear that, when $H(r) \neq 0$ for any r in the domain of problem, we may consider the homotopy (1.2.4) rather than the original homotopy (1.2.2) in order to construct the series solutions via HAM. This is why, in many papers, one simply takes $H(r) = 1$. Thus, we have that:

Theorem 1.2.16. *In the case that $H(r)$ is a non-vanishing function over the domain of the problem, it suffices to take $H(r) = 1$ in the homotopy (1.2.2) and obtain a series solution via the homotopy analysis method by an appropriate choice of auxiliary linear operator L and initial approximation $u_0(r)$, see [17].*

Example 1.2.17. *Here consider Eq.(2.5) from chapter 2 of Liao's book [27]*

$$\dot{V}(t) + V^2(t) = 1, \quad t \geq 0,$$

and its series solution in the form

$$V(t) = \sum_{n=0}^{+\infty} a_n e^{-nt},$$

where $a_n (n = 0, 1, 2, \cdots)$ are coefficients. Liao have chosen the auxiliary linear operator

$$L[\Phi(t;q)] = \frac{\partial \Phi(t;q)}{\partial t} + \Phi(t;q),$$

with the property

$$L[C_1 e^{-t}] = 0.$$

As pointed out by Liao, to obey both, the rule of solution expression and the rule of coefficient ergodicity, the corresponding auxiliary function is

$$H(t) = e^{-t},$$

the reason of this choice is stated clearly in page 28 of his book. Since $H(t) \neq 0$ for all $t \geq 0$ according to theorem (1.2.17) we can consider

$$\widetilde{L}[u] = e^t \left(\frac{\partial \Phi(t;q)}{\partial t} + \Phi(t;q)\right),$$

and use the homotopy (1.2.4). Therefore, by considering $v_0 = 1 - e^{-t}$ as an initial approximation we have

$$v_1(t) = -\frac{\hbar}{2} e^{-t} + \hbar e^{-2t} - \frac{\hbar}{2} e^{-3t},$$

$$v_2(t) = -\frac{\hbar}{2}(1+\frac{\hbar}{2})e^{-t} + \hbar(1+\frac{\hbar}{2})e^{-2t} - \frac{\hbar}{2}(1+\hbar)e^{-3t} + \frac{\hbar^2}{2}e^{-4t} - \frac{\hbar^2}{4}e^{-5t},$$

$$\vdots$$

which has full agreement with corresponding values reported at page 28 of [27].

Theorem 1.2.18. *Assume that* $0 = \hbar H(r)A[u(r)]$ *for all* $r \in K \subseteq \mathbb{R}$ *and that there exists some interval* $(a, b) \subset K$ *over which* $H(r)$ *vanishes. Then,* $u(r)$ *is not necessarily a solution to the nonlinear equation* $A[u(r)] = 0$ *over the domain* K*. [17]*

Remark 1.2.1. Thus, one should avoid using such auxiliary functions $H(r)$ which vanishes over any interval contained in the domain of the problem.

Theorem 1.2.19. *Assume that* A *is a continuous function of* u *and that* $H(r)$ *vanishes over a set of measure zero. If some continuous function* $u(r)$ *satisfies* $0 = \hbar H(r)A[u]$ *over the domain of the problem, then* $u(r)$ *is a solution to the nonlinear equation* $A[u(r)] = 0$ *over the domain of the problem. [17]*

In the homotopy given in (1.2.4), we introduced the auxiliary linear operator (1.2.3) which depends on $\frac{1}{H(r)}$. If $H(r)$ vanishes over a set of measure zero, then the auxiliary linear operator (1.2.3) will have singularities at all members of this set of measure zero. Such singularities complicate the recursive process to obtain the terms $u_m(r)$ in the mth order deformation equations. In practice these vanishing auxiliary functions will modify the particular solutions obtained when solving for the $u_m(r)$'s. So, it is convenient to avoid auxiliary functions $H(r)$ which vanish at any point over the domain of the problem, unless one has a good reason to use them.

Of course, in some papers [17, 45] it is alluded that when the set of base functions is not introduced, then one can simply choose $H(r) = 1$.

1.2.9 The convergence control parameter and discriminating the convergence region

As mentioned before Liao [24] constructed the zeroth-order deformation equation by introducing the convergence control parameter $\hbar$. Consequently, once the initial approximation, the auxiliary linear operator and the auxiliary function are properly chosen, Homotopy Analysis Method provides one with a family of solutions, dependent upon the convergence control parameter. Via an appropriate choice of the convergence control parameter $\hbar$, we can select a member of this family as the approximate solution to the nonlinear equation.

Question 1.2.20. *How is the value of $\hbar$ chosen to ensure that the solution series in a large enough region is converged?*

Such convergence region is usually obtained through trial and error. As we know, in most physical phenomena, high order derivatives at 0 is illustrative for physical quantity. So, high order derivatives of approximate analytic HAM solution at 0 depend on homotopy parameter $\hbar$, and as a consequence when we plot these derivatives versus $\hbar$ we have the so called $\hbar-$curves. Liao [27] showed that the horizontal line segment which appears in the $\hbar-$curves determines the valid region $R_{\hbar}$ for this parameter.

Question 1.2.21. *How one can check the suitability of the region $R_{\hbar}$?*

One approach which have a wide application to answer this question is plotting the residual error for some $\hbar$ inside the valid region $R_{\hbar}$. Residual error can be considered in different ways. One way is to put the approximate analytic HAM solution directly into the original nonlinear differential equation and plotting its curve for some $\hbar$ in $R_{\hbar}$, see for example chapter 2, Fig. 2.4 and Fig. 2.9.

To explain another way, consider a nonlinear differential equation $A[u(r,t)] = 0$, which is depend on two variables r and t. Let $\delta(r,t)$ denote the residual error of the mth-order homotopy-series approximation of this problem, and $\Delta = \int\int \delta^2(x,t)dxdt$ denote the integral of the residual error [54] (note that in some cases we can not achieve Δ

analytically hence we calculate it numerically). By plotting the curve of Δ versus $\hbar$, it is straightforward to find a region of $\hbar$ in which Δ decreases to zero as the order of approximation increases. So, a convergent homotopy-series solution is obtained by choosing a value in this region. Abbasbandy et al. [5] have computed norm 2 of the error for two successive approximation of $U_M(y)$, and obtained the best value for $\hbar$.

1.2.10 Diagonal Homotopy Padè

The Padè approximation is a rational function that can be thought of as a generalization of a Taylor polynomial. A rational function is the ratio of polynomials. Because these functions only use the elementary arithmetic operations, they are very easy to evaluate numerically. The polynomial in the denominator allows you to approximate functions that have singularities. More precisely, a Padè approximation of order $[n, m]$ to an analytic function $f(x)$ at a regular point or pole x_0 is the rational function $\frac{p(x)}{q(x)}$ where $p(x)$ is a polynomial of degree n, $q(x)$ is a polynomial of degree m, and formal power series of $f(x)q(x) - p(x)$ about point x_0 begins with the term $(x - x_0)^{n+m+1}$, if m is equal to n, the approximation is called a diagonal Padè approximation of order n. A Padè approximation is very accurate near the center of expansion, but the error increases rapidly as you get farther away, for example see [11]. Liao [27] have employed the Padè technique to ensure that the homotopy series is convergent at $q = 1$, he first employed the traditional $[n, m]$ Padè technique about the embeding parameter q to obtain the $[n, m]$ Padè approximate

$$\frac{\sum_{k=0}^{n} A_{n,k}(x)q^k}{\sum_{k=0}^{m} B_{m,k}(x)q^k}, \tag{1.2.5}$$

where the coefficients $A_{n,k}(x)$ and $B_{m,k}(x)$ are determined by the first several approximations

$$u_0(x), u_1(x), u_2(x), \cdots, u_{n+m}(x).$$

Then by setting $q = 1$ in (1.2.5) we have the so-called $[n, m]$ homotopy-Padè approximant

$$\frac{\sum_{k=0}^{n} A_{n,k}(x)}{\sum_{k=0}^{m} B_{m,k}(x)}.$$

You can see the application of this approach in next chapter. It is necessary to point out that in most cases the padè approximation for homotopy series will become free from convergence control parameter $\hbar$, see for example chapter 2 page 40 of [27]. This attribute, encourages us to use padè approximation.

1.2.11 Comparison Between Homotopy Analysis Method (HAM) and Homotopy Perturbation Method (HPM)

This section is an abridgment comparison of homotopy analysis method and homotopy perturbation method. J. H. He proposed the homotopy perturbation method (HPM) in 1998 about 6 years after introducing HAM. As before, consider the nonlinear differential equation $A[u(r)] = 0$ and for the sake of simplicity consider $H(r) = 1$ as an auxiliary function in HAM.

Homotopy equation for HAM is of the form:

$$(1-q)L[\phi(r;q) - u_0(r)] = q\hbar A[\phi(r;q)],$$

but homotopy equation for HPM is :

$$(1-q)L[\phi(r;q) - u_0(r)] = -qA[\phi(r;q)].$$

In fact HAM uses q to obtain Taylor series

$$\phi(r;q) = \sum_{n=0}^{\infty} \frac{q^n}{n!}\left[\frac{\partial^n \phi(r;q)}{\partial q^n}\Big|_{q=0}\right],$$

but HPM uses q as an expanding parameter

$$\phi(r;q) = u_0 + qu_1 + q^2 u_2 + \cdots .$$

Hence it can be easily proved that

$$u_n = \frac{1}{n!}\frac{\partial^n \phi(r;q)}{\partial q^n}\Big|_{q=0}.$$

This is called as the straight forward expansion into the perturbation theory. The straight forward expansion is equivalent to Taylor's expansion.

mth order deformation equation for HAM is:

$$L[u_m(r) - \chi_m u_{m-1}(r)] = \hbar R_m,$$

but the mth order equation in HPM is of the form

$$L[u_m(r) - \chi_m u_{m-1}(r)] = -R_m,$$

where in both cases

$$R_m = \frac{1}{(m-1)!} \frac{d^{m-1} A[\phi(r;q)]}{dq^{m-1}}\Big|_{q=0}.$$

One can see that HAM concludes HPM when $\hbar = -1$. It must be emphasized that HPM does not contain the auxiliary parameter $\hbar$. Thus like the traditional analytic technique, the HPM can not provide a simple way of convergence of the solution. Sajid et al. [42] showed by two examples that despite HPM is a special case of HAM in general but the HAM solution becomes divergent when $\hbar = -1$, in both examples. I think table of results of one of their examples will suffice.

Table 1.1: Comparison of the HPM and HAM results of $v''(0)$ in case of a Sisko fluid.

order of approximation	HAM results for $\hbar = -0.3$	HPM solution
2	0.64000	1.66667
4	0.60970	4.66667
6	0.60262	17.7737
8	0.60047	76.7023
10	0.59973	347.402
12	0.59945	1610.13
14	0.59935	7567.3
16	0.59930	35914.3
18	0.59928	171718.0
19	0.59928	-376338.0
20	0.59928	825872.0

Of course, there are some other papers about comparison between HAM and HPM that I suggest them to read, such as [19, 30].

