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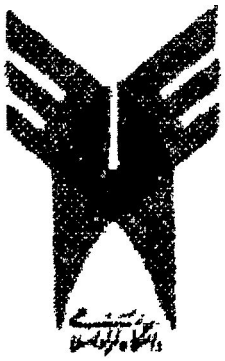

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چکیده

هدف این رساله استفاده از روشهای تفاضلی برای حل عددی معادلات دیفرانسیل فازی است. بدین منظور روش های تیلور، رانگ کوتا و یکی از روش های چند گامی برای حل عددی این نوع معادلات در نظر گرفته شده است، همچنین روش های اویلر، تیلور مرتبه دو و رانگ کوتا برای حل عددی معادلات دیفرانسیل فازی از نوع Inclusions نیز در نظر گرفته شده است. در این رساله قضایای همگرایی هر کدام از روش های فوق به تفضیل اثبات گردیده و نهایتاً با ذکر مثال های خطی و غیر خطی الگوریتم های فوق بررسی شده اند.

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SUBMITTED FOR THE DEGREE OF
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NUMERICAL SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS

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Abstract

The aim of this work is to use difference methods for solving *Fuzzy Differential Equations*, (FDE). The *Taylor*, *Runge-Kutta* and *Adams-Bashforth* methods are used for solving the FDE and also *Euler*, *Taylor of order two* and *Runge-Kutta* methods are used for solving *Fuzzy differential inclusions* and then the *Extrapolation method* for improving the solutions is discussed. The convergence theorems of all methods are proved in details and the algorithms are illustrated by solving some linear and nonlinear problems.

Publications

1. Numerical Solutions of Fuzzy Differential Equations by Taylor Method to appear in Computational Methods and Applied Mathematics.

2. Numerical Solution of Fuzzy Differential Equation to appear in Mathematical Computational and Applications.

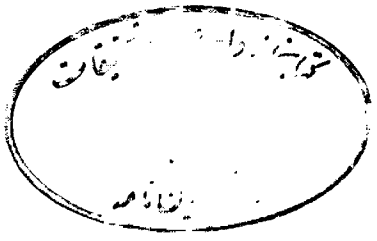
3. Extrapolation methods for fuzzy differential equations to appear in Mathematical Computational and Applications.

4. Numerical Solution of Fuzzy Differential Equation by Runge-Kutta Method of order 2 to appear in Journal of Sciences Islamic Azad University.

5. Numerical Solution of Fuzzy Differential Equation by Adams-Bashforth Two-Step Method, presented in 32'nd Iranian mathematics conference, Mazandaran.

6. Extrapolation Method For Improving The Solution Of Fuzzy Initial Value Problems, presented in 3rd'th Iranian mathematics conference, Mashhad.

7. Euler method for fuzzy differential inclusions, presented in 3'rd Iranian Fuzzy sets conference, Zahedan.



Introduction

Most of our traditional tools for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. By crisp we mean dichotomous, that is, yes-or-no type rather than more-or-less type. In conventional dual logic, for instance, a statement can be true or false and nothing in between. In set theory, an element can either belong to a set or not; and in optimization, a solution is either feasible or not. Precision assumes that the parameters of a model represent exactly either our perception of the phenomenon modeled or the features of the real system that has been modeled. Generally, precision also implies that the model is unequivocal, that is, it contains no ambiguities, [27].

A common approach to the mathematical modeling of dynamical systems in engineering the natural sciences is to characterize the system behavior by means of (ordinary) differential equations. Since these models are purely deterministic, the application of this approach requires precise knowledge about the system under investigation. However, this knowledge will rarely be available. On the contrary, parameter values, functional relationships, or initial conditions will not be often known precisely. Fuzzy initial value problem for modeling aspects of uncertainty in dynamical systems are introduced and interpreted from a probabilistic point of view. Knowledge about dynamical systems modeled by differential equations is often incomplete or vague.

For example, parameter values, functional relationships, or initial conditions, well-known methods for solving initial value problems analytically or numerically can only be used for finding selected system behavior, e.g., by fixing unknown parameters to some plausible values. However, in this way it is not possible to characterize the whole set of system behaviors compatible with our partial knowledge. We may set the fuzzy input (crisp input) somehow transformed into the fuzzy output by the corresponding crisp systems (fuzzy systems). This motivates us to refer to such systems as Fuzzy Input-Fuzzy Output (FIFO) or Crisp Input-Fuzzy Output (CIFO) systems. The development of fuzzy set theory to fuzzy technology during the first half of the 1990s has been very fast. The topics of numerical methods for solving fuzzy differential equations have been rapidly growing in recent years. The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [5]. It was followed up by D. Dubois, H. Prade in [11], who defined and used the extension principle. Other methods have been discussed by M. L. Puri, D. A. Ralescu in [20] and R. Goetschel, W. Voxman in [14]. The fuzzy differential equation and the initial value problem were regularly treated by O. Kaleva in [17] and [18], by S. Seikkala in [23],.... The numerical method for solving fuzzy differential equations is introduced by M. Ma, M. Friedman, A. Kandel in [19] by the standard Euler method. Recently Hüllermeier [15] suggested a different formulation of Fuzzy Initial Vlaue Problem (FIVP) based on a family of differential inclusions. The thesis is organized as follows:

In chapter 1 some basic definitions and results on fuzzy numbers, definition of a fuzzy derivative as well as definition of the problem, which is a fuzzy initial value problem, and fuzzy differential inclusions, that their numerical solution is the main interest of the later chapters, are discussed. In chapter 2 *Taylor method* and *Runge-Kutta*

method of order two and *Adams-Bashforth Two-Step Method* are used for solving fuzzy differential equations, with complete errors analysis and illustration of solving some linear and nonlinear examples. The *Extrapolation method* is used for improving the solutions of methods of chapter 2 which is explained in chapter 3. In chapter 4 a problem entitled *Fuzzy Differential Inclusions* is defined that their numerical solution is the main interest of this chapter. *Euler* method and *Taylor method of order two* and *Runge-Kutta method of order two* are used for solving those problems that complete errors analysis and illustration of solving some linear and nonlinear examples presented, then the *Extrapolation method* is used for improving the solutions. Finally is brought appendix.

Chapter 1

Preliminaries

Fuzziness is not a priori an obvious concept and demands some explanation. *Fuzziness* is *vagueness* i.e. to designate the kind of uncertainty which is both due to fuzziness and ambiguity. Generalizing refers to the application of a symbol to a multiplicity of objects in the field of reference, ambiguity to the association of a finite number of alternative meanings having the same phonetic form. But, the fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol applies.

1.1 Introduction

In this chapter the basic definitions of fuzzy sets and algebraic operations are defined and extension principle are provided which is one of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets.

1.2 Fuzzy sets and some basic definitions

Definition 1.2.1. If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, A(x)) | \ x \in X\}$$

$A(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in A that maps X to the membership space M (when M contains only the two points 0 and 1, A is nonfuzzy and $A(x)$ is identical to the characteristic function of nonfuzzy set).

The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite. Elements with a zero degree of membership are normally not listed. A fuzzy set is obviously a generalization of a classical set and the membership function a generalization of the characteristic function. Since we are generally referring to a universal (crisp) set X, some elements of fuzzy set may have the degree of membership zero. Often it is appropriate to consider those elements of the universe that have a nonzero degree of membership in a fuzzy set.

Definition 1.2.2. The *support* of a fuzzy set A is the ordinary subset of X:

$$supp(A) = \{x \in X | \ A(x) > 0\}.$$

Definition 1.2.3. The *height* of A is $hgt(A) = sup_{x \in X} A(x)$, i.e. the least upper bound of $A(x)$. A is said to be *normalized* iff $\exists x \in X, A(x) = 1$; this definition implies hgt(A)=1.

A more general and even more useful notion is that of an $r-$level set.

Definition 1.2.4. The set of elements that belong to the fuzzy set A at least to the degree r is called the $r-$level set or $r-$cut:

$$[A]_r = \{x \in X | \ A(x) \geq r\}$$

if nonequality is hold strictly then $[A]_r$ is called *strong $r-$level set.*

Definition 1.2.5. A triangular fuzzy number A, is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $A(x)$, the membership function of the fuzzy number A, is a triangle with base on the interval $[a_1, a_3]$ and vertex at $x = a_2$. We specify A as $(a_1/a_2/a_3)$. We will write: (1) $A > 0$ if $a_1 > 0$; (2) $A \geq 0$ if $a_1 \geq 0$; (3) $A < 0$ if $a_3 < 0$; and (4) $A \leq 0$ if $a_3 \leq 0$.

Convexity also plays an important role in fuzzy set theory. By contrast to classical set theory, however, convexity conditions are defined with reference to the membership function rather than the support of a fuzzy set.

Definition 1.2.6. A fuzzy set A is *convex* if

$$A(\lambda x_1 + (1-\lambda)x_2) \geq min\{A(x_1), A(x_2)\}, \forall x_1, x_2 \in X, \forall \lambda \in [0,1].$$

Alternatively, a fuzzy set is convex if and only if all $r-$level sets are convex, [27].

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. In its elementary form, it was already implied in Zadeh's first contribution (1965).

Definition 1.2.7. Let X be a cartesian product of universes $X = X_1 \times X_2 \times \ldots \times X_k$ and f be a mapping from X to a universe Y, $y = f(x_1, \ldots, x_k)$. Then the extension principle allows us to define a fuzzy set B in Y by

$$B = \{(y, B(y)) |\ y = f(x_1, \ldots, x_k), (x_1, \ldots, x_k) \in X\}$$

where

$$B(y) = \begin{cases} sup_{(x_1,\ldots,x_k)\in f^{-1}(y)} min\{A_1(x_1), \ldots, A_k(x_k)\} & if \quad f^{-1}(y) \neq \emptyset \\ 0 & otherwise \end{cases}$$

.

A fuzzy function is a generalization of the concept of a classical function. A classical function f is a mapping from the domain D of definition of the function into a space S; $f(D) \subseteq S$ is called the range of f. Different features of the classical concept of a function can be considered to be fuzzy rather than crisp. Therefore different *degrees* of fuzzification of the classical notion of a function are conceivable.

1. There can be a crisp mapping from a fuzzy set that carries along the fuzziness of the domain and therefore generates a fuzzy set. The image of a crisp argument would again be crisp.

2. The mapping itself can be fuzzy, thus blurring the image of a crisp argument. This we shall call a *fuzzy function*. These are called *fuzzifying function* by Dubois and Prade.

3. Ordinary functions can have fuzzy properties or be constrained by fuzzy constrains, [27].

Definition 1.2.8. A classical function $f : X \to Y$ maps a fuzzy domain A in X into a fuzzy range B in Y if and only if

$$\forall x \in X, B(f(x)) \geq A(x).$$

Given a classical function $f : X \to Y$ and a fuzzy domain A in X, the extension principle yields the fuzzy range B with the membership function

$$B(y) = sup_{x \in f^{-1}(y)} A(x),$$

hence f is a function according to the above definition, [27].

Denote by κ^n the set of all nonempty compact subsets of R^n and by κ_c^n the subset of κ^n consisting of nonempty convex compact sets.

An open ϵ-neighborhood of $\mathrm{A} \in \kappa^n$ is the set

$$\mathrm{N}(\mathrm{A}, \epsilon) = \{x \in \mathrm{R}^n : \inf_{a \in A} \|x - a\| < \epsilon\} = \mathrm{A} + \epsilon \mathrm{B}^n,$$

where B^n is the open unit ball in R^n, [9].

