In the Name of God

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Title:

Homotopy Analysis Method for Solving Fuzzy and Crisp Nonlinear Differential Equations

PH.D. Thesis

By:
Mahmoud Paripour

Supervisors:
Prof. Esmael Babolian

Prof. Saeid Abbasbandy

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To

My parents and my wife,

with my appreciation

for their encouragement, patience, support and love.

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Tehran, Mahmoud Paripour

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Declaration

All papers mentioned below have been taken from and mostly published using this thesis.

1. E. Babolian, J. Saeidian, M. Paripour, Application of the homotopy analysis equations, method for solving equal-width wave and modified equal-width wave Zeitschrift für Naturforshung A, 64 a (2009) 685-690.

2. M. Paripour, E. Babolian, J. Saeidian, Analytic solutions to diffusion equations, Mathematical and Computer Modelling, 51 (2010) 649-657.

3. E. Babolian, S. Abbasbandy, M. Paripour, Numerical solution of fuzzy differential equations by homotopy analysis method, Submitted to Information Sciences.

4. E. Babolian, M. Paripour, Fuzzy model for the air quality of a building, Proc. 4th the National Conference Environment, Hamedan, Iran,March 2010.

Abstract

Mathematical models are widely used in different fields of sciences, engineering, economic, medical and so on to formulate certain problems. Most of these models involve an unknown function and its derivations or integration with respect to independent variables. Such equations are called integral equations, differential equations or integro-differential equation. In general, such equations are called functional equations.

Because of their numerous applications, it is obvious that the solution of these kind of equations is of ultimate importance. Since the analytical solution is only available for some special kind of ordinary differential or integral equation, therefore when the analytical solution is not available (or not easy to obtain) the numerical methods are employed to get an approximated solution. In these cases, the performance of the numerical method has to be assessed and verified. To do so, the results obtained by numerical method is compared with the analytical solution for some cases where the analytical solutions are available as benchmarks. When the accuracy of numerical method is verified, the numerical model can be used to solve not available. the similar cases whose analytic solutions are in general there are only a limited number of analytical methods which are used to tackle with functional equations and they are usable just for a few special cases which mostly are far from the real problems. On the other hands, numerical methods need quite large number of computations which reduce the accuracy of the solution due to rounding errors.

Over the recent years some new methods such as Adomian decomposition, variational iteration, homotopy perturbation and homotopy analysis are introduced to solve the functional equations.

In this thesis, these methods are introduced and their performances are compared with the other alternatives. In particular, the homotopy analysis method is used to solve fuzzy differential equations, diffusion equations and equal width wave equations and the obtained solutions are compared with the mentioned methods.

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Nonlinear differential equations, Fuzzy differential equations, Adomian decomposition, Homotopy perturbation method, Variational iteration method, Homotopy analysis method.

Introduction

Nonlinear equations appear in modelling most natural phenomena and most of advancements in numerical analysis and differential equation theory are based on attempts performed to recognize and solve the nonlinear equations.

Numerical methods have been widely used to solve these equations. The limitation of numerical methods,on one side,and intrinsic interest in solving these equations analytically have led to offer different analytical methods.

Homotopy analysis method was first introduced by professor Shijun Liao and has been used by scientists of different sciences to solve kinds of different differential equations. It has been able to find a high position among researchers an effective analytic method. This combinational method is a combination of classic perturbation method and homotopy notion in topology and its ability in presenting exact approximation and mostly exact solutions have made it renown. The main idea of this method, is transform nonlinear equation to a system of linear equations which are easily solved by recursive method.

In this first chapter of this thesis, we focus on the definitions and concept of fuzzy calculus, in the second chapter containing six sections, we introduce analytic methods. In the first section, we introduce homotopy, in the second the homotopy perturbation method, in the third section Adomian decomposition method , in the fourth section homotopy analysis method and in the fifth section variational calculus and in the last section variational iteration method. We will solve quadratic Riccati equation as an example with every four method.

In the third chapter, we turn to solve fuzzy differential equation by homotopy analysis method . This chapter contains three sections. In the first section we introduce fuzzy first order differential equation and Buckly-Feuring solution, in the second section, we will present fuzzy initial value problem and finally we find an approximation of fuzzy differential equation in crisp case using homotopy analysis method then, we extend the obtained solution to a solution in fuzzy case.

In the fourth chapter, we solve the diffusion equations, an important class of parabolic equations,by homotopy analysis method. We solve all examples in the fifth chapter through equal-width wave equation and modified equal-width wave equation with the same choices by homotopy analysis method while comparing approximated solutions in homotopy perturbation, Adomian decomposition method and variational iteration method.

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Chapter 1

Preliminaries

Fuzzy set theory is a rather recent branch of mathematics invented by Iran-born professor, Mr. Lotfi Asgarzadeh. This theory has been expanded theoretically and practically since 1965. The theory of fuzzy sets provided some tools by which reasoning and decision-making could be formulated so that the mathematical patterns are used in different technologies and sciences.

In this chapter, we define some notions required to know the basics of fuzzy set theory. This chapter contains three sections: we introduce the compact convex subsets in R^n and ε^n space in the first and second sections. The derivatives used in fuzzy calculus are introduced in the last section.

1.1 Compact Convex Subsets in R^n

Attention will be focused on the following two spaces of nonempty subsets of R^n:

1) κ^n consisting of all nonempty compact(that is, closed and bounded) subsets of R^n,

2) κ_c^n consisting of all nonempty compact convex subsets of R^n. Note the strict inclusion $\kappa_c^n \subset \kappa^n$.

Definition 1.1.1. Let A and B be two nonempty subsets of R^n and let $\lambda \in R$. Define addition and scalar multiplication respectively by

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\}.$$

Theorem 1.1.2. *κ^n and κ_c^n are closed under the operation of addition and scalar multiplication.*

In fact, thesis two operations induce a linear structure on κ^n and κ_c^n with zero element $\{0\}$, But in general, $A + (-1)A \neq \{0\}$.

Example 1.1.3. Let $A = [0, 1]$ so that $(-1)A = [-1, 0]$, and so:

$$A + (-1)A = [0, 1] + [-1, 0] = [-1, 1].$$

Definition 1.1.4. we define the Hukuhara difference $A \sim_h B$ of nonempty sets A and B, provided it exists, as the nonempty set C satisfying $A = B + C$.

Example 1.1.5. From the preceding example,

$$[-1, 1] \sim_h [-1, 0] = [0, 1],$$
$$[-1, 1] \sim_h [0, 1] = [-1, 0].$$

Clearly, $A \sim_h A = \{0\}$ for all nonempty A .

It is an obvious necessary condition for the Hukuhara difference $A \sim_h B$ to exist is that some translate of B is a subset of A, $B + \{c\}$ for some $c \in R^n$. When it exist,Hukuhara difference is unique.

Example 1.1.6. $\{0\} \sim_h [0, 1]$ does not exist, since no translate of $[0, 1]$ can ever belong to the singleton $\{0\}$.

Definition 1.1.7. Let x a point in R^n and A a nonempty subset of R^n. Define the distance $d(x, A)$ from x to A by:

$$d(x, A) = \inf\{\|x - a\| : a \in A\}.$$

Thus $d(x, A) = d(x, \bar{A}) \geq 0$ and $d(x, A) = 0$ if and only if $x \in \bar{A}$ [55,2.3.2].

Definition 1.1.8. ϵ-neighborhood of nonempty set A is:

$$S_\epsilon(A) = \{x \in R^n : d(x, A) < \epsilon\},$$

Its closure is the subset:

$$\bar{S}_\epsilon(A) = \{x \in R^n : d(x, A) \leq \epsilon\}.$$

In particular, denote by $\bar{S_1}^n$ the closed unit ball in R^n. Note that $\bar{S}_\epsilon(A) = A + \epsilon\bar{S}_1^n$. for any $\epsilon > 0$ and any nonempty subset A of R^n.

Definition 1.1.9. Let A and B be nonempty subsets of R^n. Define the Hausdorff separation of B from A by

$$d_H^*(B, A) = \sup\{d(b, A) : b \in B\},$$

Equivalently

$$d_H^*(B, A) = \inf\{\epsilon > 0 : B \subset A + \epsilon \bar{S}_1^n\}.$$

Thus $d_H^*(B, A) \geq 0$ with $d_H^*(B, A) = 0$ if and only if $B \subset \bar{A}$. In addition, the triangular inequality

$$d_H^*(B, A) \leq d_H^*(B, C) + d_H^*(C, A).$$

holds for all nonempty subsets A, B and C. However, in general, $d_H^*(B, A) \neq d_H^*(A, B)$,[37,5.4.2].

Definition 1.1.10. The Hausdorff distance between nonempty subsets A and B of R^n is defined by:

$$d_H(A, B) = \max\{d_H^*(A, B), d_H^*(B, A)\},$$

Consequently,

1) $d_H(A, B) \geq 0$ with $d_H(A, B) = 0$ if and only if $\bar{A} = \bar{B}$;

2) $d_H(A, B) = d_H(B, A)$;

3) $d_H(A, B) \leq d_H(A, C) + d_H(C, B)$.

Theorem 1.1.11. [28, 3.4.5] *Both* (κ^n, d_H) *and* (κ_c^n, d_H) *are Complete Separable metric spaces.*

1.2 The Space ε^n

Definition 1.2.1. A fuzzy subset of R^n is defined in terms of a membership function which assign to each point $xinR^n$ a grade of membership in the fuzzy set. Such a membership function $\mu : R^n \longrightarrow [0, 1]$ is used synonymously to denote the corresponding fuzzy set. Denote by $\digamma^n$ the set of all fuzzy subsets of R^n.

Definition 1.2.2. For each $\alpha \in [0, 1]$ the α-level set $U[\alpha]$ of a fuzzy set U is the subset of points $x \in R^n$ with membership grade $\mu_U(x)$ of at least α , that is,

$$U[\alpha] = \{x \in R^n : \mu_U(x) \geq \alpha\}.$$

Definition 1.2.3. The Support $U[0]$ of a fuzzy set is then defined as the closure of the union of all its level sets, that is,

$$U[0] = \overline{\bigcup_{\alpha \in (0,1]} U[\alpha]}.$$

Definition 1.2.4. A fuzzy set U is said to be fuzzy convex if

$$\mu_U(\lambda x + (1-\lambda)y) \geq \min\{\mu_U(x), \mu_U(y)\}$$

for all $x, y \in U[0]$ and $\lambda \in [0,1]$.

$U[\alpha]$ satisfy the following properties:

1) For all $0 \leq \alpha \leq \beta \leq 1$:
$$U[\beta] \subseteq U[\alpha] \subseteq U[0].$$

2) $U[\alpha] \neq \emptyset$ for all $\alpha \in I$.

3) $U[0]$ is a bounded subset of R^n.

4) $U[\alpha]$ is a compact subset of R^n for all $\alpha \in I$.

5) For any nondecreasing sequence $\{\alpha_i\}$ in I, if $\alpha_i \longrightarrow \alpha$ then
$$U[\alpha] = \bigcap_{i \geq 1} U[\alpha_i].$$

6) If U is fuzzy covex, then $U[\alpha]$ is convex for each $\alpha \in I$.

Definition 1.2.5. Define the characteristic function χ_A of a subset A of R^n is defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Definition 1.2.6. Let U a fuzzy subset of R^n, and $\mu_U(x) = 1$ for $x \in R^n$ then U is normal fuzzy set.

Definition 1.2.7. Let $\mu : R^n \longrightarrow [0,1]$ and $\{x \in R^n : \mu(x) < \lambda\}$ is open set for $\lambda \in [0,1]$, then μ is a upper semicontiuous.

Example 1.2.8. Characteristic function is a upper semicontiuous in close interval,but it is not upper semicontiuous in open interval.

Definition 1.2.9. Denote by ε^n the space of all fuzzy subsets U of R^n whe satisfy properties: normal, fuzzy convex, upper semicontinuous fuzzy sets with bounded supports. When $n = 1$, elements of ε are called fuzzy numbers and the compact convex level sets are compact real intervals.

Theorem 1.2.10. [37, 8.2.6] *If $A \in \kappa_C^n$, then $\chi_A \in \varepsilon^n$.*

Definition 1.2.11. Addition and scalar multiplication of fuzzy sets in ε^n will be defined level $U, V \in \varepsilon^n$ and $c \in R - \{0\}$

$$(U + V)[\alpha] = U[\alpha] + V[\alpha],$$
$$(cU)[\alpha] = cU[\alpha].$$

for each $\alpha \in I$.

These definition are compatible with Extension Principle [64,4.3.5], that is,

$$\mu_{u+v}(Z) = \sup_{Z=x+y} \min\{\mu_u(x), \mu_v(y)\}$$
$$\mu_{cu}(x) = \mu_u(x/C).$$

Theorem 1.2.12. [28, 2.3.4] *ε^n is closed under addition and scalar multiplication.*

Definition 1.2.13. The triangular fuzzy numbers are those fuzzy sets in ε which are characterized by an ordered triple $(x_l, x_c, x_r) \in R^3$ with $x_l \leq x_c \leq x_r$, such that $U[0] = [x_l, x_r]$ and $U[1] = \{x_c\}$, for then

$$U[\alpha] = [x_c - (1 - \alpha)(x_c - x_l), x_c + (1 - \alpha)(x_r - x_c)],$$

for any $\alpha \in I$. Hence their graph (of the membership function) is a triangle. Let ε_{TG} denotes the subset of all triangular fuzzy numbers.

Definition 1.2.14. The trapezoidal fuzzy numbers are those fuzzy sets in ε which are characterized by an ordered quadruple $(x_l, x_{cl}, x_{cr}, x_r) \in R^4$ with $x_l \leq x_{cl} \leq x_{cr} \leq x_r$, such that $U[0] = [x_l, x_r]$ and $U[1] = [x_{cl}, x_{cr}]$. We denote the subset of all trapezoidal fuzzy numbers by ε_{TP} then $\varepsilon_{TG} \subset \varepsilon_{TP}$.

1.3 Some Type of Derivatives in Fuzzy Calculus

Definition 1.3.1. The parameter fuzzy number is pair (u_1, u_2) of functions $u_1(\alpha)$, $u_2(\alpha)$, $0 \leq \alpha \leq 1$ which satisfy the following requirements:

1) $u_1(\alpha)$ is a bounded monotone increasing right continuous function;

2) $u_2(\alpha)$ is a bounded monotone decreasing left continuous function;

3) $0 \leq \alpha \leq 1; u_1(\alpha) \leq u_2(\alpha)$.

This definition is given by Kaleva [52].

Let $X : I \longrightarrow D$ is a fuzzy function i.e. $X(t)$ is a fuzzy number for each $t \in I$, and I is an interval which contains zero. Also, let $X(t)[\alpha] = [x_1(t,\alpha), x_2(t,\alpha)]$ and write $x_i'(t,\alpha)$ for the partial derivatives of $x_i(t,\alpha)$ with respect to t, $i = 1, 2$. We assume that these partial derivatives always exist in this section.

1.3.1 Goetschel-Voxman Derivative

The Goetschel-Voxman Derivative of $X(t)$, written $GVDX(t)$, was defined in [44]. We first must give the metric used for this derivative. Let $X(t)$ and $Y(t)$ be two fuzzy functions for $t \in I$. Both $X(t)$ and $Y(t)$ are fuzzy numbers for each $t \in I$. Set $X(t)[\alpha] = [x_1(t,\alpha), x_2(t,\alpha)], Y(t)[\alpha] = [y_1(t,\alpha), y_2(t,\alpha)]$,for all $t \in I$ and $\alpha \in [0,1]$. Then the metric D is

$$D(X(t), Y(t)) = \max\{\sup_{0\leq\alpha\leq1} |x_1(t,\alpha) - y_1(t,\alpha)|, \sup_{0\leq\alpha\leq1} |x_2(t,\alpha) - y_2(t,\alpha)|\}.$$

The derivative of $X(t)$ at t_0 is defined as

$$GVDX(t_0) = \lim_{h\longrightarrow0}(\frac{X(t_0+h) - X(t_0)}{h})$$

provided the limit exist with respect to the metric D. However, the subtraction in limit is not standard fuzzy subtraction because

$$(X(t_0+h) - X(t_0))[\alpha] = [x_1(t_0+h,\alpha) - x_1(t_0,\alpha), x_2(t_0+h,\alpha) - x_2(t_0,\alpha)].$$

for all $t \in I$ and $\alpha \in [0,1]$.

1.3.2 Seikkala Derivative

The Seikkala Derivative of $X(t)$, written $SDX(t)$, is defined in [68]. This definition is as follows: if $[x_1'(t,\alpha), x_2'(t,\alpha)]$ are the α-cuts of a fuzzy number for each $t \in I$, then $SDX(t)$ exists and

$$SDX(t)[\alpha] = [x_1'(t,\alpha), x_2'(t,\alpha)].$$

1.3.3 Puri-Ralescu Derivative

The Puri-Ralescu Derivative of $X(t)$, written $PRDX(t)$, is defined in [41]. This definition of $PRDX(t)$ is as follows: we first specify the metric used for this derivative. $X(t)$ and $Y(t)$ are two fuzzy function and the metric D is now

$$D(X(t), Y(t)) = \sup_{0 \leq \alpha \leq 1} d_H(X(t)[\alpha], Y(t)[\alpha]),$$

for all $t \in I$, where d_H is the Hausdorff metric. Fuzzy function $X(t)$ is differentiable at $t_0 \in I$ if there exists fuzzy number $PRDX(t_0)$ such that,

$$\lim_{h \longrightarrow 0^+} (\frac{X(t_0 + h) \sim_h X(t_0)}{h}) = PRDX(t_0),$$

$$\lim_{h \longrightarrow 0^+} (\frac{X(t_0) \sim_h X(t_0 + h)}{h}) = PRDX(t_0).$$

Both limits are taken with respect to the metric D. We can see that if $PRDX(t)$ exists, then

$$PRDX(t)[\alpha] = [x_1'(t, \alpha), x_2'(t, \alpha)].$$

for all $t \in I$, all $\alpha \in [0, 1]$. $PRDX(t)$ is always a fuzzy number for each $t \in I$.