Chapter 2

Analytic solution of the generalized Zakharov equation by homotopy analysis method

In the integration of laser-plasma the system of Zakharov equation plays an important role. More recently, some authors considered the exact and explicit solutions of the system of Zakharov equations by different mehtods (for example see [43] and the references therein). Here we consider a class of nonlinear partial differential equations (NPDEs) with constant coefficients

$$iE_t + P(E_{xx} + A_1 E_{yy}) + B_1|E|^2E + C_1EF = 0, \tag{2.0.1}$$

$$A_2F_{tt} + (F_{xx} - B_2F_{yy}) + C_2(|E|^2)_{xx} = 0, \tag{2.0.2}$$

where P, A_i, B_i, C_i $(i = 1, 2)$ are real constants and

$$P, B_1, C_1, C_2 \neq 0. \tag{2.0.3}$$

By choosing suitable values for these constants we get different kinds of equations like the Davey-Stewartson (DS) and nonlinear Schrödinger (NLS) equations. If one takes

$$\begin{cases} F = F(x,t), \quad \text{i.}e\text{., } F_y = 0, \\ P = 1,\ A_1 = 0,\ B_1 = -2\lambda,\ C_1 = 2, \\ A_2 = -1,\ C_2 = -1, \end{cases} \tag{2.0.4}$$

the Eqs.(2.0.1) and (2.0.2) becomes generalized Zakharov (GZ) equations [33]

$$\begin{aligned} &iE_t + E_{xx} - 2\lambda|E|^2E + 2EF = 0, \\ &F_{tt} - F_{xx} + (|E|^2)_{xx} = 0, \end{aligned} \tag{2.0.5}$$

where E is the complex envelope of the high-frequency electric field, and the real low-frequency field F is the plasma density measured from its equlibrium value. This system is reduced to the classical Zakharov equations of plasma physics whenever $\lambda = 0$. Due to the fact that the GZE is a realistic model in plasma [16,38,56], it makes sense to study the solitary wave solutions of the GZE (see [48,50,57,58], for example). The cubic term in Eq.(2.0.1) describes nonlinear self-interaction in high-frequency subsystem, such a term corresponds to a self-focusing effect in plasma physics. The coefficient λ is a real constant that can be a positive or negative number. The sound velocity and coupling constant in Eq.(2.0.2) have been normalized to unity for simplicity. GZE is a universal model of interaction between high- and low-frequency waves simulated in detail in [18], and internal oscillations of solitary waves were analyzed basing on the variational approach in [32]. Up to now, there are many methods of constructing exact solutions, for instance, the F-expansion method [48], the Adomian decomposition method (ADM) [50] and so on that have been applied to these system of equations. Among these methods, the ADM is a powerful tool for finding the exact solutions of nonlinear partial differential equations. However, the generalized Zakharov equation is more complicated than real equations because the high-frequency electric field E in the GZE is a complex function. We eager to know whether the imaginary equations can be solved by the HAM.

2.1 Mathematical formulation

Since $E(x,t)$ in Eqs. (2.0.5) is a complex function we assume that the traveling wave solutions of Eqs. (2.0.5) are of the forms [48]

$$\begin{aligned} E(x,t) &= f(\xi)e^{i\eta}, \\ F(x,t) = \psi(\xi), \quad \eta &= \alpha x + \beta t, \qquad \xi = \kappa(x - 2\alpha t), \end{aligned} \tag{2.1.1}$$

where $f(\xi)$ and $\psi(\xi)$ are real functions, β is the constant to be determined. According to [59] suppose the exact solution $f(\xi)$ is of the form

$$f(\xi) = a w(\xi), \tag{2.1.2}$$

where $a > 0$ is an undetermined constant. Substituting (2.1.1) and (2.1.2) into Eqs.(2.0.5) and canceling $e^{i\eta}$ yields an ordinary differential equations(ODES) for $w(\xi)$ and $\psi(\xi)$

$$\kappa^2 w'' + 2w\psi - (\alpha^2 + \beta)w - 2\lambda a^2 w^3 = 0, \tag{2.1.3}$$

$$\kappa^2(4\alpha^2 - 1)\psi'' + \kappa^2(a^2w^2)'' = 0. \tag{2.1.4}$$

In order to simplify ODEs (2.1.3) and (2.1.4) further, integrating Eq. (2.1.3) once and setting integration constant to zero, and integrating it again yields

$$\psi = \frac{a^2w^2}{1 - 4\alpha^2} + C, \quad \alpha^2 \neq \frac{1}{4}, \tag{2.1.5}$$

where C is the integration constant. Substituting expression (2.1.5) into (2.1.3) we obtain an ODE for $w(\xi)$

$$\kappa^2 w'' + (2C - \alpha^2 - \beta)w + 2(\frac{1}{1 - 4\alpha^2} - \lambda)a^2w^3 = 0. \tag{2.1.6}$$

Remark 2.1.1. *Explicit form of the solution of the Eq.(2.1.6) is known, but the process of determining unknown constants a and β is difficult analytically, so we have faced to this case as a problem and we want to obtain approximate values for a and β for every κ, α and λ.*

2.2 Solution by homotopy analysis method(HAM)

Let us consider the case that $w(\xi)$ satisfies two sets of boundary conditions

$$\text{One:} \quad w(0) = 0, \quad w'(0) = 1, \quad w(\xi) \to 1, \quad \text{as} \quad \xi \to +\infty, \tag{2.2.1}$$

$$\text{Two:} \quad w(0) = 1, \quad w'(0) = 0, \quad w(\xi) \to 0, \quad \text{as} \quad \xi \to +\infty. \tag{2.2.2}$$

2.2.1 Case one

According to the Eq. (2.1.6) and the boundary conditions (2.2.1), the solitary solution can be expressed by

$$w(\xi) = \sum_{m=0}^{+\infty} d_m e^{-m\xi}, \tag{2.2.3}$$

where $d_m(m = 0, 1, \dots)$ are coefficients to be determined. According to the *Rule of Solution Expression* denoted by (2.2.3) and the boundary conditions (2.2.1), it is natural to choose $w_0(\xi) = 1 - e^{-\xi}$ as initial approximation of $w(\xi)$. We define an auxiliary linear operator L via method of complete differential with decreasing order by

$$L[\phi(\xi; q)] = (\frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi})\phi(\xi; q), \tag{2.2.4}$$

with the property

$$L[c_1 e^{-\xi} + c_2] = 0, \tag{2.2.5}$$

where c_1, c_2 are constants. From Eq. (2.1.6) we define a nonlinear operator

$$N[\phi(\xi; q), A(q), B(q)] = \kappa^2 \frac{\partial^2 \phi}{\partial \xi^2} + (2C - \alpha^2)\phi - B(q)\phi + 2(\frac{1}{1 - 4\alpha^2} - \lambda)A^2(q)\phi^3, \tag{2.2.6}$$

and then construct a homotopy

$$H[\phi(\xi; q), A(q), B(q)] = (1 - q)L[\phi(\xi; q) - w_0] - \hbar\mathcal{H}(\xi)N[\phi(\xi; q), A(q), B(q)],$$

where $\mathcal{H}(\xi)$ is an auxiliary function. Setting $H[\phi(\xi; q), A(q), B(q)] = 0$, we have the zero-order deformation equation

$$(1 - q)L[\phi(\xi; q) - w_0] = \hbar\mathcal{H}(\xi)N[\phi(\xi; q), A(q), B(q)], \tag{2.2.7}$$

subject to the boundary conditions

$$\phi(0; q) = 0, \quad \phi'(0; q) = 1, \quad \phi(+\infty; q) = 1. \tag{2.2.8}$$

When the parameter q increases from 0 to 1, the solution $\phi(\xi; q)$ varies from $w_0(\xi)$ to $w(\xi)$, also $A(q)$ and $B(q)$ vary from their initial guesses a_0 and β_0, to the wave amplitude a and constant β respectively. If this continuous variation is smooth enough, the Maclaurin's series with respect to q can be constructed for $\phi(\xi; q)$, $A(q)$ and $B(q)$. Further, if these three series are convergent at $q = 1$ we have

$$w(\xi) = w_0(\xi) + \sum_{m=1}^{+\infty} w_m(\xi), \quad a = a_0 + \sum_{m=1}^{+\infty} a_m, \quad \beta = \beta_0 + \sum_{m=1}^{+\infty} \beta_m, \tag{2.2.9}$$

where

$$w_m(\xi) = \frac{1}{m!}\frac{\partial^m \phi(\xi;q)}{\partial q^m}\Big|_{q=0}, \quad a_m = \frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\Big|_{q=0}, \quad \beta_m = \frac{1}{m!}\frac{\partial^m B(q)}{\partial q^m}\Big|_{q=0}.$$

For simplicity, define the vectors

$$\vec{w}_k = \{w_0(\xi), w_1(\xi), \cdots, w_k(\xi)\},$$

$$\vec{A}_k = \{a_0, a_1, \cdots, a_k\},$$

$$\vec{B}_k = \{\beta_0, \beta_1, \cdots, \beta_k\}.$$

Differentiating Eq. (2.2.7) m times with respect to q then setting $q = 0$ and finally dividing them by $m!$, we gain the mth-order deformation equation

$$L[w_m(\xi) - \chi_m w_{m-1}(\xi)] = \hbar \mathcal{H}(\xi) R_m(\vec{w}_{m-1}, \vec{A}_{m-1}, \vec{B}_{m-1}), \tag{2.2.10}$$

with boundary conditions

$$w_m(0) = w'_m(0) = w_m(+\infty) = 0, \tag{2.2.11}$$

where

$$\begin{aligned} R_m(\vec{w}_{m-1}, \vec{A}_{m-1}, \vec{B}_{m-1}) &= \kappa^2 w''_{m-1} + (2C - \alpha^2) w_{m-1} - \textstyle\sum_{j=0}^{m-1} \beta_j w_{m-1-j} \\ &+ 2(\tfrac{1}{1-4\alpha^2} - \lambda) \textstyle\sum_{n=0}^{m-1} \left(\sum_{j=0}^{m-n-1} a_j a_{m-n-j-1} \sum_{i=0}^{n} w_{n-i} \sum_{l=0}^{i} w_l w_{i-l} \right), \end{aligned} \tag{2.2.12}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m \ge 2. \end{cases}$$