Definition 1.2.9. A continuous function $x : I \to Y \subseteq R^n$ is said to be *absolutely continuous* if there exists a locally integrable function ν such that

$$\int_t^s \nu(\alpha) d\alpha = x(s) - x(t)$$

for all $t, s \in I$, [9].

Definition 1.2.10. A mapping $F : \mathrm{R}^n \to \kappa^n$ is *upper semicontinuous* (usc) at x_0 if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0)$ such that

$$F(x) \subset N(F(x_0), \epsilon) = F(x_0 + \epsilon B^n)$$

for all $x \in N(x_0, \delta)$.

Definition 1.2.11. Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded r-level sets. A mapping $f : T \times E^n \to E^n$ is called *levelwise continuous* at point $(t_0, x_0) \in T \times E^n$ provided, for any fixed $r \in [0,1]$ and arbitrary $\epsilon > 0$, there exists a $\delta(\epsilon, r) > 0$ such that

$$d([f(t,x)]_r, [f(t_0,x_0)]_r) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, r)$ and $d([x]_r, [x_0]_r) < \delta(\epsilon, r)$ for all $t \in T, x \in E^n$, [26].

Definition 1.2.12. Let I be a real interval. A mapping $x : I \to E$ is called a fuzzy process and its $r-$level set is denoted by

$$[x(t)]_r = [x_1(t;r), x_2(t;r)], \quad t \in I.$$

If $f : R^n \times R^n \to R^n$ is a function, then, according to Zadeh extension principle, we can extend f to $E^n \times E^n \to E^n$ by the equation

$$f(u,v)(z) = sup_{z=f(x,y)} min\{u(x), v(y)\}.$$

It is well known that

$$[f(u,v)]_r = f([u]_r, [v]_r)$$

for all $u, v \in E^n, r \in [0,1]$.

Now we define algebraic operations with $r-$level sets.

Corollary 1.2.1. *Let $v, w \in E$ and s be a scalar, then for $r \in [0,1]$;*

$$[v+w]_r = [v_1(r) + w_1(r), v_2(r) + w_2(r)],$$

$$[v-w]_r = [v_1(r) - w_2(r), v_2(r) - w_1(r)],$$

$$[v.w]_r = [\min\{v_1(r).w_1(r), v_1(r).w_2(r), v_2(r).w_1(r), v_2(r).w_2(r)\},$$

$$\max\{v_1(r).w_1(r), v_1(r).w_2(r), v_2(r).w_1(r), v_2(r).w_2(r)\}],$$

$[sv]_r = s[v]_r$, *[23]*.

William Voxman has defined the fuzzy distance function in [24] as follows:

Definition 1.2.13. A function $s : [0,1] \to [0,1]$ is a reducing function if s is increasing and $s(0) = 0, s(1) = 1$. We say that s is a regular reducing function if

$$\int_0^1 s(r)dr = \frac{1}{2}.$$

Definition 1.2.14. If μ is a fuzzy number with $r-$cut representation, $(L(r), R(r))$, and if s is a reducing function then the value of μ (with respect to s) is defined by

$$val(\mu) = \int_0^1 s(r)(L(r) + R(r))dr.$$

Given fuzzy numbers μ and v, we say that $\mu \leq v$ in case $val(\mu) \leq val(v)$.

Definition 1.2.15. The fuzzy distance function on F, $\delta : E \times E \to E$, is defined by

$$\delta(\mu, v)(z) = \sup\{\min(\mu(x), v(y))\}.$$
$$|x - y| = z$$

For notational simplicity we will let $\delta_{\mu v}$ denote the fuzzy number $\delta(\mu, v)$ for each pair of fuzzy numbers μ, v. It is not difficult to see that if $\mu, v \in E$, and if the $r-$cut representations of μ and v are $(a(r), b(r))$ and $(c(r), d(r))$, respectively, then the $r-$cut representation of $\delta_{\mu v}$, $(L(r), R(r))$, is given by

$$L(r) = \begin{cases} \max\{c(r) - b(r), 0\} & if \quad \frac{1}{2}(a(1) + b(1)) \leq \frac{1}{2}(c(1) + d(1)), \\ \max\{a(r) - d(r), 0\} & if \quad \frac{1}{2}(c(1) + d(1)) \leq \frac{1}{2}(a(1) + b(1)), \end{cases} \tag{1.2.1}$$

$R(r) = \max\{b(r) - c(r), d(r) - a(r)\}$, for more information see [24].

Definition 1.2.16. The fuzzy number $X \in E^n$ is called pyramidal if its r-level sets are n-dimensional rectangles for $0 \leq r \leq 1$.

1.3 Hausdorff metric

In this section we define a metric space by Hausdorff separation. Recall that

$$\rho(x, A) = \inf_{a \in A} \|x - a\|$$

is the distance of a point $x \in \mathrm{R}^n$ from $A \in \kappa^n$ and that the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \kappa^n$ is defined as

$$\rho(A, B) = \sup_{a \in A} \rho(a, B).$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$. The *Hausdorff metric* d_H on κ^n is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

and (κ^n, d_H) is a complete metric space. Let D^n denote the set of usc normal fuzzy sets on R^n with compact support. That is, $u \in D^n$, then $u : \mathrm{R}^n \to [0, 1]$ is usc, $supp(u)$ is compact and there exists at least one $\xi \in supp(u)$ for which $u(\xi) = 1$. The β-level set of u, $0 < \beta \leq 1$ is

$$[u]_\beta = \{x \in \mathrm{R}^n : u(x) \geq \beta\}.$$

Clearly, for $\alpha \leq \beta, [u]_\alpha \supseteq [u]_\beta$. The level sets are nonempty from normality and compact by usc and compact support. The metric d_H is defined on D^n as

$$d_\infty(u, v) = \sup\{d_H([u]_r, [v]_r) : 0 \leq r \leq 1\}, \quad u, v \in D^n$$

and (D^n, d_∞) is a complete metric space. Denote by E^n the subset of fuzzy convex elements of D^n. The metric space (E^n, d_∞) is also complete, [7].

1.4 Fuzzy derivatives

The *derivative* of a real-valued function at a fuzzy point can be interpreted as the fuzzy set $f'(x_0)$.

1.4.1 Hukuhara derivative

The Hukuhara difference between two fuzzy numbers A and B defines as follows:

Definition 1.4.1. If there exists a fuzzy number C so that $C + A = B$, then C is called the Hukuhara difference between B and A and we write this as $B -^h A = C$.

Definition 1.4.2. A mapping $x : I \to E^n$ is Hukuhara differentiable at $t \in I$ if there exists $x'(t) \in E^n$ such that the limits

$$lim_{h\to 0+}\frac{x(t+h) -^h x(t)}{h} \quad and \quad lim_{h\to 0-}\frac{x(t) -^h x(t+h)}{h}$$

exist and are equal to $x'(t)$. Here the limit is taken in the metric space (E^n, D). At the end points of I we consider only the one-side derivatives, [18].

Let $x : I \to E^n$ be a fuzzy process and differentiable on I, for each $r \in [0, 1]$;.

$$[x(t)]_r = [x_1(t;r), x_2(t;r)], \ r \in [0, 1].$$

The Hukuhara derivative $x'(t)$ of a fuzzy process x is defined by

$$[x'(t)]_r = [x_1'(t;r), x_2'(t;r)], \quad t \in I,$$

provided that this equation defines a fuzzy number for the partial of $x_i(t;r)$ with respect to $t, i = 1, 2$. [23].

Let $x(t)$ be a fuzzy number for each $t \in I$. Some kind of derivatives are defined namely Goetchel-Voxman derivative, the Seikkala derivative, the Dubois-Prade derivative, the Puri-Ralescu derivative, and the Kandel-Friedman-Ming derivative of $x(t)$. The other derivatives of fuzzy function, approaches are more abstract and therefore not directly applicable to solving the fuzzy initial value problem.

1.4.2 Goetchel-Voxman derivative

Definition 1.4.3. The Goetschel-Voxman derivative of $x(t)$, written $GVDx(t)$, was defined in [14]. The derivative of $x(t)$ at t_0 is defined as

$$GVDx(t_0) = lim_{h\to 0}\frac{x(t_0+h) - x(t_0)}{h},$$

provided that the limit exists with respect to the metric D. However the subtraction is not standard fuzzy subtraction because

$$[x(t_0+h) - x(t_0)]_r = [x_1(t_0+h;r) - x_1(t_0;r), x_2(t_0+h;r) - x_2(t_0;r)],$$

for all $t; r$. Standard fuzzy arithmetic would produce

$$[x_1(t_0+h;r) - x_2(t_0;r), x_2(t_0+h;r) - x_1(t_0;r)].$$

If $GVDx(t)$ exists then $[GVDx(t)]_r = [x_1'(t;r), x_2'(t;r)]$, for all $t \in I, r \in [0,1]$. However $GVDx(t)$ may not be a fuzzy number for some t in I.

1.4.3 Seikkala derivative

Definition 1.4.4. The Seikkala derivative of $x(t)$, written $SDx(t)$, was defined in [23]. This definition is as follows: if $[x_1'(t;r), x_2'(t;r)]$ are the $r-$cuts of a fuzzy number for each $t \in I$, then $SDx(t)$ exists and $[SDx(t)]_r = [x_1'(t;r), x_2'(t;r)]$. Notice that this is the definition of derivative of a fuzzy function that we use in this work. That is, if $\frac{dy(t)}{dt}$ exists, then $SDx(t) = \frac{dy(t)}{dt}$. Also, $SDx(t)$ is a fuzzy number for all $t \in I$.

1.4.4 Dubois-Prade derivative

Definition 1.4.5. The Dubois-Prade derivative of $x(t)$, written $DPDx(t)$, was defined in [11] and [13]. $DPDx(t)$ always exists and its membership function is given by

$$DPDx(t)(x) = sup\{r |\ x = x_1'(t;r), x = x_2'(t;r)\}.$$

However, $DPDx(t)$ may not be a fuzzy number. We may have to add something to the definition of $DPDx(t)$ to obtain a fuzzy number.

1.5 Fuzzy differential equation

In this section we define *Fuzzy Initial Value Problem* (FIVP) or *Fuzzy Cauchy Problem.* Differential equation in a fuzzy environment has been suggested as a way of modeling uncertain and incompletely specified systems. Formulation of the concept usually interprets the solution as flow on some appropriate space of fuzzy sets and has been largely concerned with existence and uniqueness problems.

Definition 1.5.1. Fuzzy initial value problem is defined as

$$\begin{cases} x'(t) &= f(t, x(t)); \quad a \leq t \leq b, \\ x(a) &= x_0. \end{cases} \tag{1.5.1}$$

Assume that $f : T \times E^n \to E^n$ is continuous and $x_0 \in E^n$. The following theorems and lemmas show that the system (1.5.1) has solution.

Lemma 1.5.1. *A mapping $x : T \to E^n$ is a solution to (1.5.1) if and only if it is continuous and satisfies the integral equation*

$$x(t) = x_0 + \int_a^t f(s, x(s))ds \tag{1.5.2}$$

for all $t \in T$, [17].

Note that we cannot extend lemma (1.5.1) for $t < a$. If f is Lipschitz continuous then (1.5.1) has a unique solution on T.

Theorem 1.5.2. *Let $f : T \times E^n \to E^n$ be continuous and assume that there exists a $k > 0$ such that*

$$D(f(t, x), f(t, y)) \leq kD(x, y)$$

for all $t \in T, x, y \in E^n$. Then (1.5.1) has a unique solution on T, [17].