1.3.4 Kandel-Fridman-Ming Derivative

The Kandel-Fridman-Ming Derivative of $X(t)$, written $KFMDX(t)$, is defined in [54]. The metric D used is

$$D_p(X(t), Y(t)) = \max\{[\int_0^1 \{x_1(t, \alpha) - y_1(t, \alpha)|^p d\alpha]^{\frac{1}{p}}, [\int_0^1 |x_2(t, \alpha) - y_2(t, \alpha)|^p d\alpha]^{\frac{1}{p}}\}.$$

for $x_1(t, \alpha)$, $x_2(t, \alpha)$, $y_1(t, \alpha)$ and $y_2(t, \alpha)$ all in $L_p[0, 1]$ for all $t \in I$. $X(t)$ is differentiable at $t_0 \in I$ if there is a fuzzy number $KFMDX(t_0)$ so that

$$\lim_{h \longrightarrow 0} D_p(\frac{X(t_0 + h) - X(t_0)}{h}, KFMDX(t_0)) = 0,$$

However,the subtraction $X(t_0 + h) - X(t_0)$ in (1.3.1) is defined. We also have,when $KFMDX(t_0)$ exists,

$$KFMDX(t)[\alpha] = [x_1'(t, \alpha), x_2'(t, \alpha)].$$

for all $t \in I$, all $\alpha \in [0, 1]$. This derivative also equals a fuzzy number for all $t \in I$ [54,3.4.7].

The following theorem investigate the relationship between these four derivative.

Theorem 1.3.2. [27] *Assume $X(t)$ fuzzy function*

1) *If $GVDX(t)$ exists and is a fuzzy number for each $t \in I$, then $SDX(t)$ exists and $GVDX(t) = SDX(t)$.*

2) *If $PRDX(t)$ exists, then $SDX(t)$ exists and $PRDX(t) = SDX(t)$.*

3) *If $KFMDX(t)$ exists, then so does $SDX(t)$ and they are equal.*

Theorem 1.3.3. [27] *Assume $x_i'(t, \alpha)$ is continuous on $I \times [0, 1]$, $i = 1, 2$. If $SDX(t)$ exists, then*

$$SDX(t) = GVDX(t) = PRDX(t) = KFMX(t).$$

Chapter 2

Analytic method in solving nonlinear function equation

2.1 Introduction

Nonlinear function equations appear to model most natural phenomena which have always been interested for different scientists. The numerical methods have widely been used to solve such equation.

Most of advances in numerical analysis and differential equation theory are based on attempts to recognize and solve nonlinear equations. The numerical methods have some limitations in that it provides us with discrete or point-wise solutions, so that it is difficult to find the general solutions. Also in some problems the equation under question is nonsingular or has a lot of solution, there by making them unreliable. The limitation of numerical methods from one side and intrinsic interest in finding analytical and closed from solution along with the growth of mathematical softwares such as Matlab, Mathematica and Maple which calculate the function values and long algebraic terms in fraction of second have caused different analytical methods to be introduced to solve function equations. We introduce some analytical methods in this chapter, homotopy concept in the first section, the homotopy method in the second section, Adomin decomposition method in the third section.Homotopy analysis method in the fourth section, variational calculus in the fifth section and the variational iteration in the last section.

2.2 Homotopy

Definition 2.2.1. A topology in X set is a collection like τ of subsets X satisfying the following conditions [64]:

1) $\emptyset$ and X belong to τ.

2) The union of members of each collection τ belongs to τ.

3) The intersection of members of each finite collection τ belong to τ.

Definition 2.2.2. Let X be a set for which τ topology is defined. Then (X, τ) is a topologic space.

Definition 2.2.3. Let X be a topologic space. Continuous function $h : [a, b] \longrightarrow X$ is a path in which $h(a)$ and $h(b)$ are starting and ending points. h is said to connect these two points.

Definition 2.2.4. Suppose $f, g : [0, 1] = I \longrightarrow X$ are paths of X in that $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. f and g are homotopic if there is a continuous map of $F : I \times I \longrightarrow X$ such that

$$F(x, 0) = f(x), \qquad F(x, 1) = g(x),$$

$$F(0, t) = x_0, \qquad F(1, t) = x_1.$$

F is a homotopy between f and g. If f and g are homotopic, we write $f \sim g$.

Now we define $F_t(x) = F(x, t)$ for $x, t \in I$, then $F_0 = f$, $F_1 = g$, $F_t(0) = x_0$ and $F_t(1) = x_1$. There fore $t \longrightarrow F_t$ is a family of paths which maps f on g . Equally, $\{F_t\}_{0 \leq t \leq 1}$ is called a homotopy.

Definition 2.2.5. Suppose $a < b < c$ and $f : [a, b] \longrightarrow X$ is a path $g : [b, c] \longrightarrow X$ is another path in which $g(b) = f(b)$, let the first path be pared first and the second path. The multiplication of these two paths are defined as the following

$$(fg)(t) = \begin{cases} f(t), & t \in [a, b], \\ g(t), & t \in [b, c], \end{cases}$$

Therefore $fg : [a, c] \longrightarrow X$.

Definition 2.2.6. Let $f, g \in X \longrightarrow Y$ be continuous mapping. f is homotopic with g a continuous mapping of $F : X \times I \longrightarrow Y$ exist such that for $x \in X$, $F(x, 1) = g(x)$ and $F(x, 0) = f(x)$.

For more explanation, we define continuous mapping $F_t : X \longrightarrow Y$ as the following for each $x \in X$, $F_t(x) = F(x, t)$. Now we consider t as the time parameter for which in $t = 0$ we have $F_0 = f$ and in $t = 1$ we have $F_1 = g$. Therefore if we change t from zero to one then f maps on to g continuously.

Theorem 2.2.7. *Suppose $X = A \cup B$ and A and B are closed in X .in addition $f : A \longrightarrow Y$ and $g : B \longrightarrow Y$ are continuous then, if for each $x \in A \cap B$ and $g(x) = f(x)$ we can lend f and g to obtain $h : X \longrightarrow Y$ which is defined as $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$.*

Lemma 2.2.8. *Homotopy relationship is an equivalence relationship.*

Proof. We know that an equivalence relationship is reflexive ,symmetric and transitive $f \sim f$ is trivial because $F(x, t) = f(x)$ is the same homotopy. If $f \sim g$, we prove $g \sim f$. Suppose f is a homotopy between g and f. We consider $G(x, t) = F(x, 1 - t)$, then G defines homotopy between f and g. Suppose $f \sim g$ and $g \sim h$, we must show that $f \sim h$. If F is a homotopy between f and g, G is a homotopy between g and h, then $H : X \times I \longrightarrow Y$ is defined as

$$H(x, t) = \begin{cases} F(x, 2t), & t \in [0, \frac{1}{2}], \\ G(x, 2t - 1), & t \in [\frac{1}{2}, 1], \end{cases}$$

now we take $t = \frac{1}{2}$

$$F(x, 2t) = F(x, 1) = g(x) = G(x, 0) = G(x, 2t - 1),$$

Therefore H is well-defined as H is continuous on two closed sets of $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ from $X \times I$, according to glue lemma it is continuous on all interval $X \times I$. Finally H is the same homotopy between f and h. □

Definition 2.2.9. Suppose $A \subseteq X$ and $f, g : X \longrightarrow Y$ and f are g given such that for each $x \in A$, $f(x) = g(x)$. Meaning f and g are equal on A. f and g are said to be homotopic intern s of A when $F : X \times I \longrightarrow Y$ such that $x \in X$, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Also for $x \in A$ and $t \in I$, $F(x, t) = f(x) = g(x)$ and we have $f \sim_A g$ which for $A = \emptyset$ is $f \sim g$.

Example 2.2.10. Both continuous functions of $f, g : X \longrightarrow \mathbb{R}^n$ are homotopic, because if we define $F(x,t) = tg(x) + (1-t)f(x)$ is continuous as f, g and their linear combination are continuous . on the other hand , $F(x,0) = f(x)$ and $F(x,1) = g(x)$, then f and g are homotopic.

Example 2.2.11. Suppose that $X = Y = \mathbb{R}^n$, $f = 1_x$ and $g = 0$ then $F(x,t) = tx$ is a homotopy from g to f.

Example 2.2.12. Suppose $X = \mathbb{R}^2 - \{(0,0)\}$, f, g and h are defined as

$$\begin{aligned} f(s) &= (\cos \pi s, \sin \pi s), \\ g(s) &= (\cos \pi s, 2 \sin \pi s), \\ h(s) &= (\cos \pi s, -\sin \pi s). \end{aligned}$$

In this case, f and g are homotopic in X but f and h are not homotopic in X, because according to figure (2.1) we can not path f through $(0,0)$ continuously.

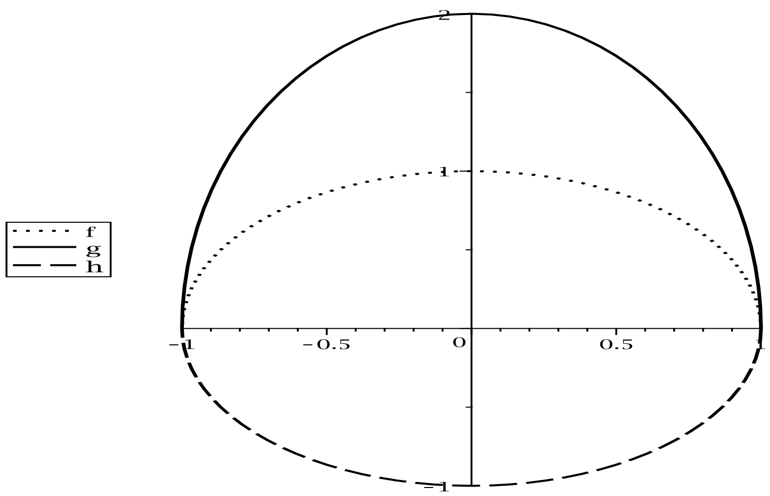

Figure 2.1.

2.3 Homotopy Perturbation Method

2.3.1 History

HPM was first introduced by professor Ji Huan He in 1998 and used to solve various problems and function equations [46, 47]. This is combination of classic perturbation and homotopy concept of topology.

The simplicity of method, theoretically and practically, and it's incredible ability in solving nonlinear equations and presenting exact approximations or exact solutions in most cases have let to the fame and popularity of this method.

This method was first applied by professor He on Light Hill [46], Duffing [48] and Blasius's equations[49]. Then this idea was used in solving nonlinear Wave equations [50] ,Riccati equations [1] and integral equations [2]. Also this method was used to calculate Laplace [3] and Fourier's transformation [22].

2.3.2 The Main Idea of Homotopy Perturbation Method

Time-dependent different equation can be considered,in general,as the following

$$A(y(r,t)) - f(r,t) = 0, \tag{2.1}$$

In which operator A is a differential operator, $y(r,t)$ is unknown function, t and r are time and place variables , respectively and $f(r,t)$ is known function. In general , we can separate A as $A = L + N$ in which L is a simple operator which can be manipulated easily (for example, the calculation of L^{-1} is easy) and N contains other parts of A. Always L is chosen to be a linear operator [14], to solve the above problem, there is considered an initial guess from the solution $\nu_0(r,t)$ in homotopy perturbation method and the homotopy equation is obtained by an embedding parameter (homotopy parameter) p, $0 \leq p \leq 1$

$$(1-p)L[\varphi(r,t;p) - \nu_0(r,t)] + p[A(\varphi(r,t;p)) - f(r,t)] = 0 \tag{2.2}$$

in which $\varphi(r,t;p)$ satisfies homotopy equation for $0 \leq p \leq 1$.

If $p = 0$ the equation can be rewritten as

$$L[\varphi(r,t;0) - \nu_0(r,t)] = 0$$

Such that if L is linear,we will have

$$\varphi(r,t;0) = \nu_0(r,t) + w(r,t), \qquad w(r,t) \in Kerl\{L\}$$

And for $p = 1$

$$A[\varphi(r,t;1)] - f(r,t) = 0$$

Which is the main equation with the solution $\varphi(r,t;1) = y(r,t)$ therefore changing the parameter from 0 to 1, we will obtain the solution of homotopy equation $\varphi(r,t;p)$

changing continuously from $\nu_0(r,t)$ to $y(r,t)$ (from the initial guess to exact solution). This process is called deformation.

If p which is relatively small is considered as perturbation parameter, we can assume that the solution of homotopy equation is expressed as a power series for p

$$\varphi(r,t;p) = u_0(r,t) + u_1(r,t)p + u_2(r,t)p^2 + \cdots . \tag{2.3}$$

If the above series is convergent for $p = 1$, then the solution of the main equation is

$$y(r,t) = u_0(r,t) + u_1(r,t) + u_2(r,t) + \cdots . \tag{2.4}$$

What is really done in homotopy perturbation method is that $\varphi(r,t;p)$ is placed from (2.3) relationship in homotopy equation (2.2), then we sort out the terms on p powers. As the term is satisfied for each p, we can take the coefficients of p powers as zero. A set of linear equation is obtained which can be solved based on initial and boundary condition to get u_1, u_2, $\cdots$, Finally we can get a solution of closed from. Since we have transformed a nonlinear equation in to a set of linear equations and calculated u_i in a pseudo-recursive process we claim the method to be an iterative method. Homotopy equation (2.2) can be rewritten as

$$L[\varphi(r,t;p) - \nu_0(r,t)] + p[N\varphi(r,t;p)] - f(r,t) + L[\nu_0(r,t)] = 0. \tag{2.5}$$

Example 2.3.1. Consider quadratic Riccati equation of ordinary nonlinear differential equation

$$\frac{dy(t)}{dt} = 2y(t) - y^2(t) + 1 \quad , y(0) = 0,$$

The exact solution of this equation is

$$y(t) = 1 + \sqrt{2}tanh(\sqrt{2}t + \frac{1}{2}log(\frac{\sqrt{2}-1}{\sqrt{2}+1})),$$

Which has Taylor's expansion at $t = 0$,

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \cdots$$

Choosing $L\varphi = \frac{\partial \varphi}{\partial t}$ and $\nu_0(t) = 0$, we can solve the equation [20]. Homotopy equation is as the following

$$\varphi_t + p[-2\varphi + \varphi^2 - 1] = 0$$

Substituting (2.3) in the above homotopy equation and balancing the p powers we get the following equation set

$$\begin{aligned}
&u_0 = 0, \quad u_0(0) = 0\\
&u_{1t} - 2u_0 + u_0^2 - 1 = 0, \quad u_1(0) = 0\\
&u_{2t} - 2u_1 + 2u_1u_0 = 0, \quad u_2(0) = 0\\
&u_{3t} - 2u_2 + u_1^2 + 2u_0u_2 = 0, \quad u_3(0) = 0\\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

Having solved these equations, we find the following values

$$\begin{aligned}
u_0(t) &= 0,\\
u_1(t) &= t,\\
u_2(t) &= t^2,\\
u_3(t) &= \frac{1}{3}t^3,\\
u_4(t) &= -\frac{1}{3}t^4,\\
&\vdots
\end{aligned}$$

Leading to the solution based on (2.4) relationship

$$y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 + \cdots .$$

The above series is in fact the Taylor's expansion of the problem.

2.4 Adomian Decomposition Method

2.4.1 History

ADM was first introduced by professor George adomian (1920-1996)in 1980 [11] which has been used to solve a wide range of function equations such as ordinary differential equations partial differential equations and integral equations of boundary or initial conditions [15-19]. In this method, the solution is expressed in terms of an infinite series usually (but not always) an approximation of the solution or exact solution.

2.4.2 The Main Idea of Adomian Decomposition Method

To solve the function equation

$$y - N(y) = f, \tag{2.6}$$

in which N is a nonlinear operator of Hilbert space H to H and f is a function on H, we search for $y \in H$ satisfying in the above equation. To solve the equation (2.6) with ADM we write y as a series of

$$y = \sum_{n=0}^{\infty} y_n, \tag{2.7}$$

and $N(y)$ as

$$N(y) = \sum_{n=0}^{\infty} A_n. \tag{2.8}$$

Which y_i is calculated the following

$$\begin{cases} y_0 = f \\ y_{n+1} = A_n(y_0, y_1, \cdots, y_n) \end{cases} \tag{2.9}$$

It is shown that A_n polynomial of y_0, y_1, $\cdots$ and y_n called Adomian polynomial [24] which is calculated in the following way

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i y_i)]_{\lambda=0}, n = 0, 1, 2, 3, \cdots.$$

The reader can refer to [16,30-35] for convergence discussion .

In general, we can use ADM to solve the following differential equation

$$Lu + Ru + Nu = g(t)$$

in which $N$is a nonlinear operator,L is the greatest derivatives of u, R is of the lower order derivative in L, therefore, the above equation is as

$$Lu = g - Ru - Nu,$$

or

$$u = f - L^{-1}(Ru) - L^{-1}(Nu).$$

in which $f = L^{-1}(g)$, $u = \sum_{n=0}^{\infty} u_n$ and $N(u) = \sum_{n=0}^{\infty} A_n$.