The general solution of Eq.(2.2.9) is

$$w_m(\xi) = \hat{w}_m(\xi) + c_1 e^{-\xi} + c_2, \tag{2.2.13}$$

where c_1, c_2 are constants and $\hat{w}_m(\xi)$ is a special solution of Eq. (2.2.10). Note that there are three unknowns $w_m(\xi)$, a_{m-1} and β_{m-1}, but we have only one differential equation for $w(\xi)$. The problem is therefore not closed and two additional algebraic equation is needed to determine a_{m-1} and β_{m-1}. Under the rule of solution expression and by choosing $\mathcal{H}(\xi) = e^{-\xi}$, we can determine β_{m-1} by vanishing the coefficient of $e^{-\xi}$

in the right hand side of Eq. (2.2.10) at each iteration. When $m = 1$, this algebraic equation is

$$2C - \alpha^2 + 2\Big(\frac{1}{1-4\alpha^2} - \lambda\Big)a_0^2 - \beta_0 = 0,$$

and hence β_0 is obtained very easily. The unknowns c_1, c_2, and a_{m-1} according to boundary conditions (2.2.10), are governed by

$$\hat{w}_m(0) + c_1 + c_2 = 0, \quad \hat{w'}_m(0) - c_1 = 0, \quad \hat{w}_m(\infty) + c_2 = 0.$$

Thus, c_1 and c_2 are given by

$$c_1 = \hat{w'}_m(0), \quad c_2 = -\hat{w}_m(\infty),$$

and the unknown a_{m-1} is obtained by solving the linear algebraic equation

$$\hat{w}_m(0) + c_1 + c_2 = 0.$$

Theorem 2.2.1. *If the series (2.2.9) are convergent, where $w_k(\xi)$ is governed by Eqns. (2.2.10) and (2.2.11) under the definitions (2.2.12) and (2.2.13) they must be the exact solution of Eqns. (2.1.6) and (2.2.1).*

Proof. If the solution series for $w(\xi)$ is convergent, it is necessary that

$$\lim_{m\to\infty} w_m(\xi) = 0, \quad \xi \in [0, +\infty)$$

Using (2.2.4), from (2.2.10), we have

$$\begin{aligned}
&\hbar H(\xi) \textstyle\sum_{k=1}^{+\infty} R_k(\vec{w}_{k-1}, \vec{A}_{k-1}, \vec{B}_{k-1}) \\
&= \lim_{m\to\infty} \textstyle\sum_{k=1}^{m} L[w_k(\xi) - \chi_k w_{k-1}(\xi)] \\
&= L\left\{\lim_{m\to\infty} \textstyle\sum_{k=1}^{m} [w_k(\xi) - \chi_k w_{k-1}(\xi)]\right\} \\
&= L\left[\lim_{m\to\infty} w_m(\xi)\right] \\
&= 0,
\end{aligned}$$

which gives, since $\hbar \neq 0$ and $H(\xi) \neq 0$,

$$\sum_{k=1}^{+\infty} R_k(\vec{w}_{k-1}, \vec{A}_{k-1}, \vec{B}_{k-1}) = 0.$$

Substituting (2.2.12) into the above expression and simplifying it, we have, due to the convergence of the series (2.2.9), that

$$\kappa^2 \frac{d^2}{d\xi^2} \sum_{k=0}^{+\infty} w_k(\xi) + (2C - \alpha^2) \sum_{k=0}^{+\infty} w_k(\xi) - \sum_{j=0}^{+\infty} \beta_j \sum_{k=0}^{+\infty} w_k(\xi)$$
$$+2(\frac{1}{1-4\alpha^2} - \lambda)(\sum_{j=0}^{+\infty} a_j)^2 (\sum_{k=0}^{+\infty} w_k(\xi))^3 = 0,$$

From (2.2.11) and the initial approximation which is defined above, it holds that

$$\sum_{k=0}^{+\infty} w_k(0) = 0, \quad \sum_{k=0}^{+\infty} w_k'(0) = \sum_{k=0}^{+\infty} w_k(+\infty) = 1.$$

Thus, as long as the solution series (2.2.9) are convergent, they must be the exact solution of Equations (2.1.6) and (2.2.1). This ends the proof. □

Henceforth we set $C = 0$. Based on this initial guess, the linear operator and the auxiliary function, we have the four-term approximation for $w(\xi)$ as

$$w(\xi) \approx w_0(\xi) + w_1(\xi) + w_2(\xi) + w_3(\xi),$$

where

$$w_0(\xi) = 1 - e^{-\xi},$$

$$w_1(\xi) = term[1],$$

$$w_2(\xi) = w_1(\xi) + term[2],$$

$$w_3(\xi) = w_1(\xi) + 2term[2] + term[3].$$

The $term[i]$'s $(i = 1, 2, 3)$ are:

$$term[1] = (-\frac{2}{3}e^{-\xi} + \frac{3}{2}e^{-2\xi} - e^{-3\xi} + \frac{1}{6}e^{-4\xi})\hbar,$$

$$term[2] = (-\frac{551}{1260}e^{-\xi} + \frac{116}{105}e^{-2\xi} - \frac{97}{70}e^{-3\xi} + \frac{449}{315}e^{-4\xi} - \frac{19}{20}e^{-5\xi} + \frac{4}{15}e^{-6\xi} - \frac{1}{42}e^{-7\xi})\hbar^2,$$

$$term[3] = (\frac{13e^{-10\xi}}{3780} - \frac{851e^{-9\xi}}{15120} + \frac{659e^{-8\xi}}{1960} - \frac{8359e^{-7\xi}}{8820} + \frac{6541e^{-6\xi}}{4200} - \frac{3212e^{-5\xi}}{1575}$$
$$+\frac{1661539e^{-4\xi}}{793800} - \frac{199729e^{-3\xi}}{132300} + \frac{228241e^{-2\xi}}{264600} - \frac{67979e^{-\xi}}{226800})\hbar^3.$$

It is very interesting that the sum of the constant coefficients is zero. Further we have the three-term approximation for a and β as

$$a \approx a_0 + a_1 + a_2,$$

where

$$a_0 = \frac{\sqrt{3}}{2},$$

$$a1 = \frac{11\hbar}{280\sqrt{3}},$$

$$a2 = \frac{5119\hbar^2}{100800\sqrt{3}} + \frac{11\hbar}{280\sqrt{3}}.$$

Also

$$\beta \approx \beta_0 + \beta_1 + \beta_2,$$

where

$$\beta 0 = -3,$$

$$\beta 1 = -\frac{11\hbar}{105},$$

$$\beta 2 = -\frac{121\hbar^2}{88200} - \frac{8\left(\frac{5119\hbar^2}{100800\sqrt{3}} + \frac{11\hbar}{280\sqrt{3}}\right)}{\sqrt{3}}.$$

Here we propound this question:

Question 2.2.2. *Why we choose $\mathcal{H}(\xi) = e^{-\xi}$?*

Before answering this question we want to see the influence of L^{-1}, the inverse of linear operator L, on 1 and $e^{-\xi}$.

$$L^{-1}(1) = \xi,$$

$$L^{-1}(e^{-\xi}) = -(1+\xi)e^{-\xi}.$$

In general when $m \neq 0, 1$ we have

$$L^{-1}(e^{m\xi}) = \frac{e^{m\xi}}{m^2 + m}.$$

Now we compose $R_m(\vec{w}_{m-1}, \vec{A}_{m-1}, \vec{B}_{m-1})$ when $m = 1$,

$$R_1(\vec{w}_0, \vec{A}_0, \vec{B}_0) = -1 + \beta_0(e^{-\xi} - 1) + a_0^2(-\frac{8}{3} + 8e^{-\xi} - 8e^{-2\xi} + \frac{8}{3}e^{-3\xi}),$$

and exert L^{-1} on the above expression,

$$L^{-1}(R_1(\vec{w}_0, \vec{A}_0, \vec{B}_0)) = -\xi + \beta_0(-(1+\xi)e^{-\xi} - \xi) + a_0^2(-\frac{8}{3}\xi - 8(1+\xi)e^{-\xi} - \frac{8}{2}e^{-2\xi} + \frac{8}{3}\frac{e^{-3\xi}}{6}).$$

As you see it contravenes the rule of solution expression, because we have secular terms ξ and $\xi e^{-\xi}$. Hence $R_m(\vec{w}_{m-1}, \vec{A}_{m-1}, \vec{B}_{m-1})$ should not contain 1 and $e^{-\xi}$, members of linear operator's kernel, at any iteration to avoid secular terms. Furthermore, we use this point for determination β_{m-1}. From above gleanings, we conclude that the choice $\mathcal{H}(\xi) = e^{-\xi}$ is suitable!

2.2.2 Case two

Now consider the Eq. (2.1.5) and boundary conditions (2.2.2), the solitary solution can be expressed by

$$w(\xi) = \sum_{m=1}^{+\infty} d_m e^{-m\xi}, \tag{2.2.14}$$

where $d_m (m = 1, 2, \ldots)$ are coefficients to be determined. According to the rule of solution expression denoted by (2.2.14) and the boundary conditions (2.2.2), it is natural to choose $w_0(\xi) = 2e^{-\xi} - e^{-2\xi}$ as an initial approximation of $w(\xi)$. Again we define an auxiliary linear operator L via method of complete differential with decreasing order by

$$L[\phi(\xi; q)] = (\frac{\partial^2}{\partial \xi^2} - 1)\phi(\xi; q), \tag{2.2.15}$$

with the property

$$L[c_1 e^{\xi} + c_2 e^{-\xi}] = 0, \tag{2.2.16}$$

where c_1, c_2 are constants. The zero-order deformation equation is

$$(1 - q)L[\phi(\xi; q) - w_0] = \hbar N[\phi(\xi; q), A(q), B(q)] \tag{2.2.17}$$

subject to the boundary conditions

$$\phi(0; q) = 1, \quad \phi'(0; q) = 0, \quad \phi(+\infty; q) = 0. \tag{2.2.18}$$