Furthermore, the solution depends continuously on the initial value.

Seppo Seikkala in [23] solves (1.5.1) as follows:

The extension principle of Zadeh leads to the following definition of $f(t, x)$ when $x = x(t)$ is a fuzzy number

$$f(t, x)(s) = \sup\{x(\tau) |\ s = f(t, \tau)\}, \quad s \in R.$$

It follows that

$$[f(t,x)]_r = [f_1(t,x;r), f_2(t,x;r)], \quad r \in (0,1].$$

where

$$\begin{aligned} f_1(t,x;r) &= min\{f(t,u)|\ u \in [x_1(t;r), x_2(t;r)]\}, \\ f_2(t,x;r) &= max\{f(t,u)|\ u \in [x_1(t;r), x_2(t;r)],\}. \end{aligned} \tag{1.5.3}$$

The function $x : R_+ \to E$ is a fuzzy solution of (1.5.1) on I, if

$$\begin{aligned} x_1'(t;r) &= min\{f(t,u)|\ u \in [x_1(r), x_2(r)]\},\ x_1(0;r) = x_{01}(r), \\ x_2'(t;r) &= max\{f(t,u)|\ u \in [x_1(r), x_2(r)]\},\ x_1(0;r) = x_{01}(r), \end{aligned} \tag{1.5.4}$$

for any $t \in I$ and $r \in [0,1]$. Thus for fixed r, we have an initial value problem in R^2. If we can solve it (uniquely), we have only to verify that the intervals $[x_1(t;r), x_2(t;r)]$, $r \in [0,1]$, define a fuzzy number $x(t)$ in E.

Theorem 1.5.3. *Let f satisfy*

$$|f(t,v) - f(t,\overline{v})| \leq g(t, |v - \overline{v}|), \quad t \geq 0, \quad v, \overline{v} \in R,$$

where g on $R_+ \times R_+$ is a continuous mapping such that $r \to g(t,r)$ is nondecreasing, the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \tag{1.5.5}$$

has a solution on R_+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (1.5.5) for $u_0 = 0$. Then the fuzzy initial value problem (1.5.1) has a unique fuzzy solution. Proof [23].

The other solution is introduced by J. Buckley and T. Feuring in [16]. They solve the FIVP (1.5.1) by fuzzifying the crisp solution to obtain fuzzy solution using the extension principle.

Classical Euler method for solving FIVP according to Hukuhara (Seikkala) derivative for FIVP is used by Ma and Friedman and Kandel in [19] as follows: Let the exact solution $[Y(t)]_r = [Y_1(t;r), Y_2(t;r)]$ is approximated by some $[y(t)]_r = [y_1(t;r), y_2(t;r)]$.

$$y_1(t_{n+1};r) = y_1(t_n;r) + hf_1(t_n, y(t_n);r)$$

$$y_2(t_{n+1};r) = y_2(t_n;r) + hf_2(t_n, y(t_n);r)$$

which f_1, f_2 are defined in (1.5.3).

1.6 Fuzzy differential inclusions

The Hukuhara derivative approach suffers a grave disadvantage in so far as the solution has the property that diam $(x(t))$ is nondecreasing in t [3], see (4.5.1) that is, the solution is irreversible in possibilistic terms. The formulation of (1.5.1) cannot really reflect any of the rich behavior of ordinary differential equations such as periodicity, stability, bifurcation and the like, and is ill-suited for modeling.

Recently, Hüllermeier [15] suggested a different formulation of the FIVP based on a family of differential inclusions at each $r-$level, $0 \leq r \leq 1$,

$$x'(t) \in [f(t, x(t))]_r, \quad x(0) \in [x_0]_r, \tag{1.6.1}$$

where now $[f(.,.)]_r : R \times R^n \to \kappa_c^n$, the space of nonempty convex compact subsets of R^n. The idea is that reachable set, the set of all such solutions $\sum_r(x_0, T)$, would be the $r-$level of a fuzzy set $\sum(x_0, T)$, in the sense that all attainable sets $A_r(x_0, t)$, $0 < t \leq T$, are levels of a fuzzy $A(x_0, t)$ on R^n. Considering $\sum(x_0, T)$, to be the solution of the 1.5.1, thus captures both uncertainty and the rich properties of differential

inclusions in one and the same technique. It has been shown that the solution set and attainability set are fuzzy sets under fairly relaxed conditions on f, [9], [8].

For example, under this interpretation, the fuzzy differential equation $x'(t) = tx(t)U + V$, where U, V are constants fuzzy numbers, is the family of inclusions

$$x'(t) \in tx(t)[U]_r + [V]_r, \quad 0 \leq r \leq 1,$$

where $[U]_r$ is the $r-$level set of U and similarly for $[V]_r$.

Let $f : \mathrm{I} \times \mathrm{E}^n \to E^n$ and consider the fuzzy initial value problem (FIVP)

$$\begin{cases} x'(t) &= f(t, x(t)), \quad t \in I = [0, T], \\ x(0) &= Y_0 \in E^n, \end{cases} \tag{1.6.2}$$

interpreted as a family of differential inclusions. Set $[f(t,x)]_r = F(t,x;r)$ and identify the FIVP with the family of differential inclusions

$$\begin{cases} x_r'(t) &\in F(t, x_r(t); r), \quad t \in I = [0, T], \\ x_r(0) &= y_0 \in [Y_0]_r, \quad 0 \leq r \leq 1, \end{cases} \tag{1.6.3}$$

where $F : \Omega \times [0,1] \to \kappa_c^n$ and Ω is an open subset of $I \times E^n$ containing $(0, [Y_0]_r)$, $r \in [0,1]$. Denote the set of all solution of (1.6.3) on I by $\sum_r([Y_0]_r, T)$ and the attainable set by $A_r([Y_0]_r, T) = \{x(T) : x(.) \in \sum_r([Y_0]_r, T)\}$. Let

$$Z_T(R^n) = \{x(.) \in C([0,T]; R^n) : x'(.) \in L^\infty([0,T]; R^n)\}.$$

These sets are not in general convex, they are acyclic which is stronger than simply connected [2]. The following theorem is a consequence of Theorem 3.1 and the definition of section 3.1 in [2]. A more complete account can be found in [2] and [9].

Theorem 1.6.1. *Let $Y_0 \in E^n$ and let Ω be an open set $R \times R^n$ containing $\{0\} \times supp(Y_0)$. Suppose that $f : \Omega \to E^n$ is usc and write $F(t,x;r) = [f(t,x)]_r \in \kappa_c^n$ for all $(t,x,r) \in R^{n+1} \times [0,1]$. Let the boundedness assumption, hold for all $y_0 \in supp(Y_0)$ and the inclusions*

$$x'(t) \in F(t,x;0),\ x(0) \in supp(Y_0).$$

Then the attainable sets $A_r([Y_0]_r, T), r \in [0,1]$, of the family of inclusions (1.6.3) are the level sets of a fuzzy set $A(Y_0,T) \in D^n$. The solution sets $\sum_r([Y_0]_r, T)$ of (1.6.3) are the level sets of a fuzzy set $\sum(Y_0,T)$ defined on $Z_T(R^n)$.

The solution of (1.6.3) is unique for each $y_0 \in supp(Y_0) \subset \mathrm{R}^n$. Under some conditions suppose $x(t) := z(t,y_0)$, is a solution of (1.6.3) for each $y_0 \in supp(Y_0)$. The authors of [25] constructed n families of r-parameterized interval-valued mappings $g^k(r) : I \to [g_1^k(t;r), g_2^k(t;r)]$ in the following way:

$$\begin{cases} g_1^k(t;r) = \min\{z^k(t,y_0) : y_0 \in [Y_0]_r\}, \\ g_2^k(t;r) = \max\{z^k(t,y_0) : y_0 \in [Y_0]_r\},\ r \in [0,1],\ k = 1,\ldots,n, \end{cases} \tag{1.6.4}$$

where $z(t,y_0) = (z^1(t,y_0),\ldots,z^n(t,y_0)) \in \mathrm{R}^n$. The minimum vector and maximum vector of $z(t,y_0)$ are respectively

$$z_{min}(t,y_0) = (g_1^1(t;r),\ldots,g_1^n(t;r)),$$

and

$$z_{max}(t,y_0) = (g_2^1(t;r),\ldots,g_2^n(t;r)),$$

where obviously

$$z(t,y_0) \in [z_{min}(t,y_0), z_{max}(t,y_0)].$$

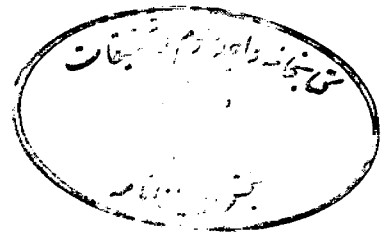
Let $X : I \to \mathrm{E}^n$ be a fuzzy process whit

$$[X(t)]_r = \prod_{k=1}^{n} [g_1^k(t;r), g_2^k(t;r)], \tag{1.6.5}$$

where $\prod$ denotes the usual set-theoretical Cartesian product, [25], then

$$\{(z^1(t;y_0), \ldots, z^n(t;y_0)) : z^i(t;y_0) \in [g_1^i(t;r), g_2^i(t;r)],\ i = 1, \ldots, n\} = [X(t)]_r,\ r \in [0,1],$$

hence the convex hull of corners of n-dimensional rectangles (1.6.5) is $[X(t)]_r$ for any $r \in [0,1]$.



Chapter 2

Numerical Solutions of Fuzzy Differential Equations

In this chapter we will propose numerical algorithms for solving *fuzzy ordinary differential equations.* A scheme based on the different methods are discussed in detail and they are followed by a complete error analysis. The algorithms are illustrated by solving some linear and nonlinear FIVP.

2.1 Introduction

In the section 1.5 we defined FIVP. Since finding the set of analytic solutions is difficult or does not exist for some problems, therefore, a numerical approach seems to be the only way of *solving* such problems.

2.2 Taylor method of order p

In this section we are going to solve FIVP using *Taylor method of order p* and then to proof the convergence theorem of method.

Let the exact solution $[Y(t)]_r = [Y_1(t;r), Y_2(t;r)]$ is approximated by some $[y(t)]_r = [y_1(t;r), y_2(t;r)]$. The Taylor method of order p is based on the

$$y_{i+1} = y_i + hT(t_i, y_i), \quad i = 0, 1, \ldots, N-1, \tag{2.2.1}$$

and

$$T(t_i, y_i) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f^{(i)}(t_i, y_i), \tag{2.2.2}$$

where

$$a = t_0 \leq t_1 \leq \ldots \leq t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{2.2.3}$$

where $x(t;r)$ is Y_1 and Y_2 alternatively.