Example 2.4.1. Example (2.3.1)is solved by ADM [41]. The Riccati equation is rewritten as the following

$$y(t) = t + 2\int_0^t y(s)ds - \int_0^t y^2(s)ds$$

With the substitution

$$y(t) = y_0(t) + \sum_{k=1}^{\infty} y_k(t).$$

In the above relationship we have

$$\begin{aligned} y_0(t) &= t, \\ y_k(t) &= 2\int_0^t y_{k-1}(s)ds - \int_0^t \sum_{i=0}^{k-1} y_i(s)y_{k-1-i}(s)ds, \qquad k \geq 1 \end{aligned}$$

Solving the recursive equation, we have

$$\begin{aligned} y(t) &= y_0(t) + y_1(t) + y_2(t) + \cdots \\ &= t + t^2 + \frac{1}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5 + \cdots . \end{aligned}$$

2.5 Homotopy Analysis Method

2.5.1 History

HAM was first presented in Liao's Ph.D. thesis in 1992 [58] with a high capability in solving nonlinear equation. The efficacy of this method has been proved in solving various problems but there isn't offered any proof, while it has found a special status in different sciences [6,8,21,57-62].

2.5.2 The Main Idea of HAM

We consider the nonlinear equation of (2.10) in a general case

$$N[u(r,t)] = 0 \tag{2.10}$$

in which N is a nonlinear operator and $u(r,t)$ is unknown function, r,t are place and time variable and the equation has the initial or boundary condition.

Suppose u_0 is an initial guess of the exact solution $u(r,t)$, then $\hbar \neq 0$ is called auxiliary parameter (or convergence control parameter [62]) $H(r,t) \neq 0$ is an auxiliary function and L is auxiliary linear operator for which

$$f(r,t) = 0 \Longrightarrow f(r,t) = w(r,t), \qquad w(r,t) \in Kerl\{L\} \tag{2.11}$$

Liao introduced homotopy equation as the following by embedding parameter $0 \leq q \leq 1$,

$$H[\phi(r,t;q); u_0(r,t), H(r,t), \hbar, q] \\ = (1-q)\{L[\phi(r,t;q) - u_0(r,t)]\} - q\hbar H(r,t) N[\phi(r,t;q)] = 0 \qquad (2.12)$$

And defined the zeroth-order deformation equation,

$$(1-q)L[\phi(r,t;q) - u_0(r,t)] = q\hbar H(r,t)N[\phi(r,t;q)], \qquad (2.13)$$

When $q = 0$ and $q = 1$, we have from (2.13),

$$\phi(r,t;0) = u_0(r,t), \\ \phi(r,t;1) = u(r,t). \qquad (2.14)$$

Therefore if parameter q changes from 0 to 1 , we can get the solution of homotopy equation $\phi(r,t;q)$ changes from $u_0(r,t)$ to $u(r,t)$.

The expansion of Taylor's function $\phi(r,t;q)$ in terms of q is

$$\phi(r,t;q) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t) q^m, \qquad (2.15)$$

In which

$$u_m(r,t) = \frac{1}{m!} \frac{\partial^m \phi(r,t;q)}{\partial q^m}\Big|_{q=0}. \qquad (2.16)$$

If auxiliary linear operator L, the initial guess u_0, auxiliary parameter $\hbar$ and auxiliary function H are chosen correctly, the series (2.14)is convergent at $q = 1$ and we have from (2.14) and (2.15)

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t). \qquad (2.17)$$

And $\vec{u_n}$ vector is defined as the following,

$$\vec{u_n} = [u_0(r,t), u_1(r,t), \cdots, u_n(r,t)].$$

Regarding (2.16), we can get $u_m(x,t)$ from equation (2.13).

To do so, we derivative both sides of(2.13) m times in terms of q, then take q equal to zero, if we divide it over $m!$, the deformation equation of m order is obtained

$$L(u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar H(r,t) R_m(\vec{u_{m-1}}, r, t), \qquad (2.18)$$

in which

$$R_m(\vec{u}_{m-1}, r, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(r,t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \qquad (2.19)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2. \end{cases}$$

The series (2.15)is substituted in zeroth-order deformation equation (2.13) and the linear system equations is solved recursively. Therefore in this method, the nonlinear equation is transformed in a linear system equations which is easily solved recursively, having solved the equation (2.18), we can substitute $u(r,t)$ in (2.17).

Theorem 2.5.1. [59] *If series (2.17) obtained from homotopy analysis method is convergent, it is the solution of main equation (2.10)*

HAM gives us the exact solution in most cases and we can control and regulate the convergence area and the velocity of the series by auxiliary parameter $\hbar$. The further discussion can be found in [59] about how to use $\hbar$.

The main problem is that the auxiliary components of HAM are determined so that the resultant series is convergent Liao claimed that is possible to do so in most cases therefore we can choose L, u_0, H and $\hbar$ such that the series becomes convergent.

Although there isn't presented any mathematical proof for this assortment, the bulk of problems solved by Liao's method shows that it is an applied guess for applied problems.

The most important issue in HAM is how to choose the initial guess u_0, linear operator L and auxiliary function $H(r,t)$ for choice of which Liao has presented in his book [59] three rules. But the reasons make this method superior versus HPM ,ADM and so on are:

1) It close n't have the limitation of perturbation method since it doesn't need large or small parameters to include in the equation.

2) This method structurally includes most of classic analytical methods such as Lyapunov's artificial small parameter, σ-expansion method or Adomian method. Liao shows that these methods are special case of homotopy analysis method. In addition, homotopy perturbation method is a special case of homotopy analysis method when we have $h = -1$ and $H(r,t) = 1$.

3) In contrast with analytical methods, HAM is an easy way to control and regulate the convergence area and the velocity of approximated series which is taken by $\hbar$-curves.

4) There is a major difference between HPM and HAM, in HAM, there is a control parameter of convergence $\hbar$ not available in HPM, which causes the service to be convergent. the evolution equation has bean examined in [57] the resultant solution for

HPM is convergent (except for specific point) but HAM can give us the exact solution by choosing a suitable value for parameter $\hbar$.

Example 2.5.2. In example (2.3.1) if we choose $y_0(t) = t$, auxiliary linear operator $L[\varphi(t;q)] = \frac{\partial\varphi(t;q)}{\partial t}$ with the property $L(c_1) = 0$ in which c_1 is the integration coefficient determined by initial conditions, and nonlinear operator N as

$$N[\varphi(t;q)] = \frac{\partial\varphi(t;q)}{\partial t} - 2\varphi(t;q) + \varphi(t;q)^2 - 1$$

And suppose $q \in [0,1]$ the zeroth-order deformation equation is turned in to,

$$(1-q)L[\varphi(t;q) - y_0(t)] = q\hbar H(t)N[\varphi(t;q)]$$

With initial condition $\varphi(0,q) = 0$ for $q = 0$ and $q = 1$, we have

$$\varphi(t;0) = y_0(t),$$
$$\varphi(t;1) = y(t).$$

Defining $y_m(t) = \frac{1}{m!}\frac{\partial^m\varphi(t;q)}{\partial q^m}\Big|_{q=0}$, we consider Taylor's series as

$$\varphi(t;q) = \varphi(t;0) + \sum_{m=1}^{\infty} y_m(t)q^m$$

having defined vector

$$\vec{y_n(t)} = [y_0(t), y_1(t), \cdots, y_n(t)].$$

The m-th order deformation will be

$$L[y_m(t) - \chi_m y_{m-1}(t)] = \hbar H(t)R_m[\vec{y_{m-1}}(t)]$$

With the initial condition $y_m(0) = 0$ and

$$R_m[y_{m-1}(t)] = y'_{m-1}(t) - 2y_{m-1}(t) + \sum_{i=0}^{m-1} y_i(t)y_{m-1-i}(t) - (1-\chi_m)$$

Having chosen $H(t) = 1$, we will have

$$y_1(t) = -\hbar t^2 + \frac{1}{3}\hbar t^3,$$
$$y_2(t) = -\hbar(1+\hbar)t^2 + \hbar(\frac{1}{3}+\hbar)t^3 - \frac{2}{3}\hbar^2 t^4 + \frac{2}{15}\hbar^2 t^5,$$
$$\vdots$$

Then the solution is

$$y(t) = y_1(t) + y_2(t) + \cdots$$

For $\hbar = -1$, $y_{HAM}(t) = y_{ADM}(t)$, To compare the solutions, the reader must refer to[71].

2.6 Variational Calculus

2.6.1 History

variational calculus dates back to 1696 when Bernoulli examined the problem of the least time. In this problem two points A and B lying in one plane but not in normal direction are considered the mass in A moves to B along the curve C under the influence of gravitation while the friction is negligible the path under question is not straight. To move from A to B at the least time, we must pave a cycloid curve. This problem was sold by Leibnitz, Newton, and Hopital but it's development as a branch of mathematics was done by Euler 1783-1807.

The major application of variational calculus is to find a curve among those connecting two points which has the minimum length or to get the minimum (or maximum) of some integrals.

As a specific problem, we want to specify a curve of equation $Y = y(x)$ with $y(x_1) = y_1$ and $y(x_2) = y_2$ such that $\int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$ is minimum. Broadly, we want to find $Y = y(x)$ with such that for function $F(x, y, y')$, the value of

$$I = \int_{x_1}^{x_2} F(x, y, y') dx \tag{2.20}$$

is minimum or maximum which is called extremum and $y(x)$ is the extremal function.

The integral (2.20)which gives an extremum value for some functions $Y(x)$ is a functional.

Lemma 2.6.1. *If for each function $\lambda(x)$ in $[a, b]$,we have*

$$\int_a^b \varphi(x)\lambda(x)dx = 0,$$

such that function $\varphi(x)$ is continuous in $[a, b]$, then

$$\forall x \in [a, b]; \varphi(x) \equiv 0.$$

Proof. We use contradiction proof. Suppose that $\varphi(x) \not\equiv 0$ in this interval, then there is a point x_0 for which $\varphi(x_0) \not\equiv 0$, let $\varphi(x_0) > 0$ due to continuity of $\varphi(x)$ in $[a, b]$ there is a neighborhood of x_0, like $[x_1, x_2]$ for which

$$\forall x \in [x_1, x_2], \varphi(x) > 0$$

Now, we define $\lambda(x)$ as

$$\lambda(x) = \begin{cases} 0, & a \leq x < x_1, \\ \varphi(x), & x_1 \leq x \leq x_2, \\ 0, & x_2 < x \leq b. \end{cases}$$

Then we will have

$$0 = \int_a^b \lambda(x)\varphi(x)dx = \int_{x_1}^{x_2} (\varphi(x))^2 dx > 0$$

Which is in contradiction with our initial supposition, then $\varphi(x) \equiv 0$. □

2.6.2 Euler Equation

To find the curve $Y(x)$ in equation (2.20), the assumption is that $Y(x) = y(x) + \varepsilon\eta(x)$ in which $\eta(x)$ is an arbitrary and ε is an arbitrary parameter. To have the curve Y in the conditions of $Y(x_1) = y_1$ and $Y(x_2) = y_2$, we need to have $\eta(x_1) = \eta(x_2) = 0$.

Now we set

$$J(y(x)) = \int_{x_1}^{x_2} F(x, y, y')dx \tag{2.21}$$

and $I(\varepsilon)$ can be defined as

$$I(\varepsilon) = J(Y(x)) = \int_{x_1}^{x_2} F(x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x))dx = \int_{x_1}^{x_2} F_\varepsilon dx \tag{2.22}$$

We will have

$$\frac{dI}{d\varepsilon}\Big|_{\varepsilon=0} = 0$$

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} [\frac{\partial F}{\partial Y}\eta(x) + \frac{\partial F}{\partial Y'}\eta'(x)]dx = 0$$

with the integration by parts from the second equation, we have:

$$\frac{dI}{d\varepsilon}\Big|_{\varepsilon=0} = \int_{x_1}^{x_2} \frac{\partial F}{\partial Y}\eta(x)dx + \frac{\partial F}{\partial Y'}\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}[\frac{\partial F}{\partial Y'}]dx = 0$$

because $\eta(x_1) = \eta(x_2) = 0$, we have:

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \eta(x)[\frac{\partial F}{\partial Y} - \frac{d}{dx}(\frac{\partial F}{\partial Y'})]dx = 0 \tag{2.23}$$

By considering lemma (2.6.1) from the equation (2.23), we conclude

$$\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0 \tag{2.24}$$

equation (2.24) is known as Euler equation.

It needs to mention that equation (2.24) for $Y(x)$ to be extermal is necessary but not sufficiency.
It is clear that the Euler equation can be written as:

$$\frac{d}{dx}[F - y'\frac{\partial F}{\partial y'}] - \frac{dF}{dx} = 0, \tag{2.25}$$

In special case when F does not contain x, then $F = F(y, y')$, we have

$$F - y'F_{y'} = c. \tag{2.26}$$

2.6.3 Variational Notation δ

A function of $F(x, y, y')$ is considered and substituted

$$\begin{aligned} \delta F &= F(x, y + \varepsilon\eta, y' + \varepsilon\eta') - F(x, y, y') \\ &= \frac{\partial F}{\partial y}\varepsilon\eta + \frac{\partial F}{\partial y'}\varepsilon\eta' + \cdots \end{aligned}$$

By substituting $\delta F = \frac{\partial F}{\partial y}\varepsilon\eta + \frac{\partial F}{\partial y'}\varepsilon\eta'$ then if $F(x, y, y') = y$ or $F(x, y, y') = y'$, we have $\delta F = \delta y = \varepsilon\eta$ or $\delta F = \delta y' = \varepsilon\eta'$. Therefore $\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$.

Also δ and $\frac{d}{dx}$ are interchangeable, because

$$\delta y' = \delta(\frac{dy}{dx}) = \varepsilon\eta' = \frac{d}{dx}(\varepsilon\eta) = \frac{d}{dx}(\delta y) = (\delta y)'.$$

Theorem 2.6.2. *Let F is continuous, then*

$$\delta \int_a^b F(x, y, y')dx = \int_a^b \delta F(x, y, y')dx.$$

Proof.

$$\begin{aligned} \delta \int_a^b F(x, y, y')dx &= \frac{\partial}{\partial y}[\int_a^b F(x, y, y')dx]\delta y \\ &+ \frac{\partial}{\partial y'}[\int_a^b F(x, y, y')dx]\delta y' \\ &= \int_a^b [\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y']dx = \int_a^b \delta F(x, y, y')dx. \end{aligned}$$

□

2.6.4 Variational Notation Properties δ

Variational notation of δ have the following properties:

$$
\begin{aligned}
&1)\delta(F_1 \pm F_2) = \delta F_1 \pm \delta F_2,\\
&2)\delta(F_1F_2) = F_1\delta F_2 + F_2\delta F_1,\\
&3)\delta(F_1/F_2) = (F_2\delta F_1 - F_1\delta F_2)/F_2^2,\\
&4)\delta(F)^n = n(F)^{n-1}\delta F,\\
&5)D(\delta F) = \delta(DF), D = \frac{d}{dx}.
\end{aligned}
$$

Now, we want to define a sufficient condition to minimize functional J in equation (2.21). If $y(x)$ minimize

$$J(Y(x)) = \int_{x_1}^{x_2} F(x, Y, Y')dx$$

then for all acceptable Y we must have:

$$J(Y) \geq J(y) \longrightarrow J(y + \varepsilon\eta) \geq J(y).$$

We rewrite the Taylor expansion $J(y + \varepsilon\eta)$ on ε

$$
\begin{aligned}
J(y+\varepsilon\eta) &= \int_{x_1}^{x_2} F(x, y+\varepsilon\eta, y'+\varepsilon\eta')dx\\
\Longrightarrow J(y+\varepsilon\eta) &= J(y) + \int_{x_1}^{x_2} (\varepsilon\eta\frac{\partial F}{\partial y} + \varepsilon\eta'\frac{\partial F}{\partial y'})dx + O(\varepsilon^2)\\
\Longrightarrow J(y+\varepsilon\eta) - J(y) &= \int_{x_1}^{x_2} (\varepsilon\eta\frac{\partial F}{\partial y} + \varepsilon\eta'\frac{\partial F}{\partial y'})dx + O(\varepsilon^2)
\end{aligned}
$$

Now according to definition, δJ can be written as:

$$\delta J = \int_{x_1}^{x_2} (\varepsilon\eta\frac{\partial F}{\partial y} + \varepsilon\eta'\frac{\partial F}{\partial y'})dx, \tag{2.27}$$

By integration by parts, we calculate the second term of equation (2.27)

$$\delta J = \int_{x_1}^{x_2} \varepsilon\eta[\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})]dx, \tag{2.28}$$

Here we define the inner products as followed:

$$< f, g >= \int_{x_1}^{x_2} fg dx \tag{2.29}$$

According to (2.29), we can write

$$\delta J =< \varepsilon\eta, J'(y) >, J'(y) = \frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}),$$

Thus, we have

$$J(y+\varepsilon\eta) - J(y) =< \varepsilon\eta, J'(y) > +O(\varepsilon^2).$$

On the other side the derivative of function f is satisfied in below equation.

$$f(a+h) - f(a) = hf'(a) + O(h^2).$$

The necessary condition for a to be an extermum of function f is that the $hf'(a) = 0$ and as the result

$$< \varepsilon\eta, J'(y) >= 0 \Longleftrightarrow \delta J = \int_{x_1}^{x_2} \varepsilon\eta[\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'})]dx$$
$$= 0,$$

Therefore according to Lemma (2.6.1), we have :

$$\frac{\partial F}{\partial y} - \frac{d}{dx}[\frac{\partial F}{\partial y'}] = 0$$

there we can write:

$$\delta \int_a^b F(x, y, y')dx = 0 \Longleftrightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0,$$

Now by considering that

$$\Delta J = J(Y) - J(y),$$

We can conclude

$$\Delta J = \delta J + \frac{\varepsilon^2}{2!}\int_{x_1}^{x_2} \frac{\partial^2 F}{\partial \varepsilon^2}(x, y+\varepsilon\eta, y'+\varepsilon\eta')dx,$$

On the other side we have:

$$\frac{\partial^2 F}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon}[\frac{\partial F}{\partial \varepsilon}] = \frac{\partial}{\partial \varepsilon}[\eta\frac{\partial F}{\partial Y} + \eta'\frac{\partial F}{\partial Y'}] = \frac{\partial}{\partial \varepsilon}[\eta F_Y + \eta F_{Y'}],$$

where $Y = y + \varepsilon\eta$ and $Y' = y' + \varepsilon\eta'$, Therefore

$$\begin{aligned}
\frac{\partial^2 F}{\partial \varepsilon^2} &= \eta\frac{\partial}{\partial \varepsilon}F_Y + \eta'\frac{\partial}{\partial \varepsilon}F_{Y'} \\
&= \eta[\frac{\partial F_Y}{\partial Y}\frac{\partial Y}{\partial \varepsilon} + \frac{\partial F_Y}{\partial Y'}\frac{\partial Y'}{\partial \varepsilon}] + \eta'[\frac{\partial F'_Y}{\partial Y}\frac{\partial Y}{\partial \varepsilon} + \frac{\partial F'_Y}{\partial Y'}\frac{\partial Y'}{\partial \varepsilon}] \\
&= \eta[\eta F_{YY} + \eta' F_{YY'}] + \eta'[\eta F_{Y'Y} + \eta' F_{Y'Y'}] \\
&= \eta^2 F_{YY} + 2\eta\eta' F_{Y'Y} + \eta'^2 F_{Y'Y'}. \qquad (2.30)
\end{aligned}$$

Now for having $J(Y) \geq J(y)$, we must have:

$$\Delta J \geq 0 \Longrightarrow \frac{\varepsilon^2}{2!}\int_{x_1}^{x_2} \frac{\partial^2 F}{\partial \varepsilon^2}(x, y+\varepsilon\eta, y'+\varepsilon\eta')dx \geq 0, \qquad (2.31)$$

According (2.30) and (2.31), we define J_2 as

$$J_2 = \frac{\varepsilon^2}{2!}\int_{x_1}^{x_2}[\eta^2 F_{YY} + 2\eta\eta' F_{Y'Y} + \eta'^2 F_{Y'Y'}]dx. \tag{2.32}$$

There we can conclude that

1) if $y(x)$ minimize the functional J, then $J_2 \geq 0$.