Here, it is necessary to mention that the mth-order deformation equation and R_m at case two are like the case one. The general solution of Eq. (2.2.10) for this case is

$$w_m(\xi) = \hat{w}_m(\xi) + c_1 e^{\xi} + c_2 e^{-\xi}, \tag{2.2.19}$$

with boundary conditions

$$w_m(0) = w'_m(0) = w_m(+\infty) = 0, \tag{2.2.20}$$

where c_1, c_2 are constants and $\hat{w}_m(\xi)$ is a special solution of Eq. (2.2.10) which contains the unknown terms a_{m-1}, β_{m-1}. Under the rule of solution expression $c_1 = 0$, and by choosing $\mathcal{H}(\xi) = 1$ we can determine β_{m-1} by vanishing the coefficient of $e^{-\xi}$ in the right hand side of Eq. (2.2.10) at each iteration. When $m = 1$, this algebraic equation is

$$-\alpha^2 + \kappa^2 - \beta_0 = 0,$$

and hence β_0 is obtained very easily and surprising, it is independent of a_0. The unknowns c_2 and a_{m-1}, according to boundary conditions (2.2.20), are governed by

$$\hat{w_m}(0) + c_2 = 0, \quad \hat{w'_m}(0) - c_2 = 0.$$

Thus, the unknown a_{m-1} is obtained by solving the linear algebraic equation

$$\hat{w_m}(0) + \hat{w'_m}(0) = 0,$$

and c_2 is given by

$$c_2 = -\hat{w_m}(0).$$

Such as the previous case we show the three-term approximation of $w(\xi)$, a and β respectively as follow:

$$w_0(\xi) = 2e^{-\xi} - e^{-2\xi},$$

$$w_1(\xi) = -\frac{1}{4}\hbar e^{-6\xi} + \frac{35}{16}\hbar e^{-5\xi} - 7\hbar e^{-4\xi} + \frac{35}{4}\hbar e^{-3\xi} - 4\hbar e^{-2\xi} + \frac{5\hbar e^{-\xi}}{16},$$

$$w_2(\xi) = w_1(\xi) - \frac{35}{528}\hbar^2 e^{-10\xi} + \frac{1071\hbar^2 e^{-9\xi}}{1024} - \frac{335}{48}\hbar^2 e^{-8\xi} + \frac{3185}{128}\hbar^2 e^{-7\xi} - \frac{1733749\hbar^2 e^{-6\xi}}{33792}$$
$$+\frac{8835575\hbar^2 e^{-5\xi}}{135168} - \frac{504931\hbar^2 e^{-4\xi}}{8448} + \frac{1388135\hbar^2 e^{-3\xi}}{33792} - 16\hbar^2 e^{-2\xi} + \frac{236053\hbar^2 e^{-\xi}}{135168}.$$

The three-term approximation of a is of the form

$$a \approx a_0 + a_1 + a_2,$$

where

$$a_0 = \frac{\sqrt{105}}{4},$$

$$a_1 = \frac{1909\sqrt{\frac{35}{3}}\hbar}{22528},$$

$$a_2 = \frac{176175823\sqrt{\frac{35}{3}}\hbar^2}{761266176} + \frac{1909\sqrt{\frac{35}{3}}\hbar}{22528},$$

also the three-term approximation of β is

$$\beta \approx \beta_0 + \beta_1 + \beta_2,$$

where

$$\beta_0 = 3, \quad \beta_1 = \beta_2 = 0.$$

2.3 Numerical results

We use the widely applied symbolic computation software MATHEMATICA to solve the first few equations (2.2.13) and (2.2.19). In this way, we derive $w_m(\xi)$, a_{m-1} and β_{m-1} for $m = 1, 2, 3, \ldots$, successively. At the Mth-order approximation, we have the approximate solution of Eq.(2.1.5), namely

$$w(\xi) \approx W_M = \sum_{m=0}^{M} w_m(\xi), \quad a \approx A_M = \sum_{m=0}^{M} a_m, \quad \beta \approx B_M = \sum_{m=0}^{M} \beta_m. \tag{2.3.1}$$

The auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the series (2.3.1) in the homotopy analysis solution at each cases (1,2). By means of so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao [27], the appropriate region for $\hbar$ is a horizontal line segment. Our solution series contain the auxiliary parameter $\hbar$. We can choose an appropriate value of $\hbar$ to ensure that the three solution series converge. In what follows, we confirm this convergence region by the aid of the residual error and the error of norm 2 curves.

2.3.1 Case one

We can investigate the influence of $\hbar$ on the convergence of a, β and $w(\xi)$ by plotting the curves of a, β and $w''(0)$ versus $\hbar$, as shown in Fig. 2.1, Fig. 2.2 and Fig. 2.3. Clearly, $a \approx 0.866025$ which agrees with the expected value $\kappa\sqrt{\frac{(1-4\alpha^2)}{\lambda(1-4\alpha^2)-1}}$ and $\beta \approx -3$ with the expected value $-2\kappa^2 - \alpha^2$ with $\kappa = \alpha = \lambda = 1$, [48]. Generally, it is found that as long as series solution for the amplitude a and parameter β are convergent, the corresponding series solution for $w(\xi)$ is also convergent [27]. The residual error is shown in Fig. 2.4, which is calculated by

$$Residual \quad Error \approx \kappa^2 \frac{\partial^2 W_M}{\partial \xi^2} - \alpha^2 W_M - B_M W_M + 2(\frac{1}{1-4\alpha^2} - \lambda) A_M^2(q) W_M^3.$$

Also in Fig. 2.5, the error of norm 2 [15], of two successive approximation, w_M, over $[0, 2]$ is calculated by

$$\sqrt{\frac{1}{31}\sum_{i=0}^{30}(w_{15}(\frac{i}{15}))^2},$$

it can be seen that the error is minimum at $\hbar = -1.20016$. This value of $\hbar$ also lies in the admissible range of $\hbar$. The so-called homotopy-Padé technique is employed, which greatly accelerates the convergence. The $[n, s]$ homotopy-Padé approximation of a is formulated by

$$\frac{\sum_{k=0}^{n} a_k}{1 + \sum_{k=1}^{s} a_{n+1+k}}.$$

In many cases, the $[n, n]$ homotopy-Padé approximation does not depend upon the auxiliary parameter $\hbar$. The value of the amplitude is shown in Table 2.1. Clearly, the amplitude converges to the exact value 0.866025.

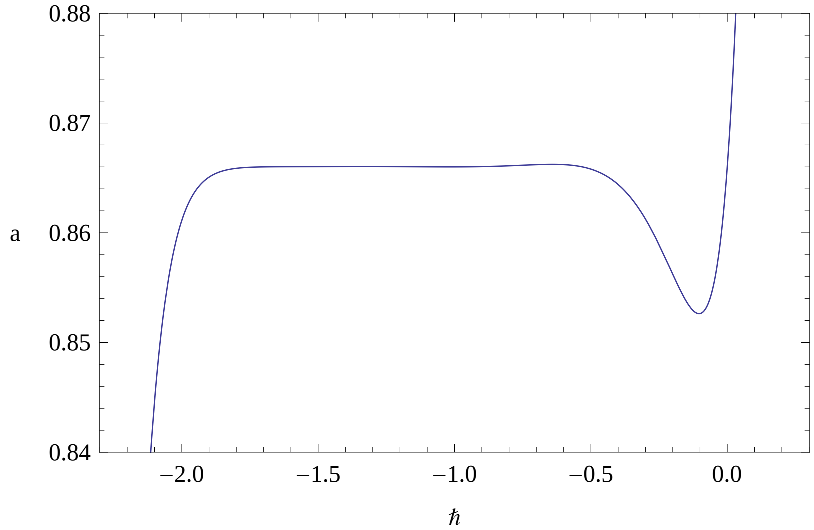

Fig. 2.1: The curve of the amplitude a versus $\hbar$ for the 15th-order approximation.

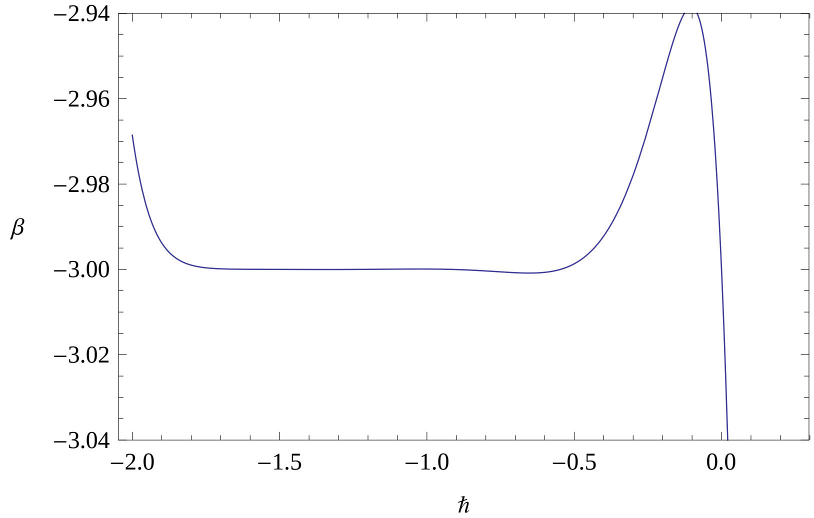

Fig. 2.2: The curve of the parameter β versus $\hbar$ for the 15th-order approximation.

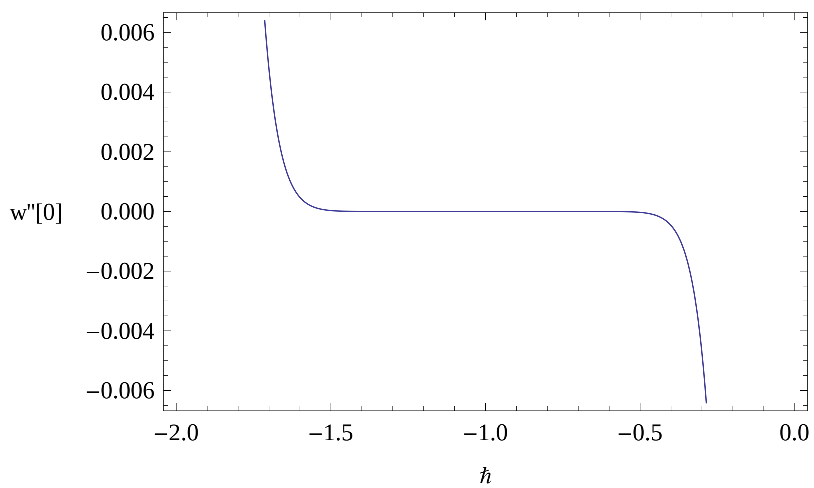

Fig. 2.3: The $\hbar$-curve of $w''(0)$ for the 15th-order approximation as $a = 0.866025$ and

$\beta = -3$.

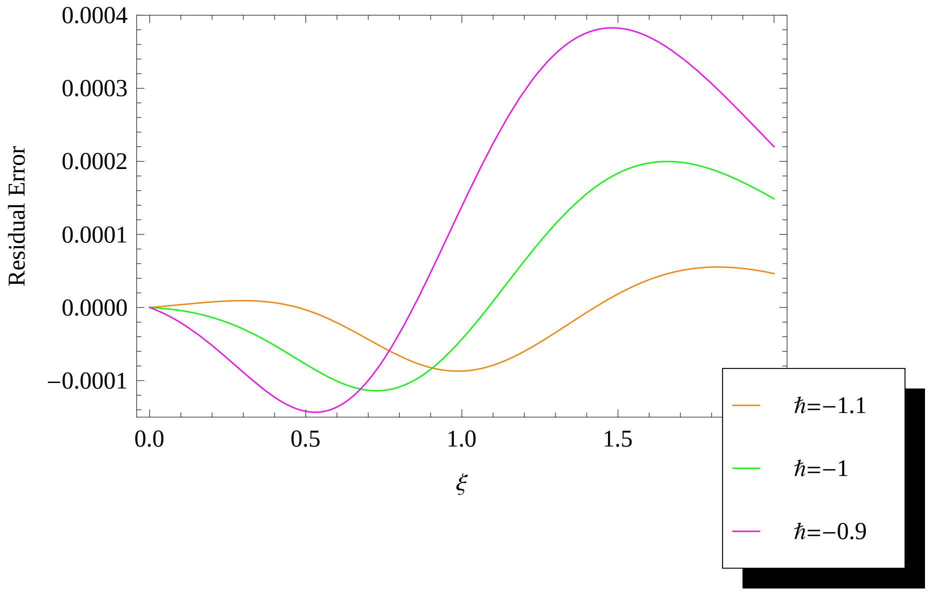

Fig. 2.4: The residual error of Eq. (2.1.5) for the 15th-order approximation.