We define

$$\begin{aligned} F[t,x;r] &= \textstyle\sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_1^{(i)}(t,x;r), \\ G[t,x;r] &= \textstyle\sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_2^{(i)}(t,x;r). \end{aligned} \tag{2.2.4}$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)]$ and $[y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$, respectively. The solution is calculated at points t_n. By Taylor method of order p and substituting Y_1 and Y_2 in (2.2.1) and considering (2.2.4) we have

$$\begin{aligned} Y_1(t_{n+1};r) &\approx Y_1(t_n;r) + hF[t_n, Y(t_n);r], \\ Y_2(t_{n+1};r) &\approx Y_2(t_n;r) + hG[t_n, Y(t_n);r]. \end{aligned} \tag{2.2.5}$$

Hence we have

$$\begin{aligned} y_1(t_{n+1};r) &= y_1(t_n;r) + hF[t_n, y(t_n);r], \\ y_2(t_{n+1};r) &= y_2(t_n;r) + hG[t_n, y(t_n);r], \end{aligned} \tag{2.2.6}$$

where

$$y_1(0;r) = x_1(0;r) \quad , \quad y_2(0;r) = x_2(0;r).$$

The lemmas (4.5.3) and (4.5.4) will be applied to show convergence of these approximates i.e.,

$$\begin{aligned} \lim_{h\to 0} y_1(t,h;r) &= Y_1(t;r), \\ \lim_{h\to 0} y_2(t,h;r) &= Y_2(t;r). \end{aligned}$$

Let $F^*(t,u,v)$ and $G^*(t,u,v)$ be the functions F and G in (2.2.4), where u and v are constants and $u \leq v$. In other words

$$F^*(t,u,v) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} min\{f_1^{(i)}(t,\tau) | \tau \in [u,v]\},$$

$$G^*(t,u,v) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} max\{f_1^{(i)}(t,\tau) | \tau \in [u,v]\},$$

i.e. $F^*(t,u,v)$ and $G^*(t,u,v)$ are obtained by substituting $[x(t)]_r = [u,v]$ in (2.2.4). The domain where F^* and G^* are defined is therefore

$$K = \{(t,u,v) |\ 0 \leq t \leq T \quad , \quad -\infty < v < \infty \quad , \quad -\infty < u \leq v\}.$$

Theorem 2.2.1. *Let $F^*(t,u,v)$ and $G^*(t,u,v)$ belong to $C^{p-1}(K)$ and let the partial derivatives of F^* and G^* in terms of u and v be bounded over K. Then, for arbitrary fixed $r : 0 \leq r \leq 1$, the approximate solutions (1.5.1) converge to the exact solutions $Y_1(t;r)$ and $Y_2(t;r)$ uniformly in t.*

Proof. It is sufficient to show

$$\begin{aligned} \lim_{h\to 0} y_1(t_N,h;r) &= Y_1(t_N;r), \\ \lim_{h\to 0} y_2(t_N,h;r) &= Y_2(t_N;r), \end{aligned}$$

where $t_N = T$. For n = 0,1, ...,N-1 , by using Taylor theorem we get

$$\begin{aligned} Y_1(t_{n+1};r) &= Y_1(t_n;r) + hF^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{h^{p+1}}{(p+1)!}Y_1^{(p+1)}(\xi_{n,1}), \\ Y_2(t_{n+1};r) &= Y_2(t_n;r) + hG^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{h^{p+1}}{(p+1)!}Y_2^{(p+1)}(\xi_{n,2}), \end{aligned} \tag{2.2.7}$$

where $\xi_{n,1}, \xi_{n,2} \in (t_n, t_{n+1})$. Denote

$$W_n = Y_1(t_n;r) - y_1(t_n;r),$$

$$V_n = Y_2(t_n;r) - y_2(t_n;r),$$

we have from (2.2.5) and (2.2.6)

$$W_{n+1} = W_n + h\{F^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] - F^*[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{h^{p+1}}{(p+1)!}Y_1^{(p+1)}(\xi_{n,1}),$$

$$V_{n+1} = V_n + h\{G^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] - G^*[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{h^{p+1}}{(p+1)!}Y_2^{(p+1)}(\xi_{n,2}).$$

Then

$$|W_{n+1}| \le |W_n| + 2Lh.max\{|W_n|, |V_n|\} + \frac{h^{p+1}}{(p+1)!}M,$$

$$|V_{n+1}| \le |V_n| + 2Lh.max\{|W_n|, |V_n|\} + \frac{h^{p+1}}{(p+1)!}M,$$

where

$$M_1 = max|Y_1^{(p+1)}(t;r)| \ , \ M_2 = max|Y_2^{(p+1)}(t;r)|,$$

for $t \in [0,T]$ and $M = \max\{M_1 \ , \ M_2\}$ and $L > 0$ is a bound for the partial derivatives of F^* and G^*. Thus by lemma 4.5.4

$$|W_n| \le (1+4Lh)^n|U_0| + \frac{2h^{p+1}}{(p+1)!}M\frac{(1+4Lh)^n - 1}{4Lh},$$

$$|V_n| \leq (1+4Lh)^n|U_0| + \frac{2h^{p+1}}{(p+1)!}M\frac{(1+4Lh)^n - 1}{4Lh},$$

where $|U_0| = |W_0| + |V_0|$. In particular

$$|W_N| \leq (1+4Lh)^N|U_0| + \frac{h^{p+1}}{(p+1)!}M\frac{(1+4Lh)^{\frac{T}{h}} - 1}{2Lh},$$

$$|V_N| \leq (1+4Lh)^N|U_0| + \frac{h^{p+1}}{(p+1)!}M\frac{(1+4Lh)^{\frac{T}{h}} - 1}{2Lh}.$$

Since $W_0 = V_0 = 0$ we obtain

$$|W_N| \leq M\frac{e^{4LT} - 1}{2L(p+1)!}h^p,$$

$$|V_N| \leq M\frac{e^{4LT} - 1}{2L(p+1)!}h^p,$$

and if $h \to 0$ we get $W_N \to 0$ and $V_N \to 0$ which completes the proof. □

Now we illustrate the algorithms by solving some kind of linear and nonlinear examples.

2.2.1 Examples

Example 2.2.1. *Consider the fuzzy initial value problem, [19].*

$$\begin{cases} y'(t) &= y(t), \quad t \in I = [0,1], \\ y(0) &= (.75 + .25r\ ,\ 1.125 - .125r), \quad 0 < r \leq 1. \end{cases}$$

By using Taylor method of order p we have

$$y_1(t_{n+1}; r) = y_1(t_n; r).\sum_{i=0}^{p}\frac{h^i}{i!},$$

$$y_2(t_{n+1};r) = y_2(t_n;r).\sum_{i=0}^{p}\frac{h^i}{i!}.$$

The exact solution is given by

$$Y_1(t;r) = y_1(0;r)e^t \quad , \quad Y_2(t;r) = y_2(0;r)e^t,$$

which at $t = 1$,

$$Y(1;r) = [(.75 + .25r)e \ , \ (1.125 - .125r)e], \quad 0 < r \leq 1.$$

The exact and approximate solutions for $p = 2$ and $p = 4$ are compared and plotted at $t = 1$ in Figures 2.2.1 and 2.2.2.

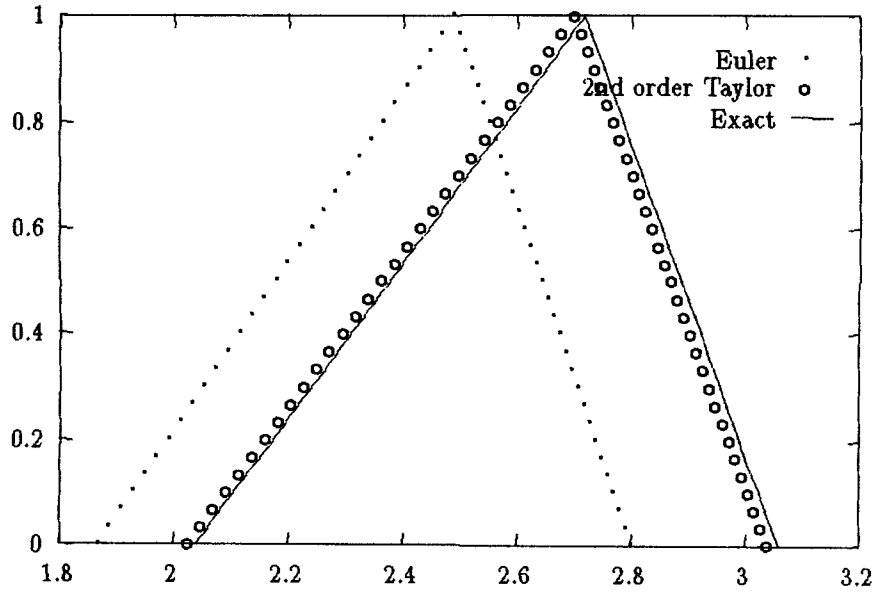


Figure 2.2.1, h=0.2

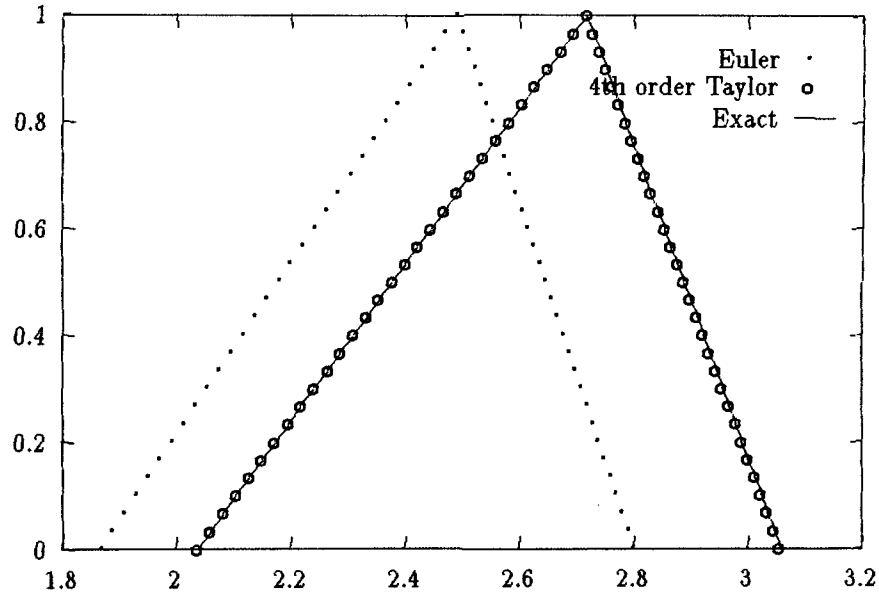


Figure 2.2.2, h=0.2

The Hausdorff distance of exact solution from Euler solution and 2nd order Taylor solution and 4th order Taylor solution is $d_\infty = 0.2587$ and $d_\infty = 0.0175$ and $d_\infty = 3.4528e-005$ respectively.

Example 2.2.2. *Consider the fuzzy initial value problem, [19].*

$$\begin{cases} y'(t) &= ty(t), \quad [a,b] = [-1,1], \\ y(-1) &= (\sqrt{e} - .5(1-r), \sqrt{e} + .5(1-r)), \quad 0 < r \leq 1. \end{cases}$$

We separate between two steps.

$(a) t < 0$: The parametric form in this case is $y_1'(t;r) = ty_2(t;r)$, $y_2'(t;r) = ty_1(t;r)$,

$y_1''(t;r) = (1+t^2)y_2(t;r)$, $y_2''(t;r) = (1+t^2)y_1(t;r)$,

$y_1^{(3)}(t;r) = min\{(1+2t+2t^2+t^4)u | u \in [y_1(t;r), y_2(t;r)]$,

$y_2^{(3)}(t;r) = max\{(1+2t+2t^2+t^4)u | u \in [y_1(t;r), y_2(t;r)]$,

$y_1^{(4)}(t;r) = min\{(2+4t^2+4t(1+t^2)+4t(1+t^2)^2+(1+t^2)^4)u | u \in [y_1(t;r), y_2(t;r)]$,

$y_2^{(4)}(t;r) = max\{(2+4t^2+4t(1+t^2)+4t(1+t^2)^2+(1+t^2)^4)u | u \in [y_1(t;r), y_2(t;r)]$,

with the initial conditions given. The unique exact solution is

$$Y_2(t;r) = \frac{A+B}{2}y_2(0;r) + \frac{A-B}{2}y_1(0;r),$$

$$Y_1(t;r) = \frac{A-B}{2}y_2(0;r) + \frac{A+B}{2}y_1(0;r),$$

where $A = e^{\frac{(t^2-a^2)}{2}}, B = \frac{1}{A}$.