2) if $y(x)$ maximize J the functional J, then $J_2 \leq 0$.

Example 2.6.3. Find the minimum for functional in below equation

$$J(y) = \int_0^1 (y'^2 + y^2)dx, y(0) = 0, y(1) = 1.$$

We have $F(x, y, y') = y'^2 + y^2$, from the Euler we will have

$$F_y - \frac{d}{dx}F_{y'} = 0 \Longrightarrow y - y'' = 0$$

By solving the second order equation from the above, we will have

$$y = c_1 \cosh(x) + c_2 \sinh(x).$$

By substituting the initial conditions, we have

$$y(0) = 0 \Longrightarrow c_1 = 0, y(1) = 1 \Longrightarrow c_2 = \frac{1}{\sinh(1)}.$$

Now we should have $J_2 \geq 0$, and there by having $F_{y'y} = 0$, $F_{yy'} = 2$, $F_{yy} = 2$ we have

$$J_2 = \varepsilon^2 \int_0^1 (\eta'^2 + \eta^2)dx \geq 0$$

as the result $y = \frac{\sinh(x)}{\sinh(1)}$ is the right solution.

2.6.5 Euler Equation For The Functional With Second Derivative

Assume $J(y(x)) = \int_a^b F(x, y, y', y'')dx$,

$$h(x_1) = h(x_2) = h'(x_1) = h'(x_2) = 0, Y(x) = y(x) + \varepsilon h(x)$$

We have

$$\Delta I = \varepsilon J_1 + \frac{\varepsilon^2}{2!}J_2 + \cdots,$$

where

$$J_1 = \int_{x_1}^{x_2} \frac{\partial F}{\partial \varepsilon}(x, y+\varepsilon h, y'+\varepsilon h', y''+\varepsilon h'')\big|_{\varepsilon=0} dx,$$
$$J_2 = \int_{x_1}^{x_2} \frac{\partial^2 F}{\partial \varepsilon^2}(x, y+\varepsilon h, y'+\varepsilon h', y''+\varepsilon h'')\big|_{\varepsilon=0} dx.$$

on the other side we have

$$\begin{aligned}\frac{\partial F}{\partial \varepsilon} &= \frac{\partial F}{\partial y}.\frac{\partial Y}{\partial \varepsilon} + \frac{\partial F}{\partial \varepsilon} + \frac{\partial F}{\partial y'}\frac{\partial Y'}{\partial \varepsilon} + \frac{\partial F}{\partial y''}\frac{\partial Y''}{\partial \varepsilon} \\ &= hF_y + h'F_{y'} + h''F_{y''},\end{aligned}$$

Therefore

$$J_1 = \int_{x_1}^{x_2} (hF_y + h'F_{y'} + h''F_{y''}) dx \tag{2.33}$$

The second and third terms of the above equation can be written as below using integration by parts:

$$\int_{x_1}^{x_2} h'F_{y'} dx = F_{y'}h\big|_{x_1}^{x_2} - \int_{x_1}^{x_2} h\frac{d}{dx}F_{y'} dx = -\int_{x_1}^{x_2} h\frac{d}{dx}F_{y'} dx,$$
$$\int_{x_1}^{x_2} h''F_{y''} dx = F_{y''}h'\big|_{x_1}^{x_2} - \int_{x_1}^{x_2} h'\frac{d}{dx}F_{y''} dx = -\int_{x_1}^{x_2} h'\frac{d}{dx}F_{y''} dx.$$

Therefore, we have

$$J_1 = \int_{x_1}^{x_2} h[F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''}] dx$$

Now according to Lemma (2.6.1), we have:

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0$$

For the functional $J(y(x)) = \int_{x_1}^{x_2} F(x, y, y', \cdots, y^{(n)}) dx$ and by assuming that $k = 0, \cdots, n-1$, $h^{(k)}(x_1) = h^{(k)}(x_2) = 0$,, we will have:

$$\sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} F_{y^{(k)}} = 0. \tag{2.34}$$

2.6.6 Euler Equations For n Functions

We consider the below functional

$$J(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n') = \int_a^b F(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n') dx$$

By assuming that the curves $y_j, (1 \leq j \leq n)$ pass the points A_j and $B_j, (1 \leq j \leq n)$ and $Y_j = y_j + \varepsilon\eta_j(x)$ by accepting that boundary condition $\eta_j(x)$ are zero at the start and finish points and by using the Taylor expansion for the variable δJ, we have:

$$\delta J = \sum_{j=1}^{n} < \varepsilon\eta, J'_{y_j}(y_1, y_2, \cdots, y_n) >, J'_{y_j} = F_{y_j} - \frac{d}{dx}F_{y'_j}, j = 1, 2, \cdots, n.$$

Now we write the extermum condition for the functional.

$$\sum_{j=1}^{n} < \varepsilon\eta_j, J'_{y_j} > = 0 \Longrightarrow < \varepsilon\eta_j, J'_{y_j} >$$

$$= 0 \Longrightarrow \int_a^b \varepsilon\eta_j(F_{y_j} - \frac{d}{dx}F_{y_j})dx = 0$$

as the result we have

$$F_{y_j} - \frac{d}{dx}F_{y_j} = 0, \quad j = 1, 2, \cdots, n. \tag{2.35}$$

2.6.7 Euler Equations For The Functional Dependent On Several Independent Variable

Assume we find the function $u(x, y)$ if the functional $I[u(x, y)] = \int\int_G F(x, y, u, u_x, u_y)dxdy$ becomes extermum in which the boundary G consider to a smooth curve of Γ. In this case the function $u(x, y)$ can be taken from a differential equation as below

$$F_u - \frac{\partial}{\partial x}F_{u_x} - \frac{\partial}{\partial y}F_{u_y} = 0. \tag{2.36}$$

this equation is know as Euler-Staragrotski equation.

2.6.8 An Extermum of a functional on a curve

In problem the extermum functional $J(y) = \int_a^b F(x, y, y')dx$ we consider a condition when the two sides the curve are not constant. In the other way, the $A(a, \alpha)$ and $B(b, \beta)$ are existing on the curves C_1 and C_2 with equation $y = \varepsilon r(x)$ and $y = \varphi(x)$, In this can the curve $y(x)$, in this can be known bye solving the equation below.

$$\begin{aligned} &F_y - \frac{d}{dx}F_{y'} = 0, \\ &F + (\varphi' - y')F_{y'}\Big|_{x=a} = 0, \\ &F + (\phi' - y')F_{y'}\Big|_{x=b} = 0. \end{aligned} \tag{2.37}$$

Example 2.6.4. The curves with equations $y = x^2$ and $y = x-2$ are known. Determine points A and B on the two curves when the length of line AB is at minimum.

Functional of the length of the curve will be considered as $J(y) = \int_A^B \sqrt{1+y'^2}dx$. By solving the Euler equation for the functional $y = c_1x + c_2$. Now by considering equation (2.37), we have:

$$F + \varphi' - y')F_{y'}\big|_A = 0 \Longrightarrow \sqrt{1+y'^2} + (1-c_1)\frac{y'}{\sqrt{1+y'^2}} = 0$$

thus $1 + c_1^2 + c_1 - c_1^2 = 0$ and $c_1 = -1$. Also

$$F + (\phi' - y')F_{y'}\big|_B = 0 \Longrightarrow \sqrt{1+y'^2} + (2x-c_1)\frac{y'}{\sqrt{1+y'^2}} = 0$$

therefore $1 + c_1^2 + (2x - c_1)c_1 = 0$, we will have $x_B = \frac{1}{2}$ and $y_B = \frac{1}{4}$.

On the other hand, we have $y = c_1x + c_2$ and by substituting (x_B, y_B) and $c_1 = -1$, we will have $c_2 = \frac{3}{4}$ and $y = -x + \frac{3}{4}$. Now from the intersection of two lines $y = x - 2$ and $y = -x + \frac{3}{4}$, we will have $x_A = \frac{11}{8}$ and $y_A = \frac{-5}{8}$. Thus $l_{AB} = \frac{7}{8}\sqrt{2}$.

Theorem 2.6.5. *Let*

$$\varphi_i(x, y_1, y_2, \cdots, y_n) = 0 \text{ and } i = 1, 2, \cdots, m; \quad m < n$$

Functions $y_1, y_2, \cdots, y_n$ *make the extermum functional of*

$$I = \int_{x_0}^{x_1} F(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n') = dx$$

in Euler equation. By using I^* *as* $I^* = \int_{x_0}^{x_1}(F + \sum_{i=1}^m \lambda_i(x)\varphi_i)dx = \int_{x_0}^{x_1} F^*dx$ *in which* $\lambda_i(x)$ *is called the Lagrange multipliers, can be satisfied under a suitable choice of* $\lambda_i(x)$*. In this case functions* $\lambda_i(x)$ *and* $y_i(x)$ *from Euler equation can be determined as followed:*

$$F^*_{y_j} - \frac{d}{dx}F^*_{y_j'} = 0, \ j = 1, 2, \cdots, n,$$
$$\varphi_i = 0, \ i = 1, 2, \cdots, m.$$

2.7 Variational Iteration Method

2.7.1 History

The variational iteration method is explained by He in 1999. This method has been by some of the engineers and mathematicians for solving the functional equations. For example the method has been used to solve Helmholtz equation [10], this technique was used to solve Burgers equation and cupled Burgers equation. Other applications of this method are mentioned in [9, 50, 72].

2.7.2 Main Idea For The Variational Iteration Method

Primarily we consider a definition and a theorem.

Definition 2.7.1. The quantity of variable v is a functional and it is dependent of the function u. If we have a v for any function u from the function $u(x)$, then variations for the function $v[u(x)]$ can be defined as:

$$\delta v[u(x)] = \frac{\partial}{\partial \alpha} v[u(x) + \alpha \delta u]\big|_{\alpha=0}$$

For more detail please refer to reference [42].

Theorem 2.7.2. *If a functional $v[u(x)]$ at the point $u = u_0(x)$ have a maximum or minimum (u_0 is an internal point defined in domain of functional v), then at $u = u_0(x)$ we have:*

$$\delta v = 0$$

Referring to (2.6.1) *for the prove.*

In variational iteration method a functional equation will be transform to a recursive sequence from functions. The limit for this can be taken a solution for the functional equation.

For explaining this method we consider a functional equation as below:

$$A[u(x,t)] - g(x,t) = 0, \tag{2.38}$$

Operator A will be considered and divided into two, L linear operator and N nonlinear operator.

According to the variational iteration method a Correction functional will considered for this equation.

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)[Lu_n(x,\xi) + N\widetilde{u}_n(x,t) - g(x,\xi)]d\xi, \tag{2.39}$$

Where $\widetilde{u}_n$ is a restriced variable and $\delta\widetilde{u}_n = 0$ and λ is the Lagrange multiplier[50]. For finding the best quantity for λ, we considered the correction functional as followed:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi)[Lu_n(x,\xi) + N\widetilde{u}_n(x,t) - g(x,\xi)]d\xi = 0, \tag{2.40}$$

In equation (2.40), we can calculate the stationary condition and find the optimum for λ.

In effect the solution for functional equation is a constant point for the recursive equation, starting with a initial and suitable amount for u_0. Therefore, λ can be identified optimally via variational theory, then by choosing a suitable quantity for u_0, we find the exact solution for the equation from the limit below:

$$u = \lim_{n \longrightarrow \infty} u_n.$$

Example 2.7.3. The solve Riccati equation using the Variational iteration method [9]. The correction functional will be considered for the equation.

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda\{\frac{d}{dx}y_n(x) - 2y_n(x) - 1 + \widetilde{y}_n^2(x)\}dx$$

By considering $\delta\widetilde{y}_n = 0$, Its stationary conditions can be obtained as follows:

$$\begin{aligned} &\delta u_n : \lambda'(x) + 2\lambda(x) = 0, \\ &\delta u_n' : 1 + \lambda(x)\big|_{x=t} = 0. \end{aligned} \tag{2.36}$$

By solving initial equation (2.36) we have:

$$\lambda = -e^{2(t-x)}.$$

There the Variational iteration formula will be as followed:

$$y_{n+1}(t) = y_n(t) - \int_0^t e^{2(t-x)}\{\frac{d}{dx}y_n(x) - 2y_n(x) - 1 + y_n^2(x)\}dx. \tag{2.37}$$

By substituting $y_0(t) = t$ in (2.37), we have:

$$y_1(t) = t + \frac{-1 + e^{2t} - 2t + 2t^2}{4}$$

$$y_2(t) = -\frac{5}{32} + \frac{3e^2t}{16} - \frac{e^4t}{32} + \frac{5t}{8} + \frac{e^{2t}t}{8} + \frac{3t^2}{4} - \frac{e^{2t}t^2}{8} + \frac{t^3}{2} - \frac{e^{2t}t^3}{12} + \frac{t^4}{8},$$

$$\begin{aligned} y_3(t) = &\frac{17425}{2048} - \frac{208505}{24576}e^{2t} - \frac{27e^{4t}}{1024} + \frac{19e^{6t}}{8192} - \frac{e^{8t}}{6144} + \frac{2303}{128}t \\ &+ \frac{15e^{2t}t}{256} + \frac{e^{4t}t}{128} + \frac{5e^{6t}}{2048} + \frac{4631t^2}{256} - \frac{25e^{2t}t^2}{256} - \frac{3e^{4t}t^2}{256} \\ &- \frac{e^{6t}t^2}{1024} + \frac{1537t^2}{128} - \frac{61e^{2t}t^3}{384} + \frac{13e^{4t}t^3}{384} - \frac{e^{6t}t^3}{768} + \frac{1487t^4}{256} \\ &- \frac{47e^{2t}t^4}{768} + \frac{5e^{4t}t^4}{768} + \frac{76t^5}{32} + \frac{23e^{2t}t^5}{960} + \frac{35t^6}{64} + \frac{7e^{2t}t^6}{192} \\ &- \frac{e^{4t}t^6}{288} + \frac{3t^7}{32} + \frac{11e^{2t}t^7}{762} + \frac{t^8}{128} + \frac{e^{2t}t^8}{384}, \end{aligned}$$

Using Taylor expansion of $y_3(t)$ at $t = 0$, we have:

$$y(t) \simeq y_3(t) = t + t^2 + \frac{t^3}{3} - \frac{t^4}{3} - \frac{7t^6}{45} + \frac{53t^7}{315} + \cdots .$$

For calculating $y_4(t), y_5(t), \cdots$, a CPU need to spend lots of time an to solve this problem, we can use the Adomian's polynomial [9].

Chapter 3

Fuzzy Differential Equations

The topic of fuzzy differential equations which attracted growing interest for some time,in particular in relation to fuzzy control, have been rapidly developed in recent years. In this chapter, we apply homotopy analysis method for solving these groups of equations. This chapter contains three sections. In the first section we introduce fuzzy first order differential equation and Buckly-Feuring solution, in the second section, we will present fuzzy initial value problem and in the last section, we present numerical results by HAM.

3.1 Fuzzy Initial Value Problem

We consider the first-order ordinary differential equation

$$\frac{dy}{dt} = f(t, y, k), \quad y(0) = c \tag{3.1}$$

where $k = (k_1, k_2, ..., k_n)$ is a vector of constants, and t is in some interval (closed and bounded) I which contains zero, we assume that f satisfies conditions [35,68] so that Eq. (3.1) has an unique solution $y = g(t, k, c)$, for $t \in I, k \in K \subset R^n, c \in C \subset R$. Let $y \in I_1$, be an interval for the y-values and set $W = I \times I_1$, a region in $\mathbb{R}^2$. Well-known sufficient conditions for Eq. (1.1) to have a unique solution are given any $k \in K$ and $c \in C$ because $(0, c)$ is in W, f is continuous in W (k is held fixed) and $\frac{\partial f}{\partial y}$ is continuous in W. If these conditions are satisfied,then there is a unique solution $y = g(t, k, c)$, for $t \in I$. Let $K = (K_1, ..., K_n)$ be a vector of triangular fuzzy numbers and let C be another triangular number. Substitute K for k and C for c in Eq. (3.1) and we get

$$\frac{dY}{dt} = f(t, Y, K),\ Y(0) = C \tag{3.2}$$

Assuming we have adopted some definition for the derivative of the unknown fuzzy function $Y(t)$. We want to solve Eq. (3.2) for $Y(t)$ and have $Y(t)$ a fuzzy number for each t in I.