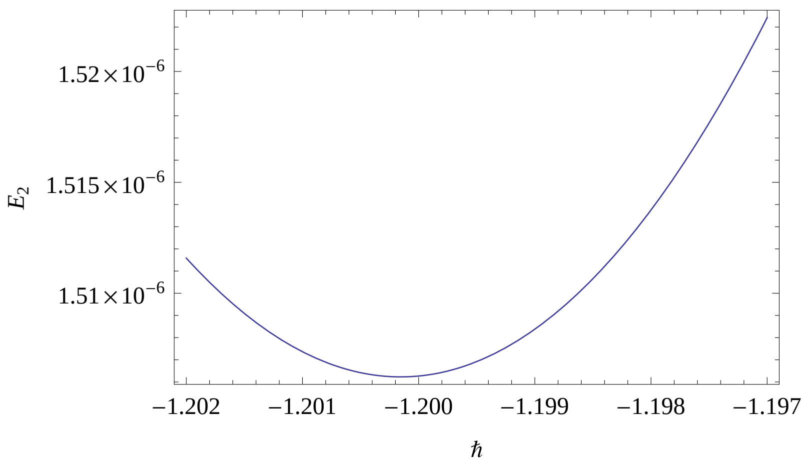

Fig. 2.5: Norm 2 of the error for the 15th-order approximation by HAM for $w(\xi)$ per $\hbar$.

2.3.2 Case two

Like the case one, we can investigate the influence of $\hbar$ on the convergence of a, β and $w(\xi)$ by plotting the curves of a, β and $w''(0)$ versus $\hbar$, as shown in Fig. 2.6, Fig. 2.7 and Fig. 2.8, respectively. Clearly, $a \approx 2.44949$ which agrees with the expected value $\kappa\sqrt{-\frac{(1-4\alpha^2)}{\lambda(1-4\alpha^2)-1}}$ and $\beta \approx 3$ with the expected value $\kappa^2 - \alpha^2$ with $\kappa = 2, \alpha = 1, \lambda = -1$

[48]. Generally, it is found that as long as series solution for the amplitude a and parameter β are convergent, the corresponding series solution for $w(\xi)$ is also convergent [27]. The residual error is shown in Fig. 2.9. The value of the amplitude is being shown at Table 2.2. Clearly, the amplitude converges to the exact value 2.44949. In Fig. 2.10, the error of norm 2 [15], of two successive approximation, w_M, over $[0, 1]$ is calculated by

$$\sqrt{\frac{1}{11}\sum_{i=0}^{10}(w_{10}(\frac{i}{10}))^2},$$

it can be seen that the error is minimum at $\hbar = -0.362969$. In this case, the value of $\hbar$ lie in the admissible range of $\hbar$ too.

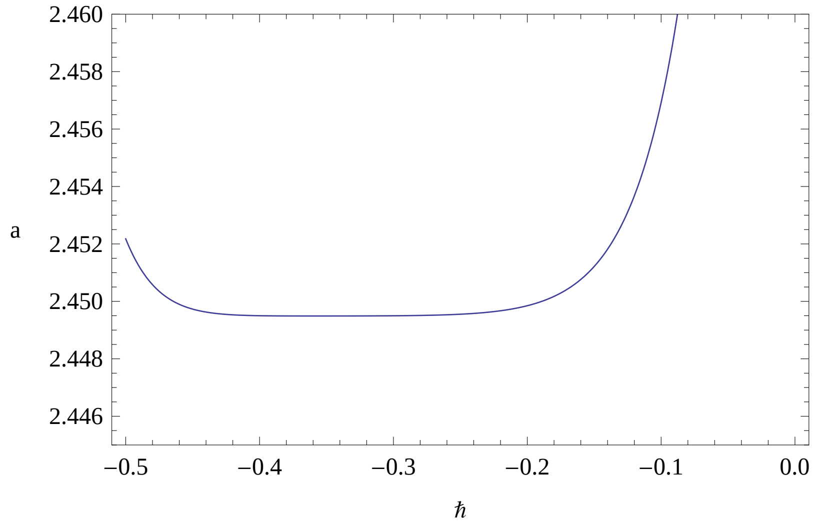

Fig. 2.6: The curve of the amplitude a versus $\hbar$ for the 10th-order approximation.

Table 2.1: Results for $[m, m]$ Homotopy-Padé approach

Order of approximation	$[m, m]$	a
2	[1,1]	0.848479
4	[2,2]	0.869361
6	[3,3]	0.866182
8	[4,4]	0.865945
10	[5,5]	0.866053
12	[6,6]	0.866027
14	[7,7]	0.866025

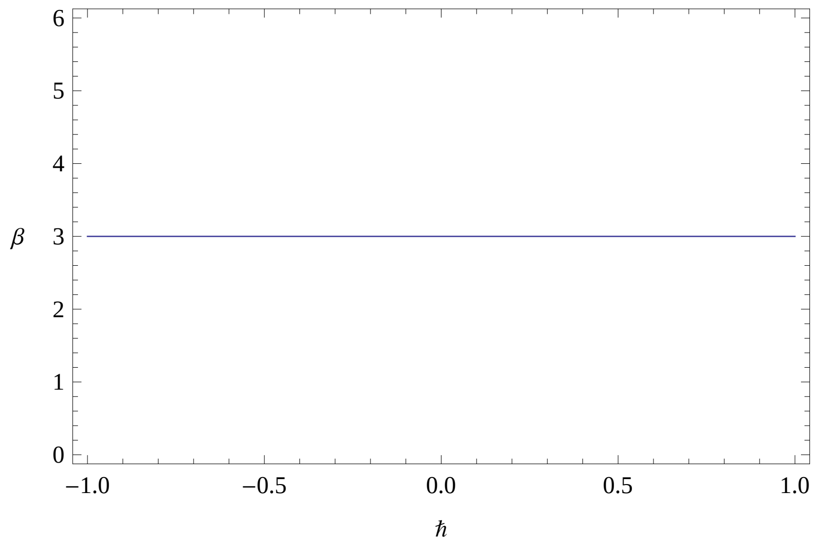

Fig. 2.7: The curve of the parameter β versus $\hbar$ for the 10th-order approximation.

2.3.3 Conclusion

As we saw during this chapter, HAM provides us with a convenient way to control the convergence of approximation series by adapting $\hbar$, which is a fundamental qualitative difference in analysis between HAM and other methods. So, this chapter shows the flexibility and potential of the homotopy analysis method for complicated nonlinear problems in science and engineering.

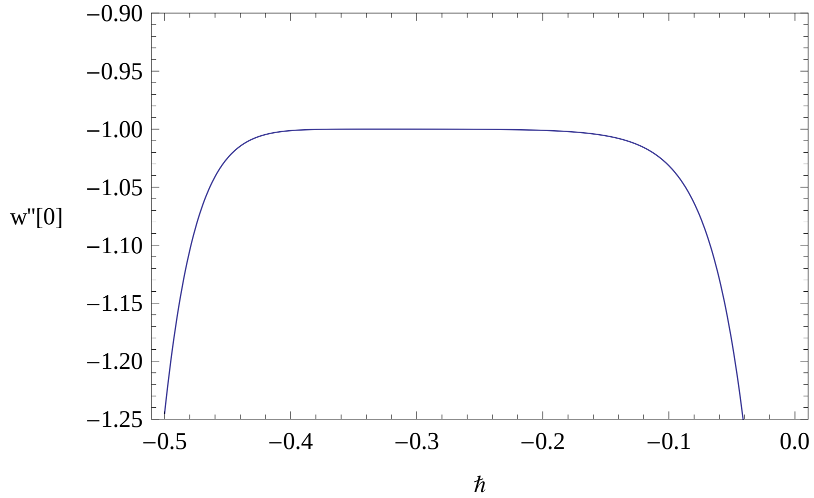

Fig. 2.8: The $\hbar$-curve of $w''(0)$ for the 10th-order approximation as $a = 2.4494898$ and $\beta = 3$.

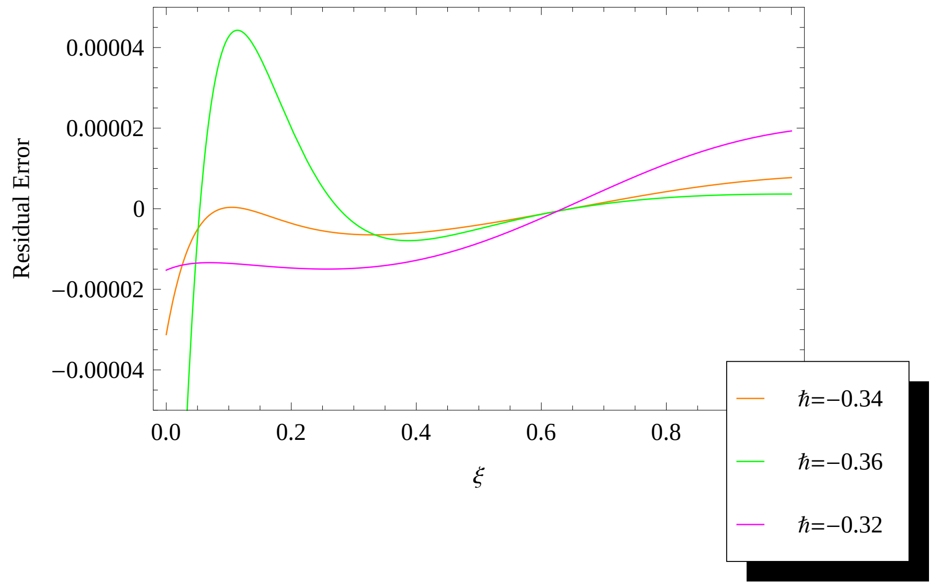

Fig. 2.9: The residual error for Eq. (2.1.5) for the 10th-order approximation.