$(b) t \geq 0$: The parametric equations are

$$y_1'(t;r) = ty_1(t;r), y_2'(t;r) = ty_2(t;r),$$

$$y_1''(t;r) = (1+t^2)y_1(t;r), y_2''(t;r) = (1+t^2)y_2(t;r),$$

with the initial conditions given. The unique exact solution at $t > 0$ is

$$Y_1(t;r) = y_1(0;r)e^{\frac{t^2}{2}}, Y_2(t;r) = y_2(0;r)e^{\frac{t^2}{2}},$$

The exact and approximate solutions for $p = 2$ and $p = 4$ are compared and plotted at the end point in Figures 2.2.3 and 2.2.4.

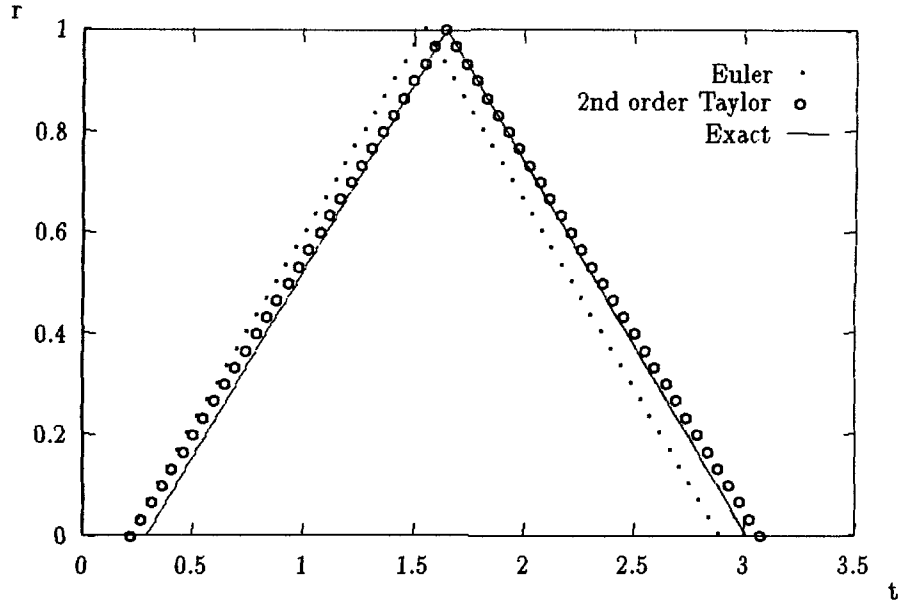


Figure 2.2.3, h=0.05

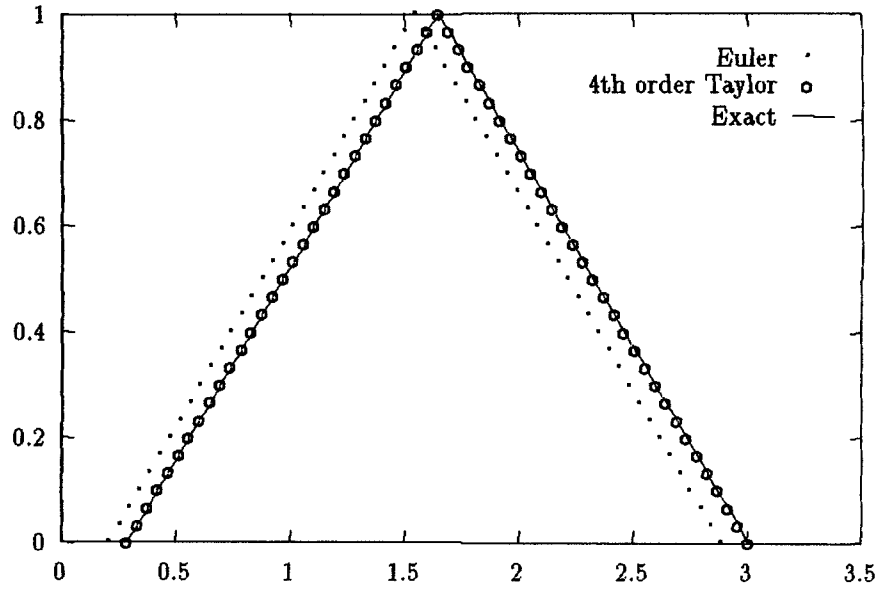


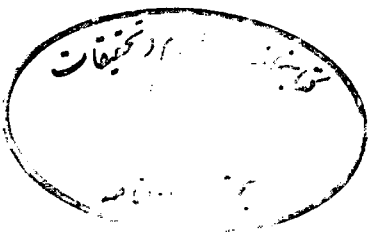
Figure 2.2.4, h=0.05

The Hausdorff distance of exact solution from Euler solution and 2nd order Taylor solution and 4th order Taylor solution is $d_\infty = 0.1284$ and $d_\infty = 0.0677$ and $d_\infty = 2.8099e-008$ respectively.

Example 2.2.3. *Consider the fuzzy initial value problem*

$$y'(t) = k_1 y^2(t) + k_2, \quad y(0) = 0,$$

where $k_i > 0$,*for* $i = 1, 2$ *are triangular fuzzy numbers, [16].*



The exact solution is given by

$$Y_1(t;r) = l_1(r)tan(w_1(r)t),$$

$$Y_2(t;r) = l_2(r)tan(w_2(r)t),$$

with

$$l_1(r) = \sqrt{k_{2,1}(r)/k_{1,1}(r)}, l_2(r) = \sqrt{k_{2,2}(r)/k_{1,2}(r)},$$
$$w_1(r) = \sqrt{k_{1,1}(r)k_{2,1}(r)}, w_2(r) = \sqrt{k_{1,2}(r)k_{2,2}(r)},$$

where

$$[k_1]_r = [k_{1,1}(r), k_{1,2}(r)] \ , \ [k_2]_r = [k_{2,1}(r), k_{2,2}(r)].$$

and

$k_{1,1}(r) = .5 + .5r$, $k_{1,2}(r) = 1.5 - .5r$ *and* $k_{2,1}(r) = .75 + .25r$, $k_{2,2}(r) = 1.25 - .25r$.

The r-level sets of $y'(t)$ are

$$Y_1'(t;r) = k_{2,1}(r)sec^2(w_1(r)t),$$

$$Y_2'(t;r) = k_{2,2}(r)sec^2(w_2(r)t),$$

which defines a fuzzy number. We have

$$f_1(t,y;r) = min\{k_1u^2 + k_2 | u \in [y_1(t;r), y_2(t;r)], k_j \in [k_{j,1}(r), k_{j,2}(r)], \ j = 1,2\},$$

$$f_2(t,y;r) = max\{k_1u^2 + k_2 | u \in [y_1(t;r), y_2(t;r)], k_j \in [k_{j,1}(r), k_{j,2}(r)], \ j = 1,2\},$$

$$f_1'(t,y;r) = min\{2k_1^2u^3 + 2uk_1k_2 | u \in [y_1(t;r), y_2(t;r)], k_j \in [k_{j,1}(r), k_{j,2}(r)], \ j = 1,2\},$$

$$f_2'(t,y;r) = max\{2k_1^2u^3 + 2uk_1k_2 | u \in [y_1(t;r), y_2(t;r)], k_j \in [k_{j,1}(r), k_{j,2}(r)],\ j = 1,2\},$$

$$f_1^{(i)}(t,y;r) = min\{\frac{\partial f^{(i-1)}(t,u)}{\partial u}.f^{(i-1)}(t,u) | u \in [y_1(t;r), y_2(t;r)], k_j \in [k_{j,1}(r), k_{j,2}(r)],\ j = 1,2\},$$

$$f_2^{(i)}(t,y;r) = max\{\frac{\partial f^{(i-1)}(t,u)}{\partial u}.f^{(i-1)}(t,u) | u \in [y_1(t;r), y_2(t;r)], k_j \in [k_{j,1}(r), k_{j,2}(r)],\ j = 1,2\},$$

for $i = 1,2$. There are two nonlinear problems which can be solved by GAMS software. Thus the suggested Taylor method in this section can be used. The exact and approximate solutions for $p = 2$ and $p = 4$ are shown in Figures 2.2.5 and 2.2.6 at $t = 1$.

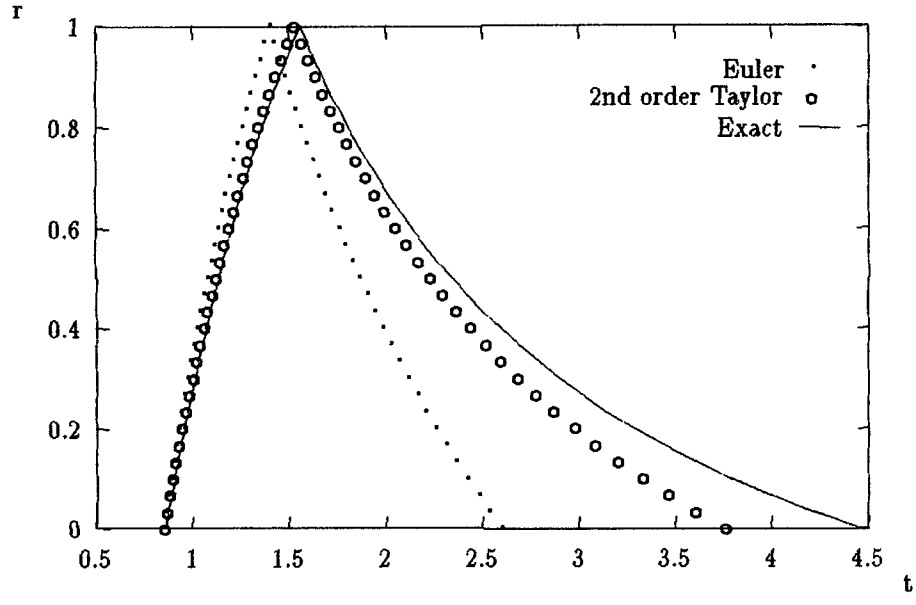


Figure 2.2.5, h=0.1

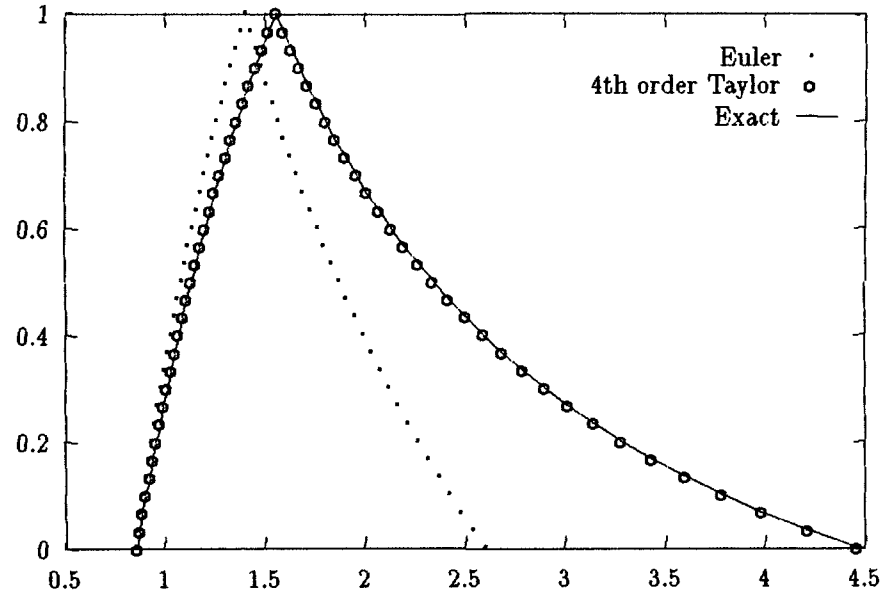


Figure 2.2.6, h=0.1

The Hausdorff distance of exact solution from Euler solution and 2nd order Taylor solution and 4th order Taylor solution is $d_\infty = 1.8711$ and $d_\infty = 0.7027$ and $d_\infty = 0.0073$ respectively.