3.1.1 Classical Solution

If we take α-cut from the two parties of Eq. (3.2), then we obtain two crisp differential equations and solve, we are attempting to get the classical solution $Y_c(t)$.

Suppose $Y_c(t) = [y_1(t,\alpha), y_2(t,\alpha)]$, for $\alpha \in [0,1]$, we need to get $Y_c(t)$ a solution is $\frac{d}{d\alpha}[y_1(t,\alpha)] > 0$, $\frac{d}{d\alpha}[y_2(t,\alpha)] < 0$ and $y_1(t,1) = y_2(t,1)$, for all $t \in I$.

Example 3.1.1. Consider the differential equation

$$\begin{cases} Y' + Y = K, \\ Y(0) = C. \end{cases}$$

Let $K = [-1+\alpha, 1-\alpha]$ and $C = [-\frac{1}{2} + 2\alpha, \frac{7}{2} - 2\alpha]$ then

$$\begin{cases} y_1'(t,\alpha) + y_1(t,\alpha) = -1+\alpha, \\ y_2'(t,\alpha) + y_2(t,\alpha) = 1-\alpha, \end{cases} \tag{3.3}$$

The solution are

$$\begin{cases} y_1(t,\alpha) = (-1+\alpha) + (\frac{1}{2}+\alpha)e^{-t}, \\ y_2(t,\alpha) = (1-\alpha) + (\frac{5}{2}-\alpha)e^{-t}, \end{cases} \tag{3.4}$$

for all $t \in I$ we get

$$\begin{cases} \frac{\partial}{\partial\alpha}[y_1(t,\alpha)] = 1 + e^{-t} > 0, \\ \frac{\partial}{\partial\alpha}[y_2(t,\alpha)] = -1 - e^{-t} < 0, \\ y_1(t,1) = y_2(t,1). \end{cases}$$

Thus the classical solution is

$$Y_C(t) = [y_1(t,\alpha), y_2(t,\alpha)].$$

Example 3.1.2. Consider the differential equation

$$\begin{cases} Y' - 2Y = K, \\ Y(0) = C. \end{cases}$$

Let $K = [\alpha, 2-\alpha]$ and $C = [-1+\alpha, 1-\alpha]$, then

$$\begin{cases} y_1'(t,\alpha) - 2y_2(t,\alpha) = \alpha, \\ y_2'(t,\alpha) - 2y_1(t,\alpha) = 2-\alpha, \end{cases} \tag{3.5}$$

The solution are

$$\begin{cases} y_1(t,\alpha) = (\frac{1}{2}\alpha - \frac{1}{2})e^{-2t} + \frac{1}{2}e^{2t} - 1 + \frac{1}{2}\alpha, \\ y_2(t,\alpha) = (\frac{1}{2} - \frac{1}{2}\alpha)e^{-2t} + \frac{1}{2}e^{2t} - \frac{1}{2}\alpha, \end{cases} \tag{3.6}$$

for all $t \in I$ we get

$$\begin{cases} \frac{\partial}{\partial\alpha}[y_1(t,\alpha)] = \frac{1}{2}e^{-2t} + \frac{1}{2} > 0, \\ \frac{\partial}{\partial\alpha}[y_2(t,\alpha)] = -\frac{1}{2}e^{-2t} - \frac{1}{2} < 0, \\ y_1(t,1) = y_2(t,1). \end{cases}$$

Thus the classical solution is

$$Y_C(t) = [y_1(t,\alpha), y_2(t,\alpha)].$$

Example 3.1.3. Consider the differential equation

$$\begin{cases} Y' + 2Y = K, \\ Y(0) = C. \end{cases}$$

Let $K = [4+\alpha, 7-2\alpha]$ and $C = [1+\frac{\alpha}{3}, 2-\frac{2}{3}\alpha]$, then

$$\begin{cases} y_1'(t,\alpha) + 2y_1(t,\alpha) = 4+\alpha, \\ y_2'(t,\alpha) + 2y_1(t,\alpha) = 7-2\alpha, \end{cases} \tag{3.7}$$

The solution are

$$\begin{cases} y_1(t,\alpha) = (-1 - \frac{\alpha}{6})e^{-2t} + 2 + \frac{\alpha}{2}, \\ y_2(t,\alpha) = (-\frac{3}{2} + \frac{1}{3}\alpha)e^{-2t} + \frac{7}{2} - \alpha, \end{cases} \tag{3.8}$$

for all $t \in I$ we get

$$\begin{cases} \frac{\partial}{\partial\alpha}[y_1(t,\alpha)] = -\frac{1}{6}e^{-2t} + \frac{1}{2}, \\ \frac{\partial}{\partial\alpha}[y_2(t,\alpha)] = \frac{1}{3}e^{-2t} - 1, \end{cases}$$

So, for some t we have $\frac{\partial}{\partial\alpha}[y_1(t,\alpha)] < 0$ and $\frac{\partial}{\partial\alpha}[y_2(t,\alpha)] > 0$. Hence, the classical solution does not exist.

3.1.2 Boxing Model Approach for the Air Quality of a Building

This simple building model considered to be a unique with its homogeneous internal air content plus the sources of air pollution [25]. In this model there are several different air pollution sources with various distribution velocities in the building. At the same time there are possibilities of having other types of air pollution entering into the building. The concentrations of these new pollution can be reduced through diffusion or by the air condition system in the building. To simplify the model, we neglect any artificial air conditioning and purification system. With respect to the above mentioned conditions and hypothesis, equation for mass of the pollution at equilibrium inside the building can be written an followed:

$Velocity\ of\ pollution\ increase\ inside\ the\ box = (Velocity\ of\ pollution\ entering) - (Velocity\ of\ pollution\ out\ gassing)$ - $(Velocity\ of\ pollution\ decomposition\ inside\ the\ box)$

For this we have

$$v\frac{dc}{dt} = s + ivc_a - ivc - kvc,$$

By knowing that

$v =$ Volume of the purified and cleaned air inside the building (m^3/ac),

$i =$ Velocity of the air exchanged (ach),

$s =$ Power of pollution sources (mg/hr),

$c =$ Pollution concentration inside the building (mg/m^3),

$c_a =$ Pollution concentration of surrounding air (mg/m^3),

$k =$ Velocity of decomposition or inactivity $(1/hr)$,

Differential equation can be solved as

$$v\frac{dc}{dt} + ivc = s,$$

if the polluting components have no effects on each other $k = 0$, the concentrations of polluting components are considered to be zero$(c_a = 0)$, and assume that the concentration of the primary pollution is zero.

Velocity of increasing the concentration of polluting materials can be determined by equation below:

$$c(t) = \frac{s}{iv}[1 - e^{-it}],$$

If we could not have determined the exact amounts k, c_a, i, v and s, then differential equation is fuzzy as followed:

$$V\frac{dC}{dt} + CIV - KCV = S + C_a IV,$$

Here for simplifying the model, we assume that the amounts for K, C_a, I and V are determined and only S is remained to be unknown. In special condition when $K = 0$ and $C_a = 0$ then the fuzzy differential equation will be as:

$$vC' + ivC = S,$$

In above equation, the assumption is that, S is a fuzzy parameter number and t is the time and it is crisp. Then the equation can be written as:

$$C' + iC = \frac{S}{v} \Longrightarrow \begin{cases} c_1'(t,\alpha) + ic_1(t,\alpha) = \frac{s_1(\alpha)}{v} \\ c_2'(t,\alpha) + ic_2(t,\alpha) = \frac{s_2(\alpha)}{v}, \end{cases}$$

By solving these two crisp differential equations, the solution will be as followed:

$$\begin{cases} c_1(t,\alpha) = \frac{s_1(\alpha)}{iv}[1 - e^{-it}], \\ c_2(t,\alpha) = \frac{s_2(\alpha)}{iv}[1 - e^{-it}]. \end{cases}$$

In this case, the solution to the equation will be as

$$c(t) = [c_1(t,\alpha), c_2(t,\alpha)].$$

Example 3.1.4. A fuel burning heater was tested under controlled lab conditions. The heater worked for 2 hours in a test chamber by gassing out $46m^3$ of CO with diffusion rate $0.25ach$ and distribution velocity of $585mg/hr$. Determine, the pollution concentration if we assume that the CO concentration in the lab and in surrounding area and negligible?

The assumption $S = (580 + 5\alpha, 590 - 5\alpha)$,
In this case we have:

$$\begin{cases} c_1(t,\alpha) = \frac{580+5\alpha}{(0.25\times 46)}[1 - e^{-2(0.25)}], \\ c_2(t,\alpha) = \frac{590-5\alpha}{(0.25\times 46)}[1 - e^{-2(0.25)}]. \end{cases}$$

$c(\alpha) = [c_1(t,\alpha), c_2(t,\alpha)]$ is illustrated in figure 3.1.

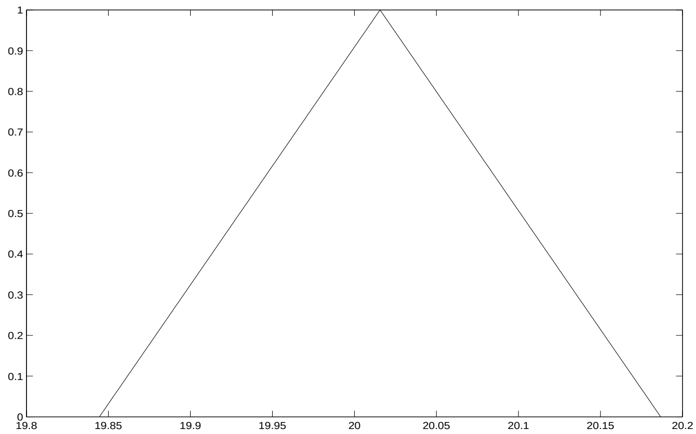

Figure 3.1 :The fuzzy number c.

3.2 Buckley -Feuring Solution

According to Buckley -Feuring's work [26]. Let $K[\alpha] = K_1[\alpha] \times ... \times K_n[\alpha]$ and $\phi(\alpha) = K[\alpha] \times C[\alpha]$, for α. We assume that $\phi(0) \subset K \times C$ so that g is continuous on $I \times \phi(\alpha)$ for all α. We first fuzzify the crisp solution $y = g(t, k, c)$ to obtain $Y(t) = g(t, K, C)$ using the extension principle. Alternatively, we get α-cuts as follows:

$$Y(t)[\alpha] = [y_1(t, \alpha), y_2(t, \alpha)] \tag{3.9}$$

with

$$Y_1(t, \alpha) = \min\{g(t, k, c)|k \in K[\alpha], c \in C[\alpha]\} \tag{3.10}$$

and

$$Y_2(t, \alpha) = \max\{g(t, k, c)|k \in K[\alpha], c \in C[\alpha]\} \tag{3.11}$$

for $t \in I$ and $\alpha \in [0, 1]$. Still another equivalent procedure to determine $Y(t)$ is to first specify, for $0 < \alpha \leq 1$, and $t \in I$

$$\Omega(\alpha) = \{g(t, k, c)|(k, c) \in \phi(\alpha)\} \tag{3.12}$$

and then define the membership function of $\overline{Y}(t)$ as follows:

$$\mu_{Y(t)}(x) = \sup\{\alpha|x \in \Omega(\alpha)\}. \tag{3.13}$$

Theorem 3.2.1. *1) $Y(t)(\alpha) = \Omega(\alpha)$ for all $\alpha \in [0, 1], t \in I$.*

2) $Y(t)$ is a fuzzy number for all $t \in I$.

Assume that $y_i(t,\alpha)$ is differentiable with respect to $t \in I$ for each α in $[0,1]$, $i=1,2$. We write the partial of $y_i(t,\alpha)$ with respect to t as $y_i'(t,\alpha)$, $i=1,2$. Let

$$\Gamma(t,\alpha) = [y_1'(t,\alpha), y_2'(t,\alpha)], \tag{3.14}$$

for $t \in I, \alpha \in [0,1]$.If $\Gamma(\alpha)$ defines the α-cut of a fuzzy number for each $t \in I$ we say that $Y(t)$ is differentiable and write

$$\frac{dY(t)}{dt} = [y_1'(t,\alpha), y_2'(t,\alpha)], \tag{3.15}$$

for $t \in I$, $\alpha \in [0,1]$. Notice that, Eq. (3.15) is just the derivative (with respect to t) of Eq. (3.9). Sufficient conditions for $\Gamma(t,\alpha)$ to define the α-cuts of a fuzzy number are [43,52]

1) $y_1'(t,\alpha)$ and $y_2'(t,\alpha)$ are continuous on $I \times [0,1]$;

2) $y_1'(t,\alpha)$ is increasing function of α for each $t \in I$;

3) $y_2'(t,\alpha)$ is decreasing function of α for each $t \in I$;

4) $y_1'(t,1) \leq y_2'(t,1)$ for each $t \in I$.

Now for $Y((t)$ to be a solution of the fuzzy initial value problem we need that $\frac{dY(t)}{dt}$ exists but also Eq. (1.3) must hold. To check Eq. (3.2) we must first compute $f(t,Y,K)$. α-cuts of $f(t,Y,K)$ can be found as follows:

$$f(t,Y,K) = [f_1(t,\alpha), f_2(t,\alpha)] \tag{3.16}$$

with

$$f_1(t,\alpha) = \min\{f(t,y,k)|y \in Y(t)[\alpha], k \in K[\alpha]\}, \tag{3.17}$$

$$f_2(t,\alpha) = \max\{f(t,y,k)|y \in Y(t)[\alpha], k \in K[\alpha]\}, \tag{3.18}$$

for α in $[0,1]$. We say that Y is a solution to Eq. (3.2) if $\frac{dY(t)}{dt}$ exists and

$$y_1'(t,\alpha) = f_1(t,\alpha), \tag{3.19}$$

$$y_2'(t,\alpha) = f_2(t,\alpha), \tag{3.20}$$

$$y_1'(0,\alpha) = c_1(\alpha), \tag{3.21}$$

$$y_2'(0,\alpha) = c_2(\alpha), \tag{3.22}$$

where $C[\alpha] = [c_1(\alpha), c_2(\alpha)]$.

3.3 Fuzzy Initial Value Problem

In this section, we look at the solutions to the fuzzy initial value problem **(FIVP)** concerning Buckley -Feuring's work [27]. Notice that we used the Sikkala definition of the derivative of a fuzzy function in the first chapter. That is, if $\frac{d}{dt}Y(t)[\alpha]$ exists, then

$$SDY(t)[\alpha] = \frac{d}{dt}Y(t)[\alpha].$$

Also, $SDY(t)$ is a fuzzy number for all $t \in I$. The Buckley -Feuring solution, written BFS for the **FIVP**. Let $BFS = Y(t)$, then $Y(t) = g(t, K, C)$ (Eqs. (3.10) and (3.11)) and $SDY(t)$ exists (Eq. (3.14) defines a fuzzy number for all t) and $SDY(t) = f(t, Y(t), k)$ and $Y(0) = C$.

Theorem 3.3.1. *Assume $SDY(t)$ exists for $t \in I$, If*

$$\frac{\partial f}{\partial y} > 0, \frac{\partial g}{\partial C} > 0 \tag{3.23}$$

and

$$\frac{\partial f}{\partial k_i} \cdot \frac{\partial g}{\partial k_i} > 0 \tag{3.24}$$

$i = 1, \cdots, n$ then $BFS = Y(t)$.

*If Eq. (3.23) or Eq. (3.24) does not hold for some i, then $Y(t)$ does not solve the **FIVP**.*

3.4 Numerical Results

In this section, we assume that $y_i'(t, \alpha)$, $i = 1, 2$, are continuous and let $I = [0, M]$, for some $M > 0$. Also, consider two nonlinear **FIVP**, to solve these problems we use the following strategy :

1) Find $\Phi_n(t, k, c)$. It is an approximation of $y = g(t, k, c)$, the solution of Eq. (3.1), then fuzzify it to $Y(t) = \Phi_n(t, K, C)$ by extension principle;

2) Checking conditions (3.23) and (3.24) for $\Phi_n(t, k, c)$;

3) Is $Y(t)$ a fuzzy number?

4) Fuzzify $f(t, y, c)$ to a fuzzy function $f(t, Y, C)$ by extension principle, where $Y(t) = \Phi_n(t, K, C)$.

Since we approximate $g(t,k,c)$ by $\Phi_n(t,k,c)$, when $y(t) = \Phi_n(t,k,c)$ is extended to fuzzy case $Y(t) = \Phi_n(t,K,C)$, then $Y(t) = \Phi_n(t,K,C)$ usually is not satisfied in Eqs. (3.9) – (3.22). We calculate the distance between $f(t,Y,C)$ and $SDY(t)$ with metric D.

Example 3.4.1. Consider initial value problem with $y(t) = \lambda \tan(\omega t)$ as exact solution in crisp case where $\lambda = \sqrt{k_2/k_1}$ and $\omega = \sqrt{k_1 k_2}$:

$$\begin{cases} y'(t) = k_1 y_1^2(t) + k_2, t \in I, \\ y(0) = 0. \end{cases}$$

where $k_i > 0$ for $i = 1, 2$.