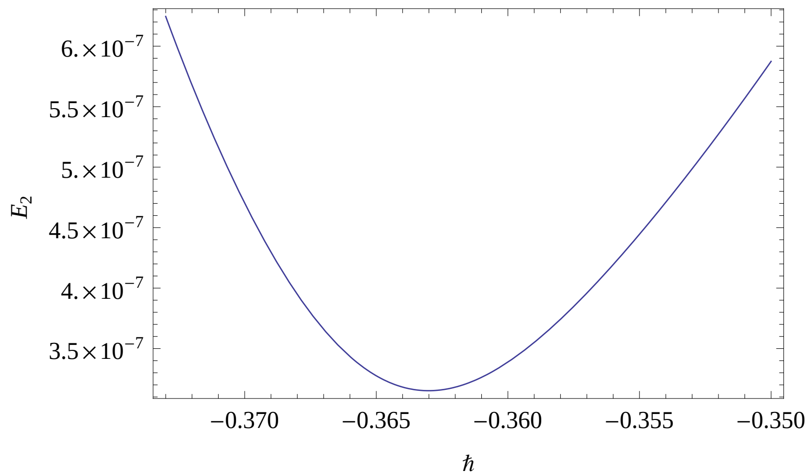

Fig. 2.10: Error of norm 2 for the 10th-order approximation by HAM for $w(\xi)$ per $\hbar$.

Table 2.2: Results for $[m, m]$ Homotopy-Padé approach

Order of approximation	$[m, m]$	a
2	[1,1]	2.45576
4	[2,2]	2.44968
6	[3,3]	2.44951
8	[4,4]	2.44949
10	[5,5]	2.44949

Chapter 3

Analytic solution of the Sharma-Tasso-Olver equation by homotopy analysis method

This chapter is concerned with the Sharma-Tasso-Olver (STO) equation,

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0, \tag{3.0.1}$$

where α is a real constant and the physical field $u(x,t)$ describes the wave motion depending on the temporal variable t and the spatial variable x. The Sharma-Tasso-Olver equation appears in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics, nonlinear optics and applied and physical sciences [9, 20]. This equation contains both linear dispersive term αu_{xxx} and the double nonlinear terms $\alpha(u^3)_x$ and $\frac{3}{2}\alpha(u^2)_{xx}$. Attention has been focused on STO equation (3.0.1) in [22,44,49,51,55] and references therein due to its scientific applications. In [55], Yan investigated equation (3.0.1) by using the Cole-Hopf transformation method. However, S. Wang et al. [49] showed the fission and fusion of the solitary wave and the soliton solutions respectively by means of the Hirota's direct method and B$\breve{a}$klund transformations method, in [51] Wazwaz analytically studied this equation by the aid of tanh method, the extended tanh method and two ansatze involving hyperbolic functions. In [22,44] the simple symmetry reduction procedure, and the extended hyperbolic function method are applied to this equation, respectively. A. J. M. Jawad et al. [20] have given the topological 1-soliton solution for the STO equation at the last part of their paper. It is not easy to define the soliton concept. But, soliton concept can be regarded with solutions of nonlinear partial differential equations. Satyanad Kichenassamy and Peter J. Olver have given a good definition of the traveling waves in the introduction of their paper [21]; " In the study of equations modeling wave phenomena, one of the fundamental objects of study is the traveling wave solution, meaning a solution of constant form moving with a fixed velocity. Determination of such solutions is accomplished by solving a reduced differential equation in fewer independent variables by one. In particular, the traveling wave solutions for a one-dimensional wave equation are found by solving a connection problem for an associated ordinary differential equation". Of particular interest are three types of traveling waves:

(i) The solitary waves, which are localized traveling waves, asymptotically zero at large distances,

(ii) the periodic waves, and

(iii) the kink waves, which rise or descend from one asymptotic state to another.

We employ the homotopy analysis method to obtain the solitary wave solutions of (3.0.1) with unknown wave speed. In what follows, the method will be reviewed briefly.

3.1 The equation for traveling waves and solution by homotopy analysis method (HAM)

A traveling wave solution is just a solution of the particular form

$$u(x,t) = w(\xi) = w(x - pt),$$

where p is the velocity of the traveling wave and $\xi = x - pt$ is the characteristic variable. Under above transformation, Eq. (3.0.1) reads

$$-pw' + \alpha(w^3)' + \frac{3}{2}\alpha(w^2)'' + \alpha w''' = 0. \tag{3.1.1}$$

Integrating Eq. (3.1.1) once and taking integration constant equal to zero yields

$$-pw + \alpha w^3 + 3\alpha w w' + \alpha w'' = 0, \tag{3.1.2}$$

where the prime denotes the differentiation with respect to ξ. The boundary conditions for the equation are

$$\begin{aligned} w(0) &= 0, \\ w(\infty) &= 1. \end{aligned} \tag{3.1.3}$$

According to the Eq. (3.1.2) and the boundary conditions (3.1.3), the solitary solution can be expressed by

$$w(\xi) = \sum_{m=0}^{+\infty} d_m e^{-m\xi}, \tag{3.1.4}$$

where $d_m (m = 0, 1, \ldots)$ are coefficients to be determined. According to the *Rule of Solution Expression* denoted by (3.1.4) and the boundary conditions (3.1.3), it is natural to choose $w_0(\xi) = 1 - e^{-\xi}$ as initial approximation of $w(\xi)$. Also, under *The rule of Solution Expression* (3.1.4) and via method of complete differential with decreasing order, it is obvious to choose the auxiliary linear operator

$$L[\phi(\xi; q)] = (\frac{\partial^2}{\partial \xi^2} - 1)\phi(\xi; q), \tag{3.1.5}$$

with the property

$$L[c_1 e^{\xi} + c_2 e^{-\xi}] = 0, \tag{3.1.6}$$

where c_1, c_2 are constants. From Eq. (3.1.2) we define a nonlinear operator

$$N[\phi(\xi; q), P(q)] = -P(q)\phi + \alpha\phi^3 + 3\alpha\phi\frac{\partial \phi}{\partial \xi} + \alpha\frac{\partial^2 \phi}{\partial \xi^2}, \tag{3.1.7}$$

and then construct such a homotopy

$$H[\phi(\xi; q), P(q)] = (1 - q)L[\phi(\xi; q) - w_0] - \hbar q \mathcal{H}(\xi) N[\phi(\xi; q), P(q)],$$

where $\mathcal{H}(\xi)$ is an auxiliary function. Setting $H[\phi(\xi; q), P(q)] = 0$, we have the zero-order deformation equation

$$(1 - q)L[\phi(\xi; q) - w_0] = q\hbar\mathcal{H}(\xi)N[\phi(\xi; q), P(q)], \tag{3.1.8}$$

subject to the boundary conditions

$$\phi(0; q) = 0, \quad \phi(\infty; q) = 1, \tag{3.1.9}$$

where $\hbar$ is a nonzero auxiliary parameter and $q \in [0, 1]$ is the homotopy-parameter. When the parameter q increases from 0 to 1, the homotopy-solution $\phi(\xi; q)$ varies from $w_0(\xi)$ to $w(\xi)$, so does the $P(q)$ from p_0, the initial guess of the wave speed, to p. If this continuous variation is smooth enough, the Maclurin's series with respect to q can be constructed for $\phi(\xi; q)$ and $P(q)$, and further, if these two series are convergent at $q = 1$, we have

$$w(\xi) = w_0(\xi) + \sum_{m=1}^{+\infty} w_m(\xi), \quad p = p_0 + \sum_{m=1}^{+\infty} p_m, \tag{3.1.10}$$

where

$$w_m(\xi) = \frac{1}{m!} \frac{\partial^m \phi(\xi; q)}{\partial q^m}\Big|_{q=0}, \quad p_m = \frac{1}{m!} \frac{\partial^m P(q)}{\partial q^m}\Big|_{q=0}, \tag{3.1.11}$$

are called mth-order homotopy-derivatives of $w(\xi)$ and p, respectively. Briefly speaking, by means of the HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations. For simplicity, define the vectors

$$\vec{w}_k = \{w_0(\xi), w_1(\xi), \cdots, w_k(\xi)\},$$

$$\vec{P}_k = \{p_0, p_1, \cdots, p_k\}.$$

Differentiating Eq. (3.1.8) and the boundary conditions (3.1.9) m times with respect to q then setting $q = 0$ and finally dividing them by $m!$, we gain mth-order deformation equation

$$L[w_m(\xi) - \chi_m w_{m-1}(\xi)] = \hbar \mathcal{H}(\xi) R_m(\vec{w}_{m-1}, \vec{P}_{m-1}), \tag{3.1.12}$$

$$w_m(0) = 0, \quad w_m(\infty) = 0, \tag{3.1.13}$$

where

$$R_m(\vec{w}_{m-1}, \vec{P}_{m-1}) = \sum_{n=0}^{m-1} \Big(-p_{m'} w_n + \alpha w_{m'} \sum_{i=0}^{n} w_i w_{n-i} + 3\alpha w_{m'} w_n' \Big) + \alpha w_{m-1}'', \tag{3.1.14}$$

$m' = m - n - 1$ and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2. \end{cases}$$

The general solution of Eq. (3.1.12) is

$$w_m(\xi) = \hat{w}_m(\xi) + c_1 e^{\xi} + c_2 e^{-\xi}, \tag{3.1.15}$$

where c_1, c_2 are constants and $\hat{w}_m(\xi)$ is a special solution of Eq. (3.1.12) which contains the unknown term p_{m-1}. They are known by solving $w_m(\xi)$, except for p_{m-1}. Under the rule of solution expression and by choosing $\mathcal{H}(\xi) = e^{\xi}$, we can determine p_{m-1} by vanishing the coefficient of e^{ξ} in $R_m(\xi)$ in each iteration. When $m = 1$, this algebraic equation is

$$\alpha - p_0 = 0.$$

As mentioned above, the general solution of Eq. (3.1.12) is Eq. (3.1.15). The unknown c_1, according to the rule of solution expression, is zero and c_2, according to boundary conditions (3.1.13), is governed by

$$c_2 = -\hat{w}_m(0),$$

in each iteration.