2.3 Runge-Kutta method of order two

The basis of all Runge-Kutta methods are to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = h \sum_{i=1}^{m} w_i k_i. \tag{2.3.1}$$

where, for $i = 1, 2, \ldots m$, the w_i's are constants and

$$k_i = f(t_n + \alpha_i h, y_n + h \sum_{j=1}^{i-1} \beta_{ij} k_j). \tag{2.3.2}$$

Equation (2.3.1) is to be exact for powers of h. Therefore, the truncation error T_m, can be written as

$$T_m = \gamma_m h^{m+1} + O(h^{m+2}),$$

The true magnitude of γ_m will generally be much less than the bound, [21]. Thus, if the $O(h^{m+2})$ term is small compared with $\gamma_m h^{m+1}$, as we expect it will be if h is small, then the bound on $\gamma_m h^{m+1}$ will usually be a bound on the error as a whole. The nonzero constants α_i , β_{ij} in Runge-Kutta method of order two, for $m = 2$ are

$$\alpha_1 = 0 \ , \ \alpha_2 = 1, \ \beta_{21} = 1.$$

Hence we have,

$$\begin{aligned} y_0 &= \alpha, \\ k_1 &= f(t_i, y_i), \\ k_2 &= f(t_i + h, y_i + hk_1), \\ y_{i+1} &= y_i + \tfrac{h}{2}(k_1 + k_2), \end{aligned} \tag{2.3.3}$$

where

$$a = t_0 \leq t_1 \leq \ldots \leq t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{2.3.4}$$

In this method from (2.3.1),(2.3.2) we define

$$\begin{aligned} y_1(t_{n+1}; r) - y_1(t_n; r) &= h \textstyle\sum_{i=1}^{2} w_i k_{i,1}(t_n, y(t_n; r)), \\ y_2(t_{n+1}; r) - y_2(t_n; r) &= h \textstyle\sum_{i=1}^{2} w_i k_{i,2}(t_n, y(t_n; r)), \end{aligned} \tag{2.3.5}$$

for (1.5.1) where the w_is are constants and

$$[k_i(t, y(t; r))]_r = [k_{i,1}(t, y(t; r)), k_{i,2}(t, y(t; r))], \quad i = 1, 2,$$

$$
\begin{aligned}
k_{i,1}(t_n, y(t_n;r)) &= f(t_n + \alpha_i h, y_1(t_n) + h\textstyle\sum_{j=1}^{i-1} \beta_{ij} k_{j,1}(t_n, y(t_n;r))),\\
k_{i,2}(t_n, y(t_n;r)) &= f(t_n + \alpha_i h, y_2(t_n) + h\textstyle\sum_{j=1}^{i-1} \beta_{ij} k_{j,2}(t_n, y(t_n;r))),
\end{aligned}
\tag{2.3.6}
$$

and

$$
\begin{aligned}
k_{1,1}(t, y(t;r)) &= min\{f(t,u) | u \in [y_1(t;r), y_2(t;r)]\},\\
k_{1,2}(t, y(t;r)) &= max\{f(t,u) | u \in [y_1(t;r), y_2(t;r)]\},\\
k_{2,1}(t, y(t;r)) &= min\{f(t+h,u) | u \in [z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r))]\},\\
k_{2,2}(t, y(t;r)) &= max\{f(t+h,u) | u \in [z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r))]\},
\end{aligned}
\tag{2.3.7}
$$

where in the Runge-Kutta method of order two,

$$
z_{1,1}(t, y(t;r)) = y_1(t;r) + hk_{1,1}(t, y(t;r)), \quad z_{1,2}(t, y(t;r)) = y_2(t;r) + hk_{1,2}(t, y(t;r)).
\tag{2.3.8}
$$

Define,

$$
\begin{aligned}
F[t, y(t;r)] &= k_{1,1}(t, y(t;r)) + k_{2,1}(t, y(t;r)),\\
G[t, y(t;r)] &= k_{1,2}(t, y(t;r)) + k_{2,2}(t, y(t;r)).
\end{aligned}
\tag{2.3.9}
$$

The exact and approximate solutions of (1.5.1) at $t_n, 0 \leq n \leq N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)]$ and $[y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$, respectively. The solutions are calculated at points t_n. By (2.3.5) and (2.3.6) we have

$$
\begin{aligned}
Y_1(t_{n+1};r) &\approx Y_1(t_n;r) + \tfrac{h}{2} F[t_n, Y(t_n;r)],\\
Y_2(t_{n+1};r) &\approx Y_2(t_n;r) + \tfrac{h}{2} G[t_n, Y(t_n;r)].
\end{aligned}
\tag{2.3.10}
$$

We define

$$\begin{aligned} y_1(t_{n+1};r) &= y_1(t_n;r) + \tfrac{h}{2}F[t_n, y(t_n;r)], \\ y_2(t_{n+1};r) &= y_2(t_n;r) + \tfrac{h}{2}G[t_n, y(t_n;r)]. \end{aligned} \tag{2.3.11}$$

The lemmas 4.5.3 and 4.5.4 will be applied to show convergence of these approximations i.e.,

$$\begin{aligned} \lim_{h\to 0} y_1(t,h;r) &= Y_1(t;r), \\ \lim_{h\to 0} y_2(t,h;r) &= Y_2(t;r). \end{aligned}$$

Let $F^*(t,u,v)$ and $G^*(t,u,v)$ are obtained by substituting $[y(t)]_r = [u,v]$ in (2.3.9),

$$\begin{aligned} F^*(t,u,v) &= k_{1,1}(t,u,v) + k_{2,1}(t,u,v), \\ G^*(t,u,v) &= k_{1,2}(t,u,v) + k_{2,2}(t,u,v). \end{aligned}$$

The domain where F^* and G^* are defined is therefore

$$K = \{(t,u,v) |\ 0 \le t \le T \ ,\ -\infty < v < \infty \ ,\ -\infty < u \le v\}.$$

Theorem 2.3.1. *Let $F^*(t,u,v)$ and $G^*(t,u,v)$ belong to $C^2(K)$ and let the partial derivatives of F^* and G^* be bounded over K. Then, for arbitrary fixed $r, 0 \le r \le 1$, the approximate solutions (1.5.1) converge to the exact solutions $Y_1(t;r)$ and $Y_2(t;r)$ uniformly in t.*

Proof. It is sufficient to show

$$\begin{aligned} \lim_{h\to 0} y_1(t_N,h;r) &= Y_1(t_N;r), \\ \lim_{h\to 0} y_2(t_N,h;r) &= Y_2(t_N;r), \end{aligned}$$

where $t_N = T$. For n = 0,1, ...,N-1 , by using Taylor theorem we get

$$\begin{aligned} Y_1(t_{n+1};r) &= Y_1(t_n;r) + \tfrac{h}{2}F^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{h^3}{4}Y^{(3)}(\xi_{n,1}), \\ Y_2(t_{n+1};r) &= Y_2(t_n;r) + \tfrac{h}{2}G^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{h^3}{4}Y^{(3)}(\xi_{n,2}), \end{aligned} \tag{2.3.12}$$

where $\xi_{n,1}, \xi_{n,2} \in (t_n, t_{n+1})$. Denote

$$W_n = Y_1(t_n;r) - y_1(t_n;r),$$

$$V_n = Y_2(t_n;r) - y_2(t_n;r),$$

we have from (2.3.11) and (2.3.12)

$$W_{n+1} = W_n + \frac{h}{2}\{F^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] - F^*[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{h^3}{4}Y^{(3)}(\xi_{n,1}),$$

$$V_{n+1} = V_n + \frac{h}{2}\{G^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] - G^*[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{h^3}{4}Y^{(3)}(\xi_{n,2}).$$

Then

$$|W_{n+1}| \le |W_n| + Lh.max\{|W_n|, |V_n|\} + \frac{h^3}{4}M_1,$$

$$|V_{n+1}| \le |V_n| + Lh.max\{|W_n|, |V_n|\} + \frac{h^3}{4}M_2,$$

where

$$M_1 = \max|Y_1^{(3)}(t;r)| \quad , \quad M_2 = \max|Y_2^{(3)}(t;r)|,$$

for $t \in [0,T]$ and $L > 0$ is a bound for the partial derivatives of F and G. Thus by lemma 4.5.4

$$|W_n| \le (1+2Lh)^n|U_0| + \frac{h^3}{2}M\frac{(1+2Lh)^n - 1}{2Lh},$$

$$|V_n| \leq (1+2Lh)^n|U_0| + \frac{h^3}{2}M\frac{(1+2Lh)^n - 1}{2Lh}$$

where $|U_0| = |W_0| + |V_0|$ and $M = \max\{M_1, M_2\}$. In particular

$$|W_n| \leq (1+2Lh)^N|U_0| + h^2M\frac{(1+2Lh)^{\frac{T}{h}} - 1}{4L},$$

$$|V_n| \leq (1+2Lh)^N|U_0| + h^2M\frac{(1+2Lh)^{\frac{T}{h}} - 1}{4L}.$$

Since $W_0 = V_0 = 0$, we obtain

$$|W_N| \leq h^2M\frac{e^{2LT} - 1}{4L},$$

$$|V_N| \leq h^2M\frac{e^{2LT} - 1}{4L},$$

and if $h \to 0$ we get $W_N \to 0$ and $V_N \to 0$ which completes the proof. □

2.3.1 Examples

Example 2.3.1. *Consider the example 2.2.1.*

Using Runge-Kutta method of order two we have

$$y_1(t_{n+1}; r) = y_1(t_n; r)[1 + h + \frac{h^2}{2}],$$

$$y_2(t_{n+1}; r) = y_2(t_n; r)[1 + h + \frac{h^2}{2}].$$

The exact and approximate solutions are plotted and compared at $t = 1$ in Figure 2.3.1.