We set the initial guess to be $y_0(t) = 0$ i.e. the initial condition, use the auxiliary linear operator $L = \frac{\partial \varphi}{\partial t}$ and put $H(t) = 1$ to be the auxiliary function. We set

$$N[\varphi(t,k;q)] = \frac{\partial \varphi(t,k;q)}{\partial t} - k_1 \varphi^2(t,k;q) - k_2$$

We have the zeroth-order deformation equation as follows:

$$(1-q)L[\varphi(t,k;q) - y_0(t)] = q\hbar N[\varphi(t,k;q)],$$

Solving the corresponding mth order deformation equations we have:

$$\begin{aligned} y_0(t) &= 0, \\ y_1(t) &= -hk_2 t, \\ y_2(t) &= -hk_2(1+h)t, \\ &\vdots \end{aligned}$$

we assume

$$g(t,k,0) \approx \varphi_3(t,k,0) = y_0(t) + y_1(t) + y_2(t).$$

Calculating $\frac{\partial f}{\partial y}$, $\frac{\partial \varphi_3}{\partial k_1}$, $\frac{\partial \varphi_3}{\partial k_2}$, $\frac{\partial f}{\partial k_1}$ and $\frac{\partial f}{\partial k_2}$, we can see that conditions (3.22) and (3.24) are satisfied for $\hbar < 0$, So we have a BFS. The α-cut corresponding to y are:

$$\begin{aligned} y_1(t,\alpha) &= -2hk_{21}(\alpha)t - h^2 k_{21}(\alpha)t, \\ y_2(t,\alpha) &= -2hk_{22}(\alpha)t - h^2 k_{22}(\alpha)t, \end{aligned}$$

where $K_i[\alpha] = [k_{i1}(\alpha), k_{i2}(\alpha)]$, for $0 \le \alpha \le 1$ and $i = 1, 2$ are α-cut corresponding to K_i, $i = 1, 2$. Then α-cut of $SDY(t)$ for $0 \le \alpha \le 1$ respect t are:

$$\begin{aligned} y_1'(t,\alpha) &= -2hk_{21}(\alpha)t - h^2 k_{21}(\alpha)t, \\ y_2'(t,\alpha) &= -2hk_{22}(\alpha)t - h^2 k_{22}(\alpha)t, \end{aligned}$$

We set $K_1(\alpha) = K_2(\alpha) = [\alpha, 2-\alpha]$ for all $\alpha \in [0,1]$. Because $\frac{\partial k_{i1}}{\partial \alpha} > 0$ and $\frac{\partial k_{i2}(\alpha)}{\partial \alpha} < 0$ for $0 \leq \alpha \leq 1$, $i = 1,2$ are satisfied and also $\frac{\partial^2 y_i}{\partial k_{ij} \partial_t} \geq 0$, $i = 1,2$, $j = 0,1,2$, therefore $y_1'(t,\alpha)$ and $y_2'(t,\alpha)$ for $t \in I$, are α-cuts of a fuzzy number.

If we use *app*15 for all $\alpha \in [0,1]$ by setting $\hbar = -1$, then we can see which is the series that leads to the exact solution (use Taylor's expansion)[14].

Results are presented in Table 3.1 and Figures 3.2 − 3.4.

Table 5.1

Absolute errors of approximation for Example 3.4.1 using HAM by $\hbar = -1$ for $0 \leq \alpha \leq 1$.

	app3	app3	app15
t	ADM	HAM	HAM
0	0	0	0
0.1	0.00007022	0.00000133	$2.1E-15$
0.2	0.00009455	0.00004336	$4.17E-14$
0.3	0.00176888	0.00033624	$2.16E-11$
0.4	0.01528122	0.00145988	$1.67E-9$
0.5	0.09074105	0.00463582	$4.94E-8$

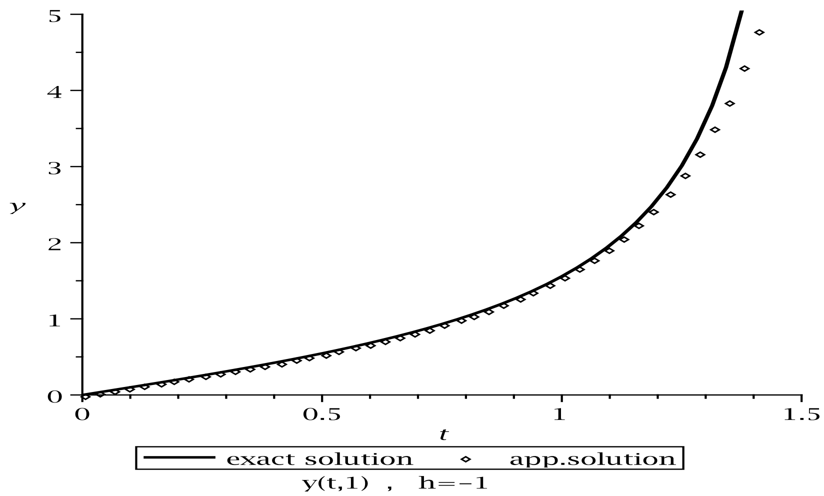

Figure 3.2 : Comparison of the exact solution with $\alpha = 1$ and approximation solution by $\hbar = -1$, $\alpha = 1$.

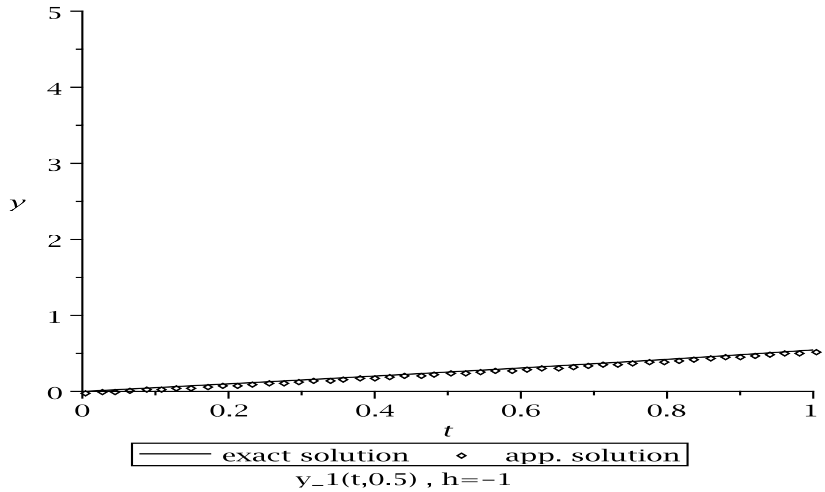

Figure 3.3 : Comparison of the exact solution $y_{1exact}(t, o.5)$ and approximation solution $y_1(t, o.5)$ by $\hbar = -1$.

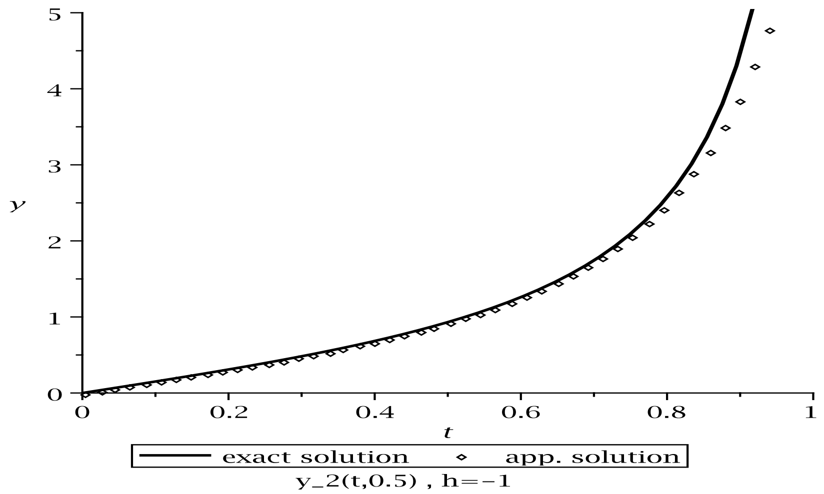

Figure 3.4 : Comparison of the exact solution $y_{2exact}(t, o.5)$ and approximation solution $y_2(t, o.5)$ by $\hbar = -1$.

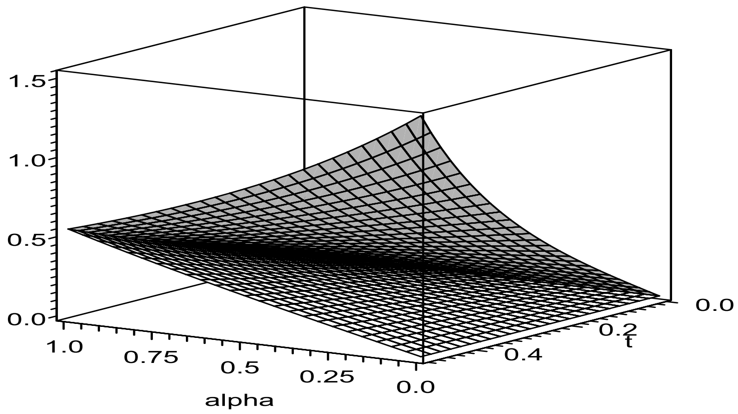

Figure 3.5 : $\varphi_3(t, K)$ by $\hbar = -1$.

Example 3.4.2. Let $c > 0$ and consider the following initial value problem [19]:

$$\begin{cases} y'(t) = \frac{y^3(t)}{1+y^2(t)}, t \in I, \\ y(0) = c. \end{cases}$$

We set the initial guess to be $y_0(t) = 0$ i.e. the initial condition, use the auxiliary linear operator $L = \frac{\partial\varphi}{\partial t}$ and put $H(t) = 1$ to be the auxiliary function. We set

$$N[\varphi(t, k; q)] = \frac{\partial\varphi(t, k; q)}{\partial t} - \frac{\varphi^3(t, k; q)}{1 + \varphi^2(t, k; q)},$$

We have the zeroth-order deformation equation as follows:

$$\begin{aligned} y_0(t) &= c \\ y_1(t) &= -\frac{\hbar c^3}{1+c^2}t \\ y_2(t) &= -\frac{c^3}{1+c^2}(\hbar^2 + \hbar)t + \frac{\hbar^2}{2}[\frac{3c^5}{(1+c^2)^2} - \frac{2c^7}{(1+c^2)^3}]t^2, \\ &\vdots \end{aligned}$$

we assume:

$$g(t, c) \approx \varphi_3(t, c) = y_0(t) + y_1(t) + y_2(t).$$

We can see that conditions (3.22) and (3.24) are satisfied, So we have a BFS. The

α-cut corresponding to y with $c[\alpha] = [c_1(\alpha), c_2(\alpha)]$ ($\alpha \in [0,1]$) are :

$$y_1(t,\alpha) = c_1(\alpha) - \frac{(c_1(\alpha))^3}{1+(c_1(\alpha))^2}(2\hbar+\hbar^2)t+ \\ \frac{\hbar^2}{2}[\frac{3(c_1(\alpha))^5}{(1+(c_1(\alpha))^2)^2} - \frac{2((c_1(\alpha))^7}{(1+(c_1(\alpha))^2)^3}]t^2,$$

$$y_2(t,\alpha) = c_2(\alpha) - \frac{(c_2(\alpha))^3}{1+(c_2(\alpha))^2}(2\hbar+\hbar^2)t+ \\ \frac{\hbar^2}{2}[\frac{3(c_2(\alpha))^5}{(1+(c_2(\alpha))^2)^2} - \frac{2((c_2(\alpha))^7}{(1+(c_2(\alpha))^2)^3}]t^2,$$

Then α-cut of $SDY(t)$ for $0 \le \alpha \le 1$ respect t are:

$$y_1'(t,\alpha) = c_1(\alpha) - \frac{(c_1(\alpha))^3}{1+(c_1(\alpha))^2}(2\hbar+\hbar^2)t+ \\ \frac{\hbar^2}{2}[\frac{3(c_1(\alpha))^5}{(1+(c_1(\alpha))^2)^2} - \frac{2((c_1(\alpha))^7}{(1+(c_1(\alpha))^2)^3}]t^2,$$

$$y_2'(t,\alpha) = c_2(\alpha) - \frac{(c_2(\alpha))^3}{1+(c_2(\alpha))^2}(2\hbar+\hbar^2)t+ \\ \frac{\hbar^2}{2}[\frac{3(c_2(\alpha))^5}{(1+(c_2(\alpha))^2)^2} - \frac{2((c_2(\alpha))^7}{(1+(c_2(\alpha))^2)^3}]t^2.$$

We set $c(\alpha) = [\alpha, 2-\alpha]$ for all $\alpha \in [0,1]$, and we have

$$Y_{HAM}(t) = Y_{ADM}(t).$$

for $\hbar = -1$ [14]. Results are presented in Table 3.2 and Figures 3.6 and 3.7.

Table 5.1

Computed Distance between $D(SD\varphi_i(t,C), f(t,\varphi_i(t,C),C))$ for Example 3.4.2 using HAM for $h = -1.01$, $0 \le \alpha \le 1$ and $i = 3, 5$.

	$i=3$	$i=3$	$i=5$
t	ADM	HAM	HAM
0	0	0	0
0.1	0.00953337	0.00627423	0.00000022
0.2	0.03776878	0.03148632	0.00000696
0.3	0.08419103	0.07527114	0.00005621
0.4	0.14835599	0.13717415	0.00024733
0.5	0.22990967	0.21683211	0.00077531

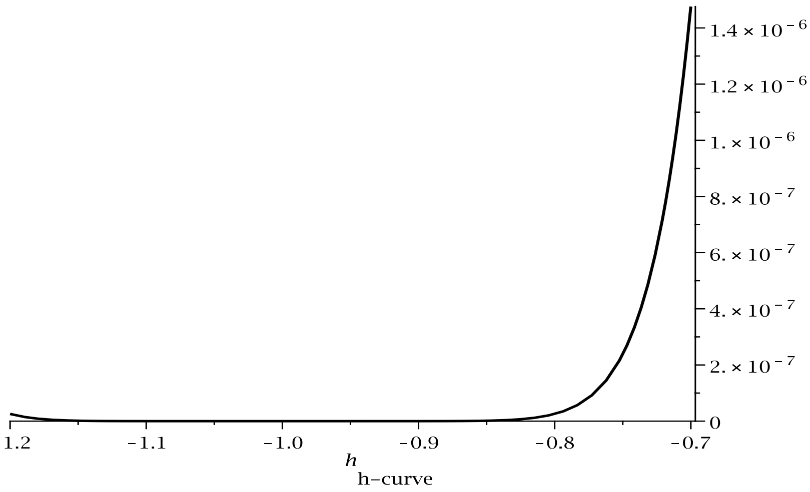

Figure 3.6 : h-curves according $y_t(0)$ to app4 of Example 3.4.2.

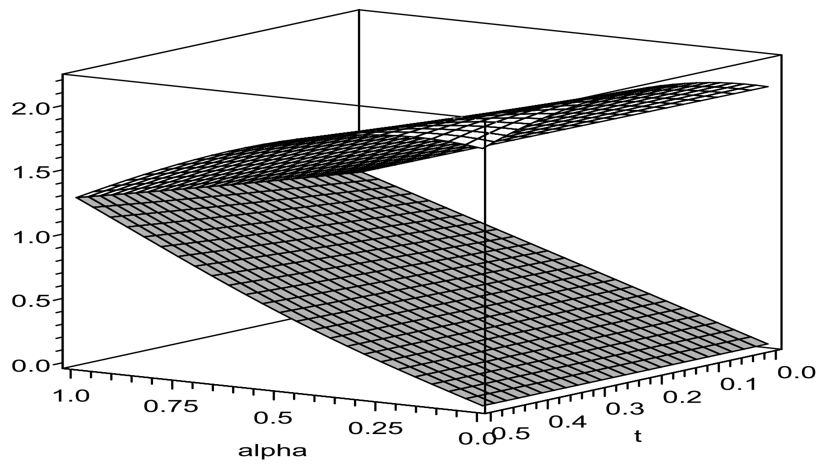

Figure 3.7 : $\varphi_5(t, c)$ by $\hbar = -1.01$.

3.5 Conclusion

In this chapter the numerical approximation of fuzzy first -order initial value problem is considered. We use homotopy analysis method (HAM) to find the approximate solution in crisp case and then extend it to fuzzy case. Using this method, we solve the problems that the classical methods could not be applied in crisp case.

Chapter 4

Diffusion equations

Diffusion equations an important class of parabolic equations, came from a variety of diffusion phenomena which appear widely in nature. They are suggested as mathematical models of physical problems in many fields, such as filtration, phase transition, biochemistry and dynamics of biological groups [1]. Because of these wide variety of applications many scientists, especially applied mathematicians, were attracted to work on new methods and ideas for efficiently solving diffusion equations. In many cases these equations possess singularity. The appearance of singularity makes the study more involved and challenging. Many ideas and methods have been developed to overcome special difficulties caused by singularity, which enrich the theory of PDEs.

4.1 Diffusion Equations

The general form diffusion equations is

$$u_t = (D(u)u_x)_x, \tag{4.1}$$

where D is the diffusion term. The diffusion term can appear in several functional forms, such as power law and exponential forms [40]. We focus on the following types:
1. Slow diffusion process:
If the diffusion term has the form

$$D(u) = u^n \ , \quad n > 0,$$

then we have the slow diffusion equations. For $n = 1$, equation (4.1) arises in isothermal percolation of a perfect gas through a micro-porous medium [40,68]. For $n = 2$, equation (4.1) is used to model a process of melting and evaporation of metals [68,72].