3.2 Numerical Results

Liao [27] proved that, as long as a series solution given by the HAM converges, it must be one of the exact solutions. So, it is important to ensure that the solution series (3.1.10) are convergent. We use the widely applied symbolic computation software MATHEMATICA to solve the first few equations (3.1.15). At the Mth-order approximation, we have the approximate solution of Eq.(3.1.8), namely

$$w(\xi) \approx W_M(\xi) = \sum_{m=0}^{M} w_m(\xi), \quad p \approx P_M = \sum_{m=0}^{M} p_m. \tag{3.2.1}$$

for $\alpha = 1$ we show a few terms of the series solution:

$$w_1(\xi) = \frac{\hbar}{3}e^{-\xi} - \frac{1}{3}\hbar e^{-2\xi},$$

$$w_2(\xi) = (\frac{\hbar}{3} + \frac{\hbar^2}{8})e^{-\xi} - \frac{1}{3}\hbar e^{-2\xi} - \frac{\hbar^2}{8}e^{-3\xi},$$

$$w_3(\xi) = (\frac{\hbar}{3} + \frac{\hbar^2}{4} + \frac{83\hbar^3}{1440})e^{-\xi} + (\frac{-\hbar}{3} + \frac{\hbar^3}{24})e^{-2\xi} + (\frac{-\hbar^2}{4} - \frac{5\hbar^3}{96})e^{-3\xi} - \frac{17\hbar^3}{360}e^{-4\xi}$$

$$\vdots$$

Our solution series contain the auxiliary parameter $\hbar$. As mentioned before the auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the series (3.1.10) in the homotopy analysis solution. By means of so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao [27], the appropriate region for $\hbar$ is a horizontal line segment. We can investigate the influence of $\hbar$ on the convergence of p by plotting the curves of p versus $\hbar$, as shown in Fig. 3.1. Generally, it is found that as long as series solution for the wave speed p is convergent, the corresponding series solution for $w(\xi)$ is also convergent [27]. For instance, our analytic solution converges, as shown by residual error in Fig. 3.2. It shows the efficiency of HAM. Residual error is defined as follows

$$Residual \quad Error \approx \alpha W''_M + 3\alpha W_M W'_M + \alpha W_M^3 - P_M W_M.$$

In first five figures we set $\alpha = 1$ for convenience. We can investigate the influence of $\hbar$ on the convergence of $w'(0)$ and $w''(0)$, by plotting the curve of them versus $\hbar$, as shown in Figure. 3.3, the series $w'(0)$ and $w''(0)$ given by the solution series (3.2.1) are convergent when $-1.5 \leq \hbar \leq -0.5$. The series solution for wave speed is convergent, and corresponding series for $w(\xi)$ is also convergent. For instance when $\hbar = -1.2$, our approximate solution converges, as shown in Fig. 3.4.

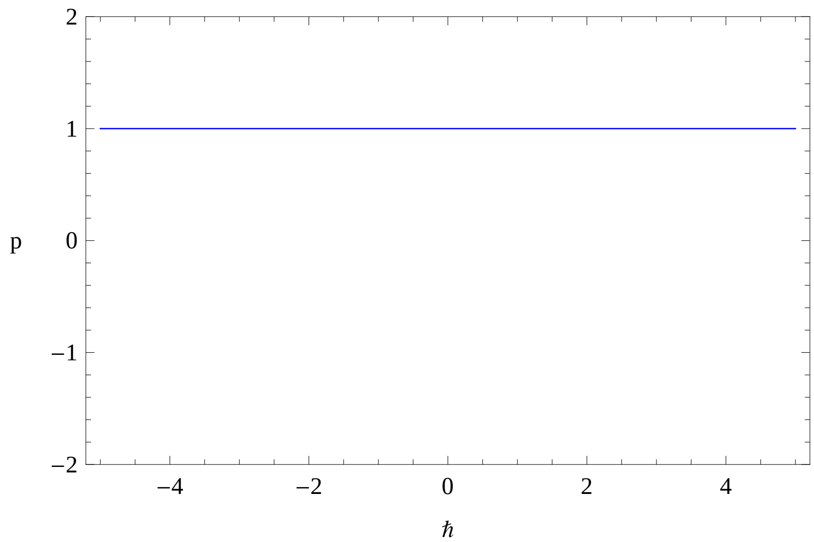

Fig. 3.1: The curve of the approximation of wave speed p versus $\hbar$ for the 20th-order approximation.

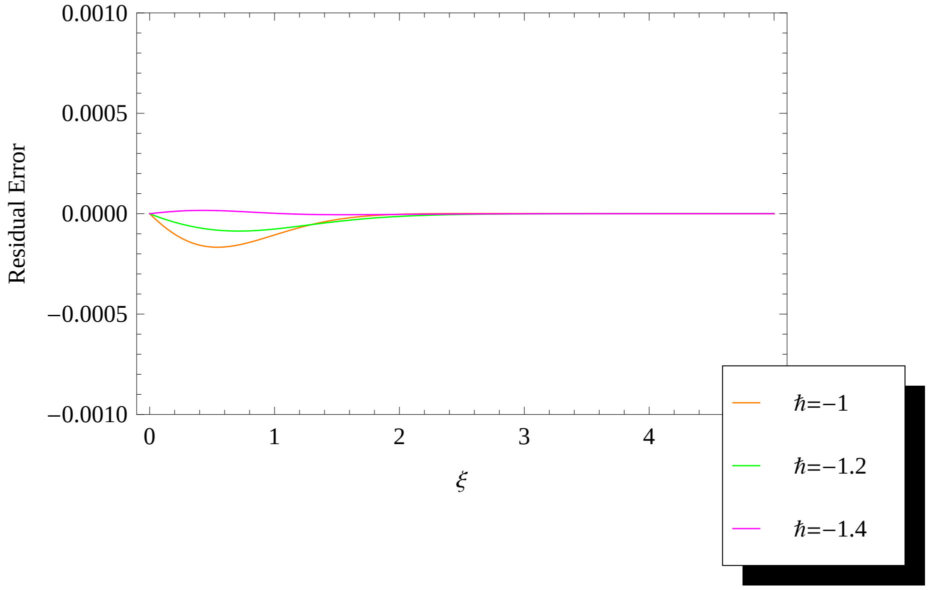

Fig. 3.2: The residual error of Eq. (3.1.2) for the 20th-order approximation.

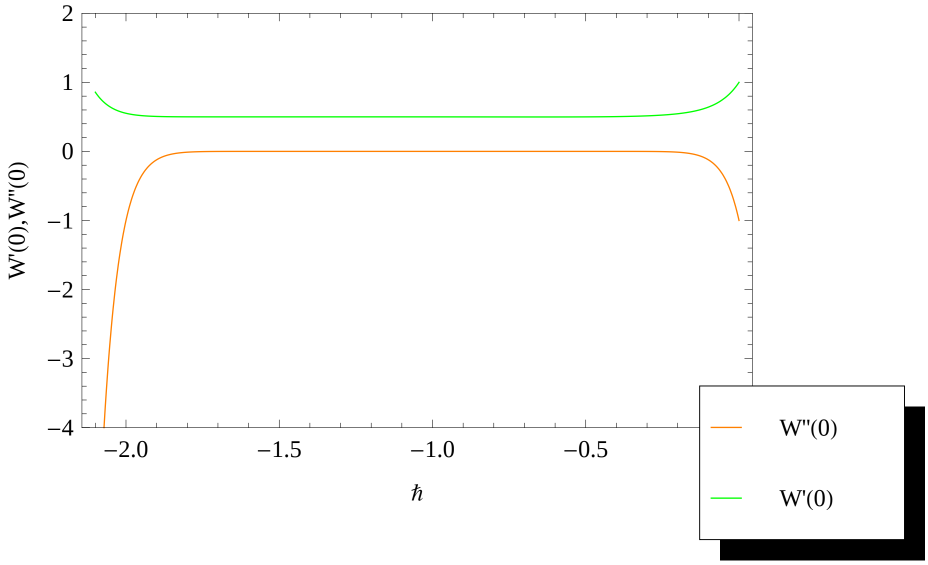

Fig. 3.3: The curves of approximation of $w'(0)$ and $w''(0)$ versus $\hbar$ for the 20th-order approximation.

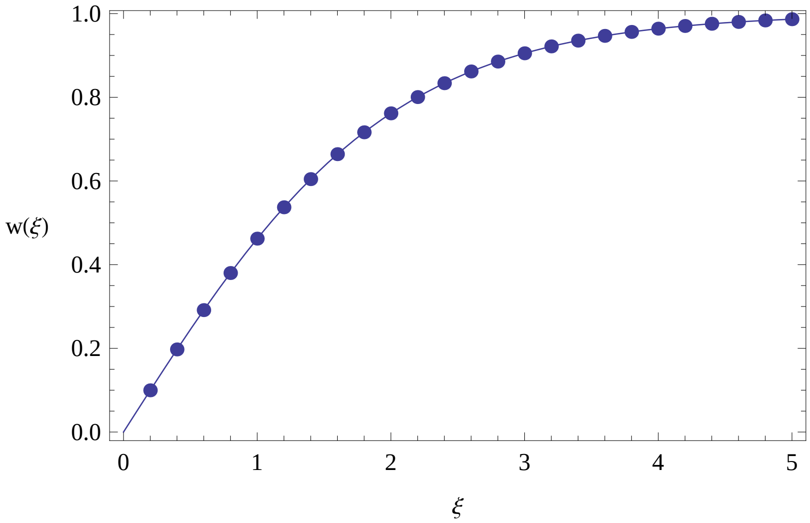

Fig. 3.4: The approximation for $w(\xi)$ when $\hbar = -1.2$ and the kink solution [51] $w(\xi) = \tanh(\frac{\xi}{2})$. Solid curve: kink solution; Symbols: 20-th order approximation.

In Fig. 3.5, norm 2 of the error [15], of two successive approximations, w_M, over $[0, 5]$ is calculated by

$$\sqrt{\frac{1}{101}\sum_{i=0}^{100}(w_{20}(\frac{i}{20}))^2},$$

it can be seen that the error is minimum at $\hbar = -1.26793$. In this case, the value of $\hbar$ lies in the admissible range of $\hbar$ too.

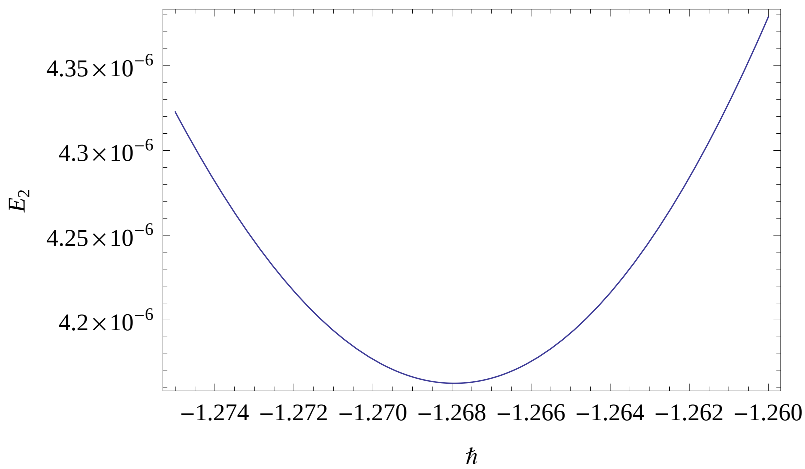

Fig. 3.5: Norm 2 of the Error for the 20th-order approximation by HAM for $w(\xi)$ per $\hbar$.