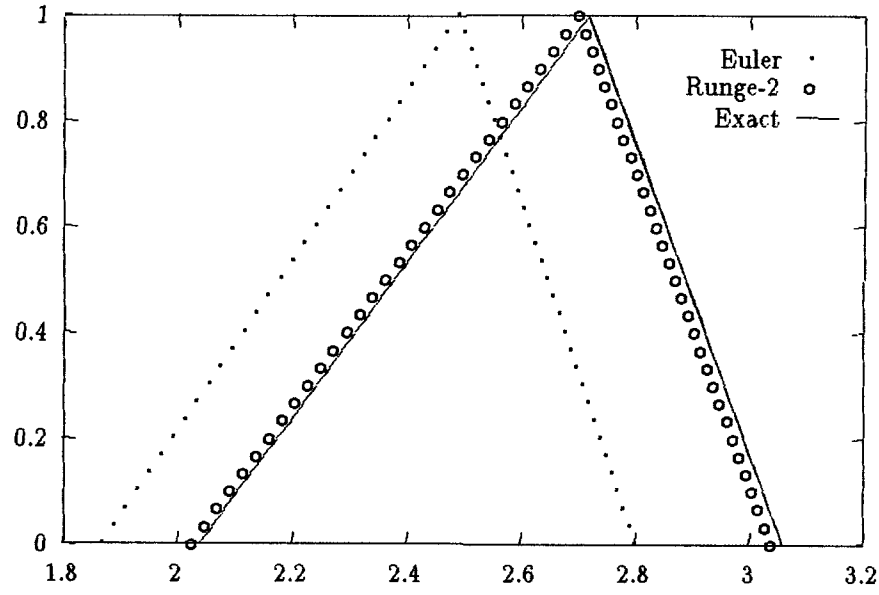


Figure 2.3.1, $h = .5$

Example 2.3.2. *Consider the example (2.2.1).*

By using Runge-Kutta method of order two at $t_n, 0 \leq n \leq N$ we have

$$k_{1,1}(t_n;r) = min\{t.u|u \in [y_1(t_n;r), y_2(t_n;r)]\},$$

$$k_{1,2}(t_n;r) = max\{t.u|u \in [y_1(t_n;r), y_2(t_n;r)]\},$$

$$k_{2,1}(t_n;r) = min\{(t+\frac{h}{2}).u|u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},$$

$$k_{2,2}(t_n;r) = max\{(t+\frac{h}{2}).u|u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},$$

where

$$z_{1,1}(t_n;r) = y_1(t_n;r) + \frac{h}{2}k_{1,1}(t_n;r), \quad z_{1,2}(t_n;r) = y_2(t_n;r) + \frac{h}{2}k_{1,2}(t_n;r).$$

By considering $t > 0$ and $t < 0$, the above minimization and maximization problems can be solved by GAMS software. The exact and approximate solutions are compared and plotted in figure 2.3.2.

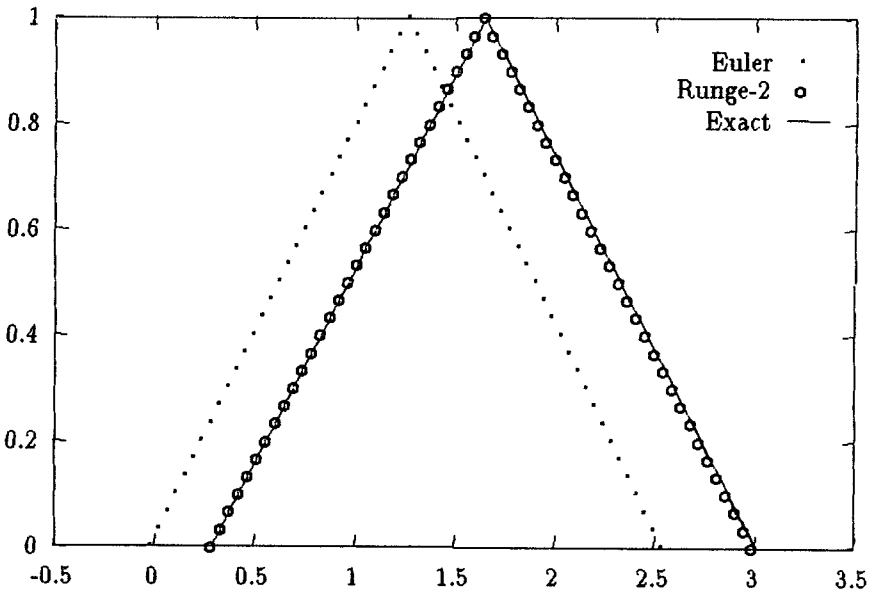


figure 2.3.2, $h = .4$

Example 2.3.3. *Consider the example 2.2.3.*

By using Runge-Kutta method at $t_n, 0 \leq n \leq N$

$$k_{1,1}(t_n; r) = (c_{1,1}(r).y_1^2(t_n; r) + c_{2,1}(r))$$

$$k_{1,2}(t_n; r) = (c_{1,2}(r).y_2^2(t_n; r) + c_{2,2}(r))$$

$$k_{2,1}(t_n; r) = (c_{1,1}(r).z_{1,1}^2(t_n; r) + c_{2,1}(r))$$

$$k_{2,2}(t_n; r) = (c_{1,2}(r).z_{1,2}^2(t_n; r) + c_{2,2}(r))$$

where

$$z_{1,1}(t_n; r) = y_1(t_n; r) + \frac{h}{2}k_{1,1}(t_n; r), \quad z_{1,2}(t_n; r) = y_2(t_n; r) + \frac{h}{2}k_{1,2}(t_n; r).$$

There are several nonlinear programming and can be solved by GAMS software. Thus the suggested Runge-Kutta method of order two of this paper can be used. The exact and approximate solutions are shown in Figure 2.3.3 at $t = 1$.

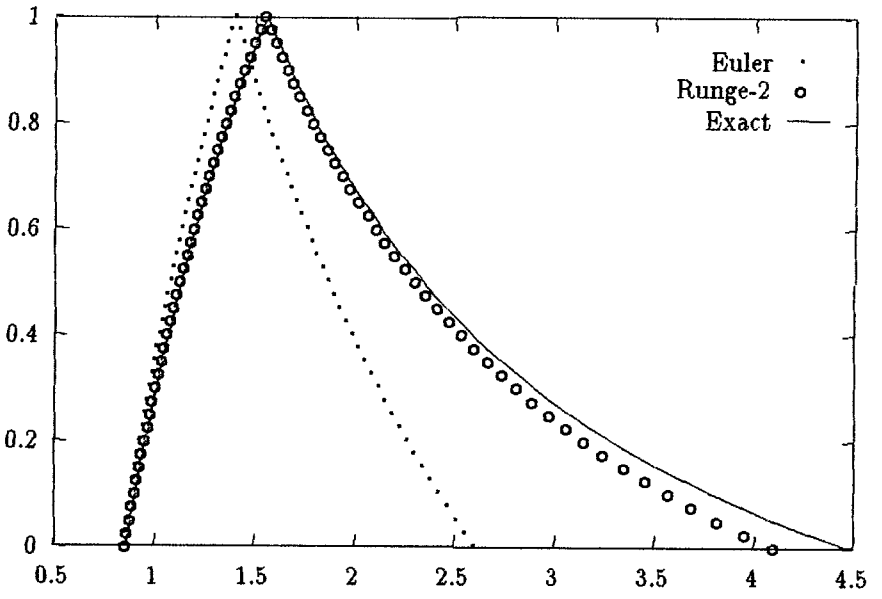


Figure 2.3.3, $h = .5$

2.4 Runge-Kutta method of order four

The famous nonzero constants α_i , β_{ij} in 4th order Runge-Kutta method are

$$\alpha_1 = 0 \ , \ \alpha_2 = \alpha_3 = 1/2 \ , \ \alpha_4 = 1 \ , \ \beta_{21} = 1/2 \ , \ \beta_{32} = 1/2 \ , \ \beta_{43} = 1,$$

and we have, see [21]

$$\begin{aligned} y_0 &= \alpha, \\ k_1 &= f(t_i, y_i), \\ k_2 &= f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2}k_1), \\ k_3 &= f(t_i + \tfrac{h}{2}, y_i + \tfrac{h}{2}k_2), \\ k_4 &= f(t_i + h, y_i + hk_3), \\ y_{i+1} &= y_i + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{aligned} \tag{2.4.1}$$

where

$$a = t_0 \leq t_1 \leq \ldots \leq t_N = b \text{ and } h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{2.4.2}$$

Theorem 2.4.1. *Let $f(t,y)$ belong to $C^4[a,b]$ and let it's partial derivatives are bounded and assume there exist, positive numbers, P, M, where*

$$|f(t,y)| < M \ , \ |\frac{\partial^{i+j} f}{\partial t^i \partial y^j}| < \frac{P^{i+j}}{M^{j-1}} \ , \ i+j \leq m$$

then in the 4th order Runge-Kutta method $y(t_{i+1}) - y_{i+1} = \frac{73}{720} h^5 M P^4 + O(h^6)$, See [21].

In this method from (2.3.1),(2.3.2) we define

$$\begin{aligned} y_1(t_{n+1};r) - y_1(t_n;r) &= h \textstyle\sum_{i=1}^{4} w_i k_{i,1}(t_n, y(t_n;r)), \\ y_2(t_{n+1};r) - y_2(t_n;r) &= h \textstyle\sum_{i=1}^{4} w_i k_{i,2}(t_n, y(t_n;r)), \end{aligned} \tag{2.4.3}$$

where the w_i's are constants and

$$\begin{aligned} [k_i(t, y(t;r))]_r &= [k_{i,1}(t, y(t;r)), k_{i,2}(t, y(t;r))], \quad i = 1,2,3,4 \\ k_{i,1}(t_n, y(t_n;r)) &= f(t_n + \alpha_i h, y_1(t_n) + h \textstyle\sum_{j=1}^{i-1} \beta_{ij} k_{j,1}(t_n, y(t_n;r))), \\ k_{i,2}(t_n, y(t_n;r)) &= f(t_n + \alpha_i h, y_2(t_n) + h \textstyle\sum_{j=1}^{i-1} \beta_{ij} k_{j,2}(t_n, y(t_n;r))), \end{aligned} \tag{2.4.4}$$

and

$$
\begin{aligned}
k_{1,1}(t, y(t;r)) &= min\{f(t,u)|u \in [y_1(t;r), y_2(t;r)]\},\\
k_{1,2}(t, y(t;r)) &= max\{f(t,u)|u \in [y_1(t;r), y_2(t;r)]\},\\
k_{2,1}(t, y(t;r)) &= min\{f(t+\tfrac{h}{2}, u)|u \in [z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r))]\},\\
k_{2,2}(t, y(t;r)) &= max\{f(t+\tfrac{h}{2}, u)|u \in [z_{1,1}(t, y(t;r)), z_{1,2}(t, y(t;r))]\},\\
k_{3,1}(t, y(t;r)) &= min\{f(t+\tfrac{h}{2}, u)|u \in [z_{2,1}(t, y(t;r)), z_{2,2}(t, y(t;r))]\},\\
k_{3,2}(t, y(t;r)) &= max\{f(t+\tfrac{h}{2}, u)|u \in [z_{2,1}(t, y(t;r)), z_{2,2}(t, y(t;r))]\},\\
k_{4,1}(t, y(t;r)) &= min\{f(t+h, u)|u \in [z_{3,1}(t, y(t;r)), z_{3,2}(t, y(t;r))]\},\\
k_{4,2}(t, y(t;r)) &= max\{f(t+h, u)|u \in [z_{3,1}(t, y(t;r)), z_{3,2}(t, y(t;r))]\}.
\end{aligned}
\tag{2.4.5}
$$