2. Fast diffusion process:
In this family, diffusion term has the general form

$$D(u) = u^n \ , \quad n < 0.$$

For $n = -1$, equation (4.1) appears in the thermal limit approximation of Carlemans model of the Boltzman equation and the expansion into a vacuum of a thermalize electron cloud described by the isothermal Maxwellian distribution [30,72]. For $n = -2$, equation (4.1) is considered as a model of diffusion in high-polymeric systems [40,72].
3. Other diffusion processes:
We study two other cases of diffusion terms as follows:

$$D(u) = \frac{1}{1+u^2} \ , \quad D(u) = \frac{1}{u^2-1}.$$

As well as these general types, which have nonconstant diffusion terms, we consider two linear cases of diffusion equations, one homogenous and the other nonhomogenous.

4.2 Application of HAM

In this section we apply HAM to solve the aforementioned types of diffusion equations. In all cases, we set the initial guess to be $v_0(x,t) = u(x,0)$, i.e. the initial condition, and use the auxiliary linear operator $L = \dfrac{\partial}{\partial t}$ and the auxiliary function $H(x,t) = 1$.we have

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t), \tag{4.2}$$

which must be one of the solutions of the original nonlinear equation. This is actually an advantage of HAM that treats different equations in one common framework [66].

Example 4.2.1. Consider the fast diffusion equation as follows:

$$u_t = (u^{-1}u_x)_x \quad , \qquad u(x,0) = \frac{2c}{(a+x)^2} \ ,$$

where a and $c \neq 0$ are arbitrary constants.
Applying HAM with the considered elements we have the *zero*-order deformation equation

$$(1-q)\phi_t = \hbar q(\phi_t - (\phi_x\phi^{-1})_x).$$

Subsequently solving the m-th order deformation equations one has

$$\begin{aligned}
u_0(x,t) &= \frac{2c}{(a+x)^2},\\
u_1(x,t) &= \frac{-2\hbar t}{(a+x)^2},\\
u_2(x,t) &= \frac{-2\hbar(1+\hbar)t}{(a+x)^2},\\
&\vdots\\
u_n(x,t) &= \frac{-2\hbar(1+\hbar)^{n-1}t}{(a+x)^2}.
\end{aligned}$$

So according to (5), we have the approximate solution

$$\begin{aligned}
u(x,t) &= \frac{2c}{(a+x)^2} + \frac{-2\hbar t}{(a+x)^2} + \frac{-2\hbar(1+\hbar)t}{(a+x)^2} + \cdots\\
&= \frac{2c}{(a+x)^2} + (\frac{-2\hbar t}{(a+x)^2})\sum_{n=1}^{\infty}(1+\hbar)^{n-1},
\end{aligned}$$

provided $|1+\hbar| < 1$ the series converges to

$$\begin{aligned}
&= \frac{2c}{(a+x)^2} + \frac{-2\hbar t}{(a+x)^2}(\frac{1}{1-(1+\hbar)})\\
&= \frac{2(c+t)}{(a+x)^2},
\end{aligned}$$

which is the exact solution.

Example 4.2.2. Consider the fast diffusion equation

$$u_t = (u^{-2}u_x)_x \quad , \qquad u(x,0) = \frac{1}{\sqrt{1+x^2}} \ ,$$

which has the exact solution $u(x,t) = \frac{1}{\sqrt{e^{2t}+x^2}}$. Applying HAM with the considered elements we have the *zero*-order deformation equation

$$(1-q)\phi_t = \hbar q(\phi_t - (\phi_x\phi^{-2})_x).$$

Subsequently solving the m-th order deformation equations one has

$$\begin{aligned}
u_0(x,t) &= \frac{1}{\sqrt{1+x^2}},\\
u_1(x,t) &= \frac{\hbar t}{(1+x^2)\sqrt{1+x^2}},\\
u_2(x,t) &= \frac{\hbar(1+\hbar)t}{(1+x^2)\sqrt{1+x^2}} + \frac{\hbar^2t^2(\frac{1}{2}-x^2)}{(1+x^2)^2\sqrt{1+x^2}},\\
&\vdots
\end{aligned}$$

We use a 9-term approximation and set

$$app8 := u_0 + u_1 + \cdots + u_8.$$

Plotting the corresponding $\hbar$-curves, we have the interval $[-1.8, -0.2]$ as the valid region for $\hbar$, (see Figure 4.1,4.2,4.3 and 4.4)

This means that for these values of $\hbar$ the series converges to the exact solution to the considered example. We test different values of $\hbar$ in the valid region and conclude that the value $\hbar = -.8$ is the one which results in an approximation with the minimum error. Figures 4.5 and 4.6 show the values of the error function $er8 := u_{exact} - app8$ at two different times, i.e. $t = .5$ and $t = 1$. When $t = 1$ the error of our approximation is about 10^{-4}.

According to Figure , the maximum error occurs in the x-interval $(-4, 4)$, so we have tabulated the relative errors for various times on this interval, in Table 4.1.

Table 4.1

Relative errors of $app8$ for fast diffusion equation of Example 4.2.2 using HAM by $\hbar = -.8$.

x	t=0.5	t= 0.7	t=1	t=1.2	t=1.5
0.5	1.050E-6	8.222E-6	5.167E-4	2.678E-3	8.989E-3
1	1.658E-5	9.352E-6	2.712E-3	1.281E-2	6.034E-2
1.5	2.404E-6	2.335E-4	2.256E-3	3.282E-3	3.681E-2
2	2.629E-5	5.411E-6	2.412E-3	1.177E-3	6.651E-2
2.5	2.725E-5	1.478E-4	1.032E-4	4.209E-3	4.573E-2
3	1.686E-5	1.571E-4	1.038E-3	1.431E-3	1.096E-2
4	1.653E-7	6.910E-5	1.007E-3	3.518E-3	1.387E-2

Moreover by setting $\hbar = -1$, in the corresponding approximation (5), one has

$$\begin{aligned} y(x,t) &= (x^2+1)^{-\frac{1}{2}} - (x^2+1)^{-\frac{3}{2}}t + (x^2+1)^{-\frac{5}{2}}(1-2x^2)\tfrac{t^2}{2!} \\ &\quad - (x^2+1)^{-\frac{7}{2}}(1-10x^2+4x^4)\tfrac{t^3}{3!} \\ &\quad + (x^2+1)^{-\frac{9}{2}}(1-36x^2+60x^4-8x^6)\tfrac{t^4}{4!} + \cdots , \end{aligned}$$

which is the series that leads to the exact solution (use Taylor's expansion).

2

2

2

Example 4.2.3. Consider the slow diffusion equation

$$u_t = (uu_x)_x \quad , \quad u(x,0) = x^2 \ .$$

Applying HAM with the considered elements we have the *zero*-order deformation equation

$$(1-q)\phi_t = \hbar q(\phi_t - (\phi_x \phi)_x).$$

Subsequently solving the m-th order deformation equations one has

$$\begin{aligned}
u_0(x,t) &= x^2, \\
u_1(x,t) &= -6\hbar t x^2, \\
u_2(x,t) &= -6\hbar t x^2(-6\hbar t + (1+\hbar)), \\
u_3(x,t) &= -6\hbar t x^2(-6\hbar t + (1+\hbar))^2, \\
&\vdots \\
u_n(x,t) &= -6\hbar t x^2(-6\hbar t + (1+\hbar))^{n-1} \ , \quad n \geq 1 \quad .
\end{aligned}$$

So according to (4.2), we have the approximate solution

$$\begin{aligned}
u(x,t) &= x^2 + \sum_{n=1}^{\infty} -6\hbar t x^2(-6\hbar t + (1+\hbar))^{n-1} \\
&= x^2 - 6\hbar t x^2 \sum_{n=1}^{\infty} (-6\hbar t + (1+\hbar))^{n-1}.
\end{aligned}$$

This series is convergent provided that $|1+\hbar-6\hbar t| < 1$, in this case the series will converge to

$$x^2 - 6\hbar t x^2 \frac{1}{1-(1+\hbar-6\hbar t)} = \frac{x^2}{1-6t},$$

which is the exact solution.
Let us discuss the convergence condition $|1+\hbar-6\hbar t| < 1$, analyzing this condition we have two cases:

$$1) \text{ if } \ \hbar > 0, \ \text{ then the convergence region is: } \ \frac{1}{6} + \frac{1}{3\hbar} > t > \frac{1}{6},$$

$$2) \text{ if } \ \hbar < 0, \ \text{ then the convergence region is: } \ \frac{1}{6} + \frac{1}{3\hbar} < t < \frac{1}{6}.$$

So we conclude that for every $\hbar$ we have a convergence region for the solution series. For example, setting $\hbar = 1$, the convergence region would be $\frac{1}{6} < t < \frac{1}{2}$, which is a small interval. Imposing $\hbar$ to obtain very small values ($\hbar \longrightarrow 0^+ \quad \text{or} \quad \hbar \longrightarrow 0^-$) one can extend the convergence region of the solution series. If $t = \frac{1}{6}$ then

$$\begin{aligned} u_0 &= x^2 \\ u_1 &= -\hbar x^2 \\ u_n &= -\hbar x^2, n = 1, \cdots \end{aligned}$$

thus

$$\begin{aligned} u_0 + \cdots + u_n = x^2 - n\hbar x^2 &\xrightarrow[n \longrightarrow \infty]{x \neq 0} \begin{cases} -\infty & \hbar > 0 \\ +\infty & \hbar < 0 \end{cases} \\ &= 0, \quad x = 0 \end{aligned}$$

this show $\frac{x^2}{1-6t}$ solution of equation for $t = \frac{1}{6}$.

Example 4.2.4. Consider the slow diffusion equation

$$u_t = (u^2 u_x)_x \quad , \quad u(x, 0) = x + 1 \ .$$

Applying HAM with the considered elements we have the *zero*-order deformation equation

$$(1 - q)\phi_t = \hbar q(\phi_t - (\phi^2 \phi_x)_x).$$

Subsequently solving the m-th order deformation equations one has

$$\begin{aligned} u_0(x, t) &= x + 1, \\ u_1(x, t) &= -2\hbar t(x + 1), \\ u_2(x, t) &= (x + 1)(6\hbar^2 t^2 - 2\hbar(1 + \hbar)t), \\ u_3(x, t) &= (x + 1)(-20\hbar^3 t^3 + 12\hbar^2(1 + \hbar)t^2 - 2(1 + \hbar)^2 \hbar t), \\ &\vdots \end{aligned}$$

The series terms don't have a simple general form, reordering in a suitable form, will result in

$$\begin{aligned} u(x, t) &= u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots \\ &= \{-2\hbar(1 + (1 + \hbar) + (1 + \hbar)^2 + (1 + \hbar)^3 + \cdots)\}t(x + 1), \\ &\quad + \{6\hbar^2(1 + 2(1 + \hbar) + 3(1 + \hbar)^2 + 4(1 + \hbar)^3 + \cdots)\}t^2(x + 1), \\ &\quad + \{-20\hbar^3(1 + 3(1 + \hbar) + 6(1 + \hbar)^2 + 10(1 + \hbar)^3 + \cdots)\}t^3(x + 1), \\ &\quad \vdots \end{aligned}$$

Imposing $|1+\hbar| < 1$, the series converges to

$$u(x,t) = (x+1)\Big(\begin{array}{l} -2\hbar\frac{1}{1-(1+\hbar)}t \\ +6\hbar^2\frac{1}{(1-(1+\hbar))^2}t^2 \\ -20\hbar^3\frac{1}{(1-(1+\hbar))^3}t^3 \\ \vdots \\ \quad\Big), \end{array}$$

leading to

$$u(x,t) = (x+1)(1+2t+6t^2+20t^3+70t^4+\cdots),$$

which is the Taylor expansion of the exact solution $u(x,t) = \frac{x+1}{\sqrt{1-4t}}$. So as far as we keep the time in a valid region we would gain the exact solution by HAM.

Example 4.2.5. Consider the diffusion equation

$$u_t = (\frac{1}{1+u^2}u_x)_x \quad , \quad u(x,0) = \tan(x) \ .$$

Applying HAM with considered elements we have the *zero*-order deformation equation

$$(1-q)\phi_t = \hbar q(\phi_t - (\frac{1}{1+\phi^2}\phi_x)_x).$$

Subsequently solving the m-th order deformation equations one has

$$\begin{aligned} u_0(x,t) &= \tan(x), \\ u_1(x,t) &= 0, \\ u_2(x,t) &= 0, \\ u_3(x,t) &= 0, \\ &\vdots \end{aligned}$$

according to (4.2), the solution is $u(x,t) = \tan(x)$, which is the exact solution.

Example 4.2.6. Consider the homogeneous linear diffusion equation

$$u_t = u_{xx} - u \quad , \quad 0 < x < \pi \quad , t > 0,$$

with the initial condition $u(x,0) = \sin(x)$.
Applying HAM with considered elements we have the *zero*-order deformation equation

$$(1-q)\phi_t = \hbar q(\phi_t - \phi_{xx} + \phi).$$

Subsequently solving the m-th order deformation equations one has

$$\begin{aligned}
u_0(x,t) &= \sin(x),\\
u_1(x,t) &= \sin(x)(2\hbar t),\\
u_2(x,t) &= \sin(x)(2\hbar^2 t^2 + 2\hbar(1+\hbar)t),\\
u_3(x,t) &= \sin(x)(\tfrac{4}{3}\hbar^3 t^3 + 4\hbar^2(1+\hbar)t^2 + 2\hbar(1+\hbar)^2 t),\\
&\vdots
\end{aligned}$$

Again, like Example 4.3.4, we don't have a general form for the series terms, but reordering in a suitable form, will result in

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots\\
&= \sin(x)\\
&\quad + \{(1 + (1+\hbar) + (1+\hbar)^2 + (1+\hbar)^3 + \cdots)\}2\hbar t\sin(x),\\
&\quad + \{(1 + 2(1+\hbar) + 3(1+\hbar)^2 + 4(1+\hbar)^3 + \cdots)\}\hbar^2\frac{(2t)^2}{2!}\sin(x),\\
&\quad + \{(1 + 3(1+\hbar) + 6(1+\hbar)^2 + 10(1+\hbar)^3 + \cdots)\}\hbar^3\frac{(2t)^3}{3!}\sin(x),\\
&\vdots
\end{aligned}$$

Imposing $|1+\hbar| < 1$, the series converges to

$$u(x,t) = \sin(x)(1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots),$$

leading to

$$u(x,t) = \sin(x)e^{-2t},$$

which is the exact solution.

Example 4.2.7. Consider the inhomogeneous linear diffusion equation

$$u_t = u_{xx} + \cos(x) \quad , \quad 0 < x < \pi \quad , t > 0,$$

with the initial condition $u(x,0) = 0$.
Applying HAM with considered elements we have the *zero*-order deformation equation

$$(1-q)\phi_t = \hbar q(\phi_t - \phi_{xx} - \cos(x)).$$

Subsequently solving the m-th order deformation equations one has

$$\begin{aligned}
u_0(x,t) &= 0,\\
u_1(x,t) &= \cos(x)(-\hbar t),\\
u_2(x,t) &= \cos(x)(\tfrac{-1}{2}\hbar^2 t^2 - \hbar(1+\hbar)t),\\
u_3(x,t) &= \cos(x)(\tfrac{-1}{6}\hbar^3 t^3 - \hbar^2(1+\hbar)t^2 - \hbar(1+\hbar)^2 t),\\
&\vdots
\end{aligned}$$

Again, we don't have a general form for series terms, just like Examples 4.3.4 and 4.3.6, but collecting according to (4.2), then reordering in a suitable form, will result in

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots\\
&= \{(1 + (1+\hbar) + (1+\hbar)^2 + (1+\hbar)^3 + \cdots)\}(-\hbar t)\cos(x),\\
&\quad + \{(1 + 2(1+\hbar) + 3(1+\hbar)^2 + 4(1+\hbar)^3 + \cdots)\}\hbar^2 \frac{-t^2}{2!}\cos(x),\\
&\quad + \{(1 + 3(1+\hbar) + 6(1+\hbar)^2 + 10(1+\hbar)^3 + \cdots)\}\hbar^3 \frac{-t^3}{3!}\cos(x),\\
&\vdots
\end{aligned}$$

Imposing $|1+\hbar| < 1$, the series converges to

$$u(x,t) = \cos(x)(t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \cdots),$$

leading to

$$u(x,t) = \cos(x)(1 - e^{-t}),$$

which is the exact solution.

4.3 Conclusion

In this chapter, we apply the method to different types of diffusion equations. Although in most examples the exact solution (in closed form) is obtained, there are ones which have a solution series where we get the approximate solution by truncating the series. The presented examples are good ones which illustrate different procedures for finding the convergence region.

Chapter 5

Wave equations

In this chapter, by using HAM to solve equal width wave (EW) and modified equal width wave (MEW) equations, we have made a new contribution to this field of research. Our goal is to emphasize on two points: one is the efficiency of HAM in handling these important family of equations and its superiority over other analytic methods like HPM, VIM and ADM. Other point is that although the considered two equations have different nonlinear terms, we have used the same auxiliary elements to solve them.

This chapter contains second sections. In the first section we introduce EW, in the second section, we will present MEW. We solve all equations by HAM while comparing results in HPM, ADM and VIM.

5.1 Equal Width Wave Equation

The equal-width wave (EW) equation plays a major role in the study of nonlinear dispersive waves since it describes a broad class of physical phenomena such as shallow water waves and ion acoustic plasma waves. The EW equation, derived for long waves propagating in the positive x-direction has the form

$$u_t + uu_x - u_{xxt} = 0, \tag{5.1}$$

with the initial condition

$$u(x, 0) = 3 \operatorname{sech}^2(\frac{x - 15}{2}). \tag{5.2}$$

In the fluid problem u is related to the vertical displacement of the water surface, while in the plasma application u is the negative of the electrostatic potential.