Now we want to investigate the effect of parameter α on the solutions. By setting different values for α we achieve different bounds for $\hbar$. For example set $\alpha = 3$, once again we can investigate the influence of $\hbar$ on the convergence of p by plotting the curves of p versus $\hbar$, as shown in Fig. 3.6. Consequently, it is found that as long as series solution for the wave speed p is convergent, the corresponding series solution for $w(\xi)$ is also convergent. For instance, our analytic solution converges, as shown by residual error in Fig. 3.7. Further, from Fig. 3.8 it is clear that the series $w'(0)$ and $w''(0)$ given by the solution series (3.2.1) are convergent when $-0.6 \leq \hbar \leq -0.1$.

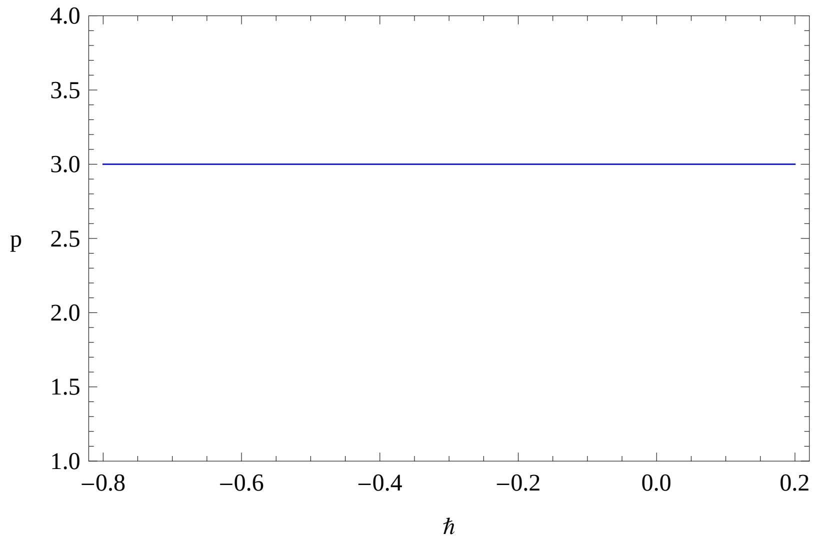

Fig. 3.6: The curve of the approximation of wave speed p versus $\hbar$ for the 20th-order approximation.

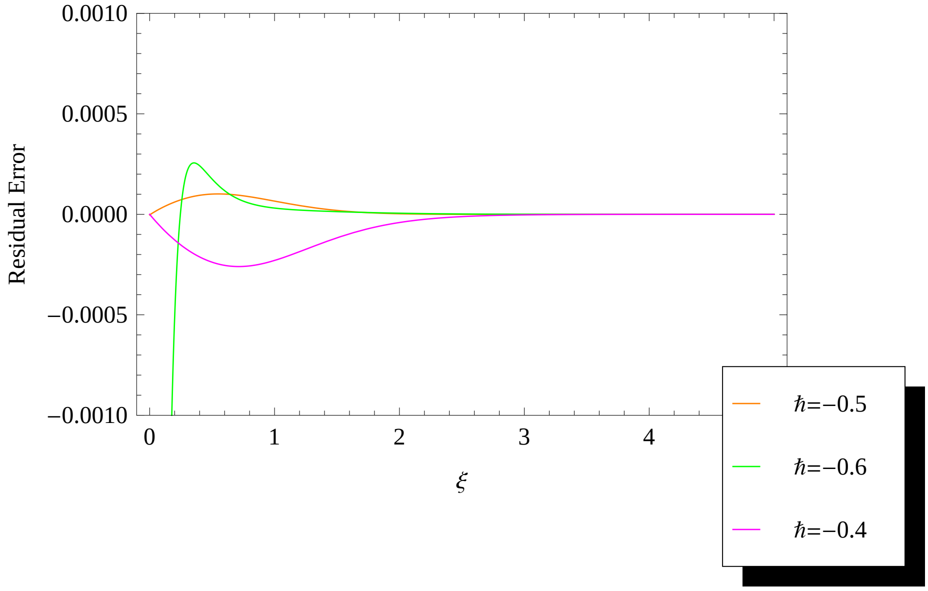

Fig. 3.7: The residual error of Eq. (3.1.2) for the 20th-order approximation.

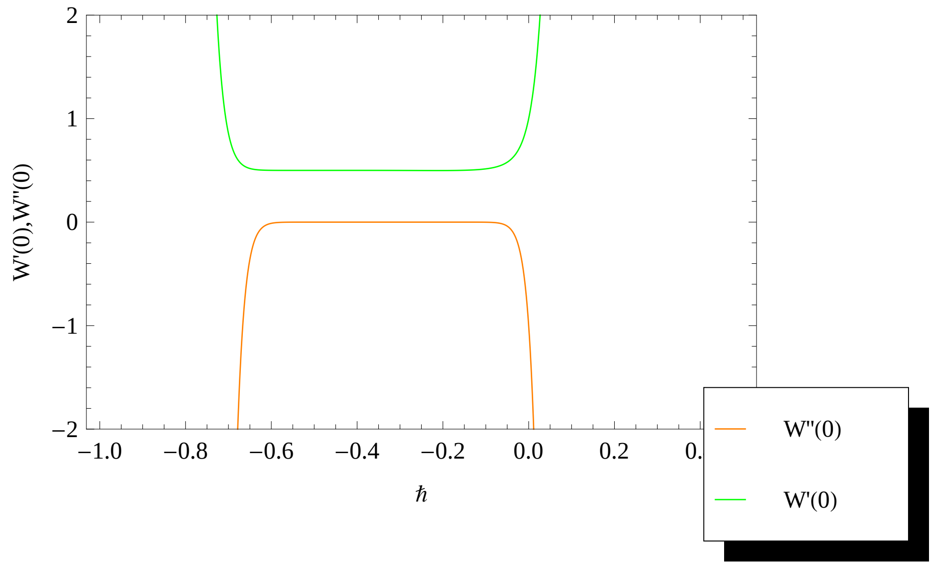

Fig. 3.8: The curves of approximation of $w'(0)$ and $w''(0)$ versus $\hbar$ for the 20th-order approximation.

From Fig. 3.9, it can be seen that norm 2 of the error is minimum at $\hbar = -0.422648$. In this case, the value of $\hbar$ lies in the admissible range of $\hbar$ too.

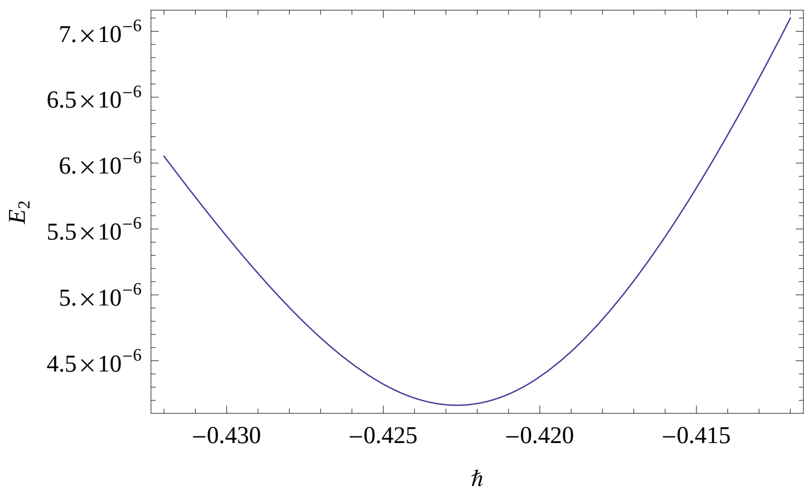

Fig. 3.9: Norm 2 of the error for the 20th-order approximation by HAM for $w(\xi)$ per $\hbar$.

3.3 Conclusion

So, this chapter, again, shows the flexibility and potential of the homotopy analysis method for nonlinear problems in science and engineering.

Chapter 4

Further work

In chapter 1, we gave some guidelines which can be useful to perform HAM on some nonlinear problems but there is some limitations yet on solving a given problem that is completely new for us (means we have not any prior on it). In this chapter we contemplate to introduce some problems which we have not given any HAM approximate solution for them, yet. We invite anybody who is solicitous HAM, to experiment these problems.

4.1 First problem

This problem is suggested by John Parkes to Dr. Abbasbandy in April 28, 2008. Waiver the physical scenario and mathematical formulation, the problem which is formulated for the HAM is

$$\phi'' = e^{\phi} - \frac{M}{\sqrt{M^2 - 2\phi}} \tag{4.1.1}$$

subject to the boundary conditions

$$\phi(0) = a, \quad \phi'(0) = 0, \quad \phi(\infty) = 0, \tag{4.1.2}$$

where prime denotes the derivative with respect to ξ which $\xi := x - Mt$ (M is the Mach number), a and M are related by

$$M^2 = \frac{(e^a - 1)^2}{2(e^a - a - 1)}. \tag{4.1.3}$$

Firstly, a value of M is chosen from the interval $1 < M \leq M_{max}$. The HAM should then deliver approximations to ϕ (the solitary wave) and to a (the amplitude of the wave). Liao in [29] has used HAM to solve multiple solution of Gelfand equation. This equation has the term $e^{u(x)}$ and he used the transformation $w(x) = e^{-u(x)}$ then apply the HAM on the new formulation of this equation. But this transformation is not effective here and the essential trouble is finding the terms of series solution $\phi(\xi) = \sum_{k=0}^{+\infty} \phi_k(\xi)$ by the aid of mth-order deformation equation. Parkes has told that he knows several similar problems in plasma physics, but the details are more complicated!. The following problems were suggested by Dr. Abbasbandy.

4.2 Second problem

The reduced form of the Navier-Stokes equations is of the form [47]

$$\eta f''' + f'' + R[1 + ff'' - (f')^2] = 0, \tag{4.2.1}$$

$$\eta g'' + g' + R[fg' - gf'] = 0, \tag{4.2.2}$$

where R is a Reynold number. The boundary conditions are influenced by

$$f'(1) = \lambda f''(1), \quad g(1) = 1 + \lambda g'(1), \quad f(1) = 0, \quad f'(\infty) = 1, \quad g(\infty) = 0, \tag{4.2.3}$$

where λ is a normalized slip factor.

4.3 Third problem

The electro-magnetohydrodynamic (EMHD) free-convection flow of a weakly conducting fluid (e.g. seawater) from an electromagnetic actuator is considered. The actuator is a so called *Riga-plate* which consists of a spanwise aligned array of alternating electrodes and permanent magnets mounted on a plane surface [39]. The non-dimentional forms of these equations are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{4.3.1}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} + e^{-Y}, \tag{4.3.2}$$

with the boundary conditions of the form

$$U\Big|_{Y=0} = 0, \quad V\Big|_{Y=0} = -V_0, \quad U\Big|_{Y\to\infty} \to 0. \tag{4.3.3}$$

It seems that we need some mathematical transformation here to perform HAM on this problem but the lack of knowledge have fettered us.

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