A numerical simulation and explicit solution of the EW equation were obtained by Raslan [68], he used a combination of collocation method using quadratic B-splines and Runge-Kutta method. Dogan applied Galerkin method to this equation [39]. Recently Yusufoglu and Bekir solved this equation using VIM and ADM [75].

we set the initial guess to be $v_0(x,t) = u(x,0)$ i.e. the initial condition, use the auxiliary linear operator $L = \frac{\partial}{\partial t}$ and put $H = 1$ to be the auxiliary function. We have the zeroth-order deformation equation as follows:

$$(1-q)(\phi_t - v_{0t}) = q\hbar(\phi_t + \phi\phi_x - \phi_{xxt}). \tag{5.1.1}$$

Solving the corresponding mth order deformation equations we have

$$\begin{aligned}
u_0(x,t) = \;& 3\mathrm{sech}^2(\frac{x-15}{2}),\\
u_1(x,t) = \;& -9\mathrm{sech}^4(\frac{x-15}{2})\tanh(\frac{x-15}{2})\hbar t,\\
u_2(x,t) = \;& \{\frac{-189}{4}\mathrm{sech}^8(\frac{x-15}{2}) + \frac{81}{2}\mathrm{sech}^6(\frac{x-15}{2})\}\hbar^2 t^2\\
& +\{9(3\hbar-1)\mathrm{sech}^4(\frac{x-15}{2})\tanh(\frac{x-15}{2})\\
& -\frac{135}{2}\mathrm{sech}^6(\frac{x-15}{2})\tanh(\frac{x-15}{2})\}\hbar t,\\
u_3(x,t) = \;& \{81\mathrm{sech}^6(\frac{x-15}{2}) - \frac{189}{2}\mathrm{sech}^8(\frac{x-15}{2})\\
& +\frac{135}{4}\mathrm{sech}^{10}(\frac{x-15}{2})\tanh(\frac{x-15}{2})\\
& -24\mathrm{sech}^8(\frac{x-15}{2})\tanh(\frac{x-15}{2})\}\hbar^2 t^2\\
& +\{-\frac{99}{2}\hbar\mathrm{sech}^6(\frac{x-15}{2}) + \frac{747}{4}\hbar\mathrm{sech}^8(\frac{x-15}{2})\\
& -\frac{1161}{4}\hbar\mathrm{sech}^{10}(\frac{x-15}{2}) - 9\mathrm{sech}^4(\frac{x-15}{2})\tanh(\frac{x-15}{2})\}\hbar t
\end{aligned}$$

$$+\{54\text{sech}^4(\frac{x-15}{2})\tanh(\frac{x-15}{2}) - 9\hbar\text{sech}^4(\frac{x-15}{2})\tanh(\frac{x-15}{2})$$

$$-15\text{sech}^6(\frac{x-15}{2})\tanh(\frac{x-15}{2}) - \frac{165}{2}\hbar\text{sech}^6(\frac{x-15}{2})\tanh(\frac{x-15}{2})$$

$$-105\hbar\text{sech}^8(\frac{x-15}{2})\tanh(\frac{x-15}{2})\}\hbar$$

$$\vdots$$

If we set $\hbar = -1$ in these terms we have exactly the terms obtained by the ADM [75], so we see that the ADM is only a special case of HAM. We use these 4 terms to construct the approximate solution

$$app_4(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t). \tag{5.1.2}$$

We plot the $\hbar$-curves corresponding to $u_t(0,0)$, $u_{tt}(0,0)$ and $u_{ttt}(0,0)$, we have the valid region for $\hbar$ as $R_\hbar = [0.1, 0.5]$ (see figures 5.1, 5.2 and 5.3). Testing different values of $\hbar$ in this valid region, $R_\hbar$, we conclude that the value $\hbar = 0.15$ has the minimum error. We have tabulated the absolute errors of HAM approximation for $\hbar = 0.15$ in Table 5.1. It is seen that, although we have used only 4 terms in constructing the approximate solution, it is very close to the exact solution in the time interval discussed in references, see [75].

Table 5.1

Absolute errors of approximation for EW equation using HAM by $\hbar = 0.15$

t	x=0	x= 5	x=10	x=15	x=20	x=25
0.001	3.66E-9	5.44E-7	8.04E-5	5.52E-8	8.05E-5	5.45E-7
0.002	7.33E-9	1.08E-6	1.60E-4	2.20E-7	1.61E-4	1.09E-6
0.003	1.09E-8	1.63E-6	2.41E-4	4.96E-7	2.41E-4	1.63E-6
0.004	1.46E-8	2.17E-6	3.21E-4	8.83E-7	3.22E-4	2.18E-6
0.01	3.65E-8	5.42E-6	8.01E-4	5.51E-6	8.08E-4	5.47E-6

For EW equation, there are three conservation laws, corresponding to conservation of mass, momentum and energy; they are [76]

$$I_1 = \int_a^b u dx,$$

$$I_2 = \int_a^b (u^2 + u_x^2) dx,$$

$$I_3 = \int_a^b u^3 dx.$$

For computational reasons we chose the interval to be $[0, 80]$, as was chosen in [76]. The efficiency of HAM and its superiority in comparison with VIM and HPM can easily be checked in Table 5.2, where we have computed the values of I_1, I_2 and I_3 for different values of t, see [75].

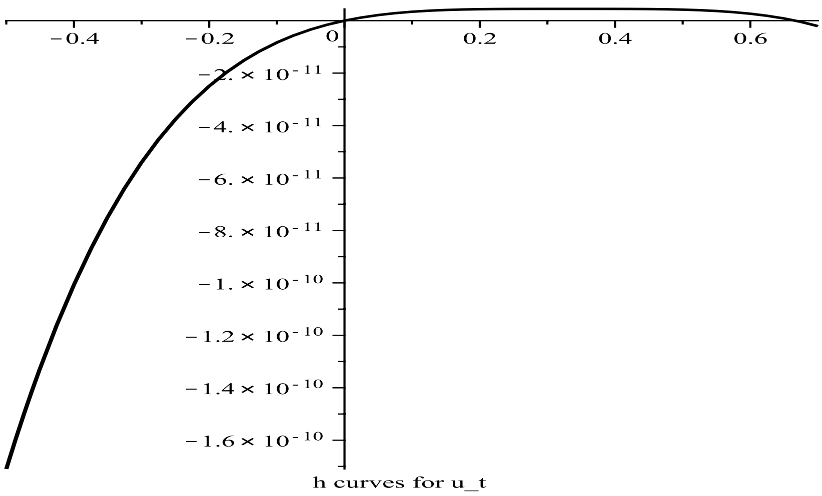

Figure 5.1 :$\hbar$-curves according $u_t(0,0)$ to app4 of EW.

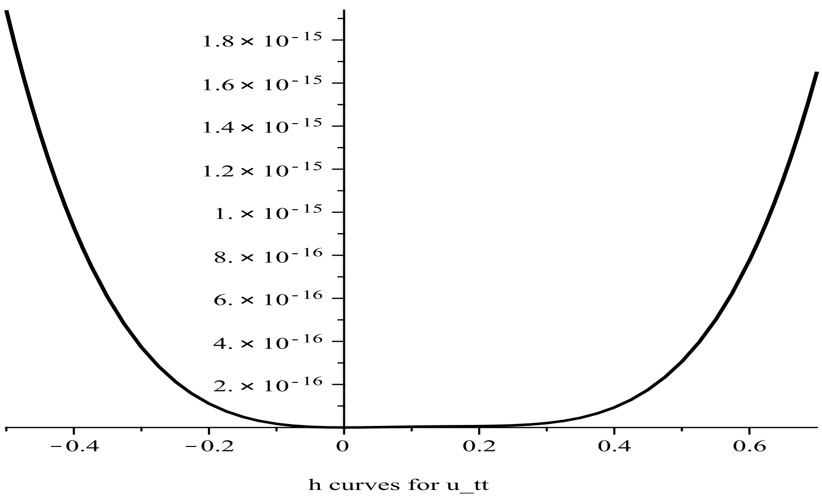

Figure 5.2 :$\hbar$-curves according $u_{tt}(0,0)$ to app4 of EW.

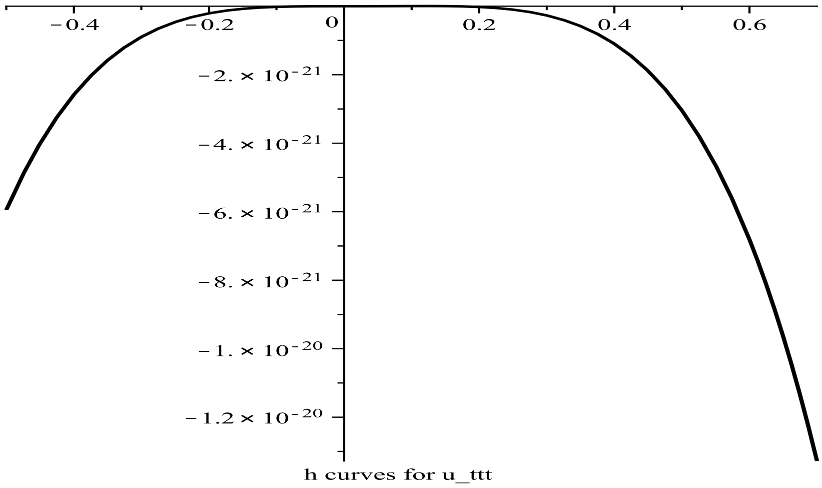

Figure 5.3 :$\hbar$-curves according $u_{ttt}(0,0)$ to app4 of EW.

Table 5.2

Computed quantities I_1,I_2 and I_3 for EW by HAM, HPM, ADM and VIM

t	I_1exact	I_1 HAM	I_1 HPM	I_1 ADM	I_1 VIM
0.001	11.99999633	11.99999633	11.99999633	11.99999637	11.99999636
0.002	11.99999633	11.99999633	11.99999633	11.99999637	11.99999636
0.003	11.99999634	11.99999633	11.99999633	11.99999636	11.99999636
0.004	11.99999634	11.99999633	11.99999633	11.99999636	11.99999636
0.005	11.99999634	11.99999633	11.99999633	11.99999636	11.99999636
0.01	11.99999637	11.99999633	11.99999633	11.99999636	11.99999636
t	I_2exact	I_2 HAM	I_2 HPM	I_2 ADM	I_2 VIM
0.001	28.80000000	28.80001839	28.80913120	28.80913169	28.80014584
0.002	28.80000000	28.80007357	28.83652503	28.83652549	28.80058191
0.003	28.79999998	28.80016554	28.88218190	28.88218237	28.80130870
0.004	28.80000001	28.80029429	28.94610254	28.94610301	28.80232622
0.005	28.80000000	28.80045984	29.02828799	29.02828846	28.80363453
0.01	28.79999999	28.80183935	29.71324466	29.71324514	28.81453942
t	I_3exact	I_3 HAM	I_3 HPM	I_3 ADM	I_3 VIM
0.001	57.60000001	57.60004061	57.60863394	57.60863445	57.60009994
0.002	57.60000001	57.60016243	57.63453609	57.63453659	57.60039822
0.003	57.60000000	57.60036546	57.67770728	57.67770778	57.60089544
0.004	57.60000004	57.60064971	57.73814894	57.73814947	57.60159154
0.005	57.60000004	57.60101517	57.81586322	57.81586374	57.60248656
0.01	57.60000001	57.60406070	58.46363371	58.46363419	57.60994790

5.2 Modified Equal-Width Wave Equation

The modified equal-width wave (MEW) equation is formulated as follows:

$$u_t + \varepsilon u^2 u_x - \mu u_{xxt} = 0. \tag{5.3}$$

This equation has a solitary wave solution of the form

$$u(x,t) = A\ \mathrm{sech}(\frac{1}{\sqrt{\mu}}(x - ct - x_0)), \tag{5.4}$$

where $A = \sqrt{\dfrac{6c}{\varepsilon}}$. Authors have used various kinds of numerical methods to solve equation MEW. Zaki [76] used a quintic B-spline collocation method to investigate the motion of a single solitary wave, interaction of two solitary waves and birth of solitons for the MEW equation. Hamdi et al. [46] derived exact solitary wave solutions of the MEW. Evans and Raslan [43] solved the MEW equation by a collocation finite element method using quadratic B-splines to obtain the numerical solutions of the single solitary wave, solitary waves interaction and birth of solitons. Also, a linearized numerical scheme based on finite difference method has been used by Esen and Kutlay [42]. Wazwaz [73] investigated the MEW equation and two of its variants by the tanh and the sine-cosine methods.

Considering equation (5.3), we study the case where $\mu = 1$, $\varepsilon = 3$ and $A = 0.25$. Other cases can be treated in a similar way. The initial condition is

$$u(x,0) = \frac{1}{4}\ \mathrm{sech}(x - 30). \tag{5.5}$$

The *zeroth-order deformation equation* is constructed as follows

$$(1-q)(\phi_t - v_{0t}) = q\hbar(\phi_t + 3\phi^2\phi_x - \phi_{xxt}). \tag{5.2.1}$$

Solving the corresponding mth order deformation equation we have

$$u_0(x,t) = \frac{1}{4}\mathrm{sech}(x-30),$$

$$u_1(x,t) = -\frac{3}{64}\mathrm{sech}^3(x-30)\tanh(x-30)\hbar t,$$

$$u_2(x,t) = \{\frac{45}{2048}\text{sech}^5(x-30) - \frac{27}{1024}\text{sech}^7(x-30)\}\hbar^2t^2$$
$$+\{-\frac{3}{64}\text{sech}^3(x-30)\tanh(x-30) + \frac{3}{8}\hbar\text{sech}^3(x-30)\tanh(x-30)$$
$$-\frac{15}{16}\hbar\text{sech}^5(x-30)\tanh(x-30)\}\hbar t,$$
$$\vdots$$

We use these five terms to construct the approximate solution

$$app_3(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t). \tag{5.2.2}$$

The maximum error occurs on point $x = 30$, so we search for a value of $\hbar$ which reduces the error at this point (which will consequently reduce the error in other points). Searching for a good value of $\hbar$, from $\hbar$-curves, it is seen that $\hbar = 0.1$ is the most suitable one. If one chooses $\hbar = -1$ will have the HPM results, which is not as good as our choice at the point $x = 30$. Choosing $\hbar = 0.1$ we have tabulated the relative errors of HAM approximation (15) in Table 5.3 The points in table are chosen according to [40].

Table 3

Relative errors of approximation for MEW equation using HAM by $\hbar = 0.1$

t	x=20	x=25	x= 30	x=35
0.01	1.25E-3	1.25E-3	2.58E-7	1.25E-3
0.05	6.27E-3	6.27E-3	6.45E-6	6.23E-3
0.1	1.26E-2	1.26E-2	2.58E-5	1.24E-2
0.5	6.45E-2	6.45E-2	6.43E-4	6.06E-2

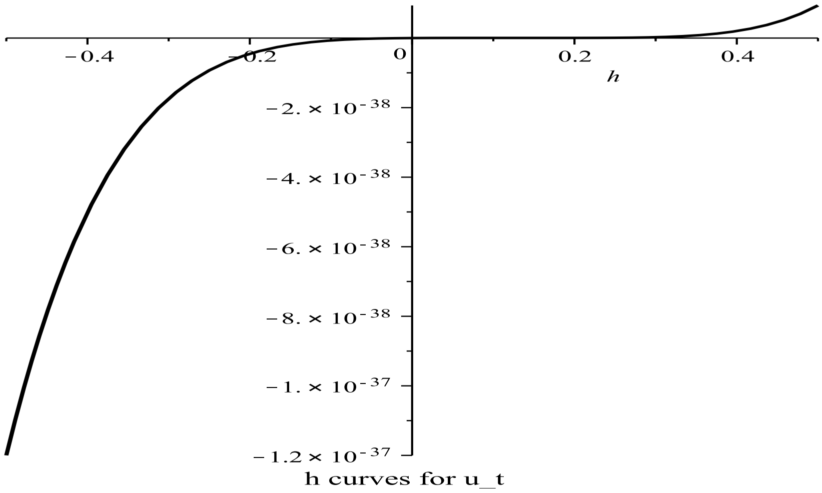

Figure 5.4: $\hbar$-curves according $u_t(0,0)$ to app5 of MEW.

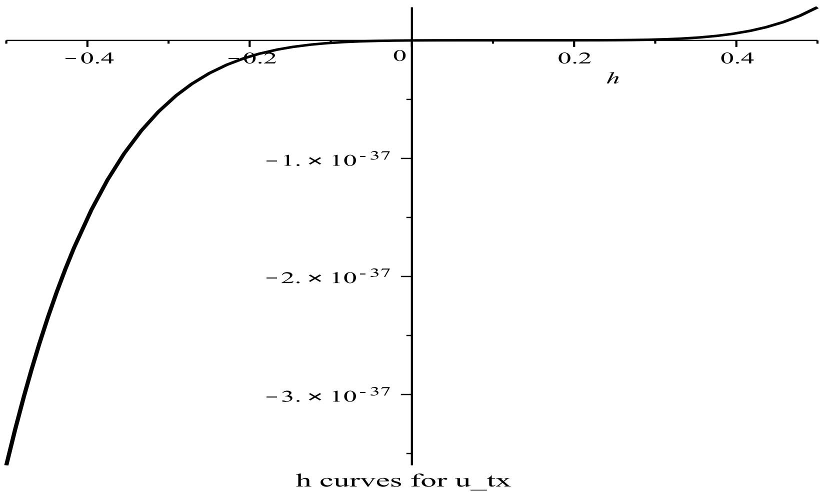

Figure 5.5:$\hbar$-curves according $u_{tx}(0,0)$ to app5 of MEW.

5.3 Conclusions

In this chapter, we have solved the equal width and modified equal width wave equations using HAM. We used only 4 and 3 terms , respectively, to construct the approximations. These approximations are close enough to the exact solutions as can be easily checked in Tables 5.1,5.2 and 5.3. The results are valuable because we have a continuous approximation, that is useful for computational purposes.

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