Where in the 4th order Rung-Kutta method,

$$
\begin{aligned}
z_{1,1}(t, y(t;r)) &= y_1(t;r) + \tfrac{h}{2}k_{1,1}(t, y(t;r)), \quad z_{1,2}(t, y(t;r)) = y_2(t;r) + \tfrac{h}{2}k_{1,2}(t, y(t;r)),\\
z_{2,1}(t, y(t;r)) &= y_1(t;r) + \tfrac{h}{2}k_{2,1}(t, y(t;r)), \quad z_{2,2}(t, y(t;r)) = y_2(t;r) + \tfrac{h}{2}k_{2,2}(t, y(t;r)),\\
z_{3,1}(t, y(t;r)) &= y_1(t;r) + hk_{3,1}(t, y(t;r)), \quad z_{3,2}(t, y(t;r)) = y_2(t;r) + hk_{3,2}(t, y(t;r)).
\end{aligned}
\tag{2.4.6}
$$

Define,

$$
\begin{aligned}
F[t, y(t;r)] = &\quad k_{1,1}(t, y(t;r)) + 2k_{2,1}(t, y(t;r)) + 2k_{3,1}(t, y(t;r)) + k_{4,1}(t, y(t;r)),\\
G[t, y(t;r)] = &\quad k_{1,2}(t, y(t;r)) + 2k_{2,2}(t, y(t;r)) + 2k_{3,2}(t, y(t;r)) + k_{4,2}(t, y(t;r)).
\end{aligned}
\tag{2.4.7}
$$

The exact and approximate solutions of (1.5.1) at $t_n, 0 \leq n \leq N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)]$ and $[y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)]$, respectively. The solution is calculated at points t_n. By (2.4.3) and (2.4.4) we have

$$
\begin{aligned}
Y_1(t_{n+1};r) &\approx Y_1(t_n;r) + \tfrac{h}{6}F[t_n, Y(t_n;r)],\\
Y_2(t_{n+1};r) &\approx Y_2(t_n;r) + \tfrac{h}{6}G[t_n, Y(t_n;r)].
\end{aligned}
\tag{2.4.8}
$$

We define

$$\begin{aligned} y_1(t_{n+1};r) &= y_1(t_n;r) + \tfrac{h}{6}F[t_n, y(t_n;r)], \\ y_2(t_{n+1};r) &= y_2(t_n;r) + \tfrac{h}{6}G[t_n, y(t_n;r)]. \end{aligned} \tag{2.4.9}$$

The lemmas (4.5.3) and (4.5.4) and (2.4.1) will be applied to show convergence of these approximations i.e.,

$$\begin{aligned} \lim_{h\to 0} y_1(t,h;r) &= Y_1(t;r), \\ \lim_{h\to 0} y_2(t,h;r) &= Y_2(t;r). \end{aligned}$$

Let $F^*(t,u,v)$ and $G^*(t,u,v)$ are obtained by substituting $[y(t)]_r = [u,v]$ in (2.4.7),

$$\begin{aligned} F^*(t,u,v) &= k_{1,1}(t,u,v) + 2k_{2,1}(t,u,v) + 2k_{3,1}(t,u,v) + k_{4,1}(t,u,v), \\ G^*(t,u,v) &= k_{1,2}(t,u,v) + 2k_{2,2}(t,u,v) + 2k_{3,2}(t,u,v) + k_{4,2}(t,u,v). \end{aligned} \tag{2.4.10}$$

The domain where F^* and G^* are defined is therefore

$$K = \{(t,u,v) |\ 0 \le t \le T \ ,\ -\infty < v < \infty \ ,\ -\infty < u \le v\}.$$

Theorem 2.4.2. *Let $F^*(t,u,v)$ and $G^*(t,u,v)$ belong to $C^4(K)$ and let the partial derivatives of F^* and G^* be bounded over K. Then, for arbitrary fixed $r, 0 \le r \le 1$, the approximate solutions (1.5.1) converge to the exact solutions $Y_1(t;r)$ and $Y_2(t;r)$ uniformly in t.*

Proof. It is sufficient to show

$$\begin{aligned} \lim_{h\to 0} y_1(t_N,h;r) &= Y_1(t_N;r), \\ \lim_{h\to 0} y_2(t_N,h;r) &= Y_2(t_N;r), \end{aligned}$$

where $t_N = T$. For n = 0,1, ...,N-1 , by using Taylor theorem we get

$$\begin{aligned} Y_1(t_{n+1};r) &= Y_1(t_n;r) + \tfrac{h}{6}F^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{73}{720}h^5MP^4 + O(h^6), \\ Y_2(t_{n+1};r) &= Y_2(t_n;r) + \tfrac{h}{6}G^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \tfrac{73}{720}h^5MP^4 + O(h^6), \end{aligned} \tag{2.4.11}$$

denote

$$W_n = Y_1(t_n;r) - y_1(t_n;r),$$

$$V_n = Y_2(t_n;r) - y_2(t_n;r),$$

we have from (2.4.9) and (2.4.11)

$$W_{n+1} \approx W_n + \frac{h}{6}\{F^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] - F^*[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{73}{720}h^5MP^4 + O(h^6),$$

$$V_{n+1} \approx V_n + \frac{h}{6}\{G^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] - G^*[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{73}{720}h^5MP^4 + O(h^6).$$

Then

$$|W_{n+1}| \le |W_n| + \frac{1}{3}Lh.max\{|W_n|, |V_n|\} + \frac{73}{720}h^5MP^4 + O(h^6),$$

$$|V_{n+1}| \le |V_n| + \frac{1}{3}Lh.max\{|W_n|, |V_n|\} + \frac{73}{720}h^5MP^4 + O(h^6),$$

for $t \in [0,T]$ and $L > 0$ is a bound for the partial derivatives of F^* and G^*. Thus by lemma (2.4.1)

$$|W_n| \le (1 + \frac{2}{3}Lh)^n|U_0| + (\frac{73}{360}h^5MP^4 + O(h^6))\frac{(1 + \frac{2}{3}Lh)^n - 1}{\frac{2}{3}Lh},$$

$$|V_n| \le (1 + \frac{2}{3}Lh)^n|U_0| + (\frac{73}{360}h^5MP^4 + O(h^6))\frac{(1 + \frac{2}{3}Lh)^n - 1}{\frac{2}{3}Lh},$$

where $|U_0| = |W_0| + |V_0|$. In particular

$$|W_N| \le (1 + \frac{2}{3}Lh)^N|U_0| + (\frac{73}{240}h^4MP^4 + O(h^5))\frac{(1 + \frac{2}{3}Lh)^{\frac{T}{h}} - 1}{L},$$

$$|V_N| \le (1 + \frac{2}{3}Lh)^N|U_0| + (\frac{73}{240}h^4MP^4 + O(h^5))\frac{(1 + \frac{2}{3}Lh)^{\frac{T}{h}} - 1}{L}.$$

Since $W_0 = V_0 = 0$, we obtain

$$|W_N| \le (\frac{73}{240}MP^4)\frac{e^{\frac{2}{3}LT} - 1}{L}h^4 + O(h^5),$$

$$|V_N| \le (\frac{73}{240}MP^4)\frac{e^{\frac{2}{3}LT} - 1}{L}h^4 + O(h^5),$$

and if $h \to 0$ we get $W_N \to 0$ and $V_N \to 0$ which completes the proof. □

2.4.1 Examples

Example 2.4.1. *Consider the example 2.2.1.*

By using 4th order Runge-Kutta method we have

$$y_1(t_{n+1};r) = y_1(t_n;r)[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}],$$
$$y_2(t_{n+1};r) = y_2(t_n;r)[1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \frac{h^4}{24}].$$

The exact and approximate solutions we obtained by Euler method [19], and 4th order Runge-Kutta method are compared and plotted at $t = 1$ in Figure 2.4.1.

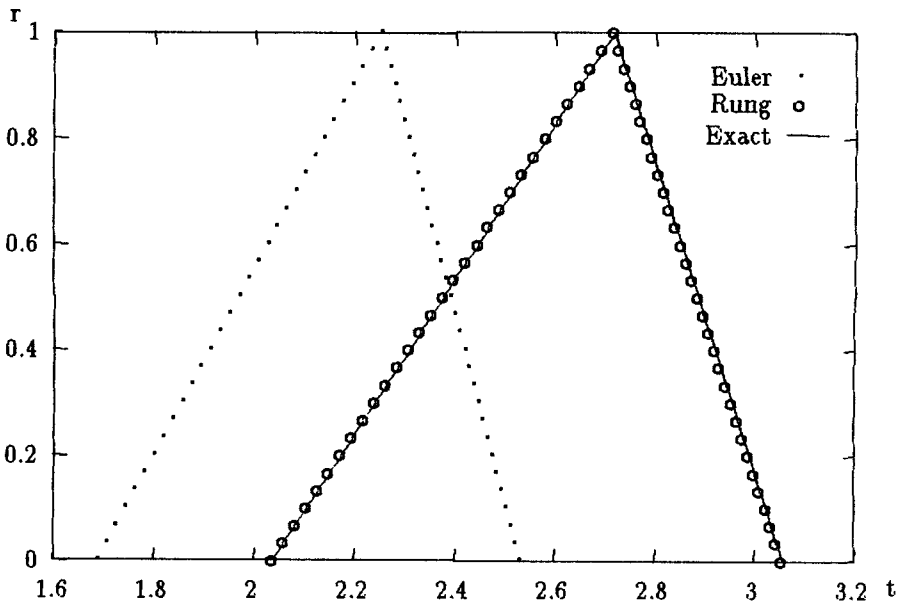


Figure 2.4.1, $h = .5$

Example 2.4.2. *Consider the example 2.2.1.*

By using 4th order Runge-Kutta method at $t_n, 0 \leq n \leq N$ we have

$$k_{1,1}(t_n;r) = min\{t.u|u \in [y_1(t_n;r), y_2(t_n;r)]\},$$

$$k_{1,2}(t_n;r) = max\{t.u|u \in [y_1(t_n;r), y_2(t_n;r)]\},$$

$$k_{2,1}(t_n;r) = min\{(t+\frac{h}{2}).u|u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},$$

$$k_{2,2}(t_n;r) = max\{(t+\frac{h}{2}).u|u \in [z_{1,1}(t_n;r), z_{1,2}(t_n;r)]\},$$

$$k_{3,1}(t_n;r) = min\{(t+\frac{h}{2}).u|u \in [z_{2,1}(t_n;r), z_{2,2}(t_n;r)]\},$$

$$k_{3,2}(t_n;r) = max\{(t+\frac{h}{2}).u|u \in [z_{2,1}(t_n;r), z_{2,2}(t_n;r)]\},$$

$$k_{4,1}(t_n;r) = min\{(t+h).u|u \in [z_{3,1}(t_n;r), z_{3,2}(t_n;r)]\},$$

$$k_{4,2}(t_n;r) = max\{(t+h).u|u \in [z_{3,1}(t_n;r), z_{3,2}(t_n;r)]\}.$$

Where

$$z_{1,1}(t_n;r) = y_1(t_n;r) + \frac{h}{2}k_{1,1}(t_n;r), \quad z_{1,2}(t_n;r) = y_2(t_n;r) + \frac{h}{2}k_{1,2}(t_n;r),$$

$$z_{2,1}(t_n;r) = y_1(t_n;r) + \frac{h}{2}k_{2,1}(t_n;r), \quad z_{2,2}(t_n;r) = y_2(t_n;r) + \frac{h}{2}k_{2,2}(t_n;r),$$

$$z_{3,1}(t_n;r) = y_1(t_n;r) + hk_{3,1}(t_n;r), \quad z_{3,2}(t_n;r) = y_2(t_n;r) + hk_{3,2}(t_n;r).$$

By considering $t > 0$ and $t < 0$,the above minimizing and maximizing problems can be solved by GAMS software. The exact and approximate solutions are compared and plotted in Figure 2.4.2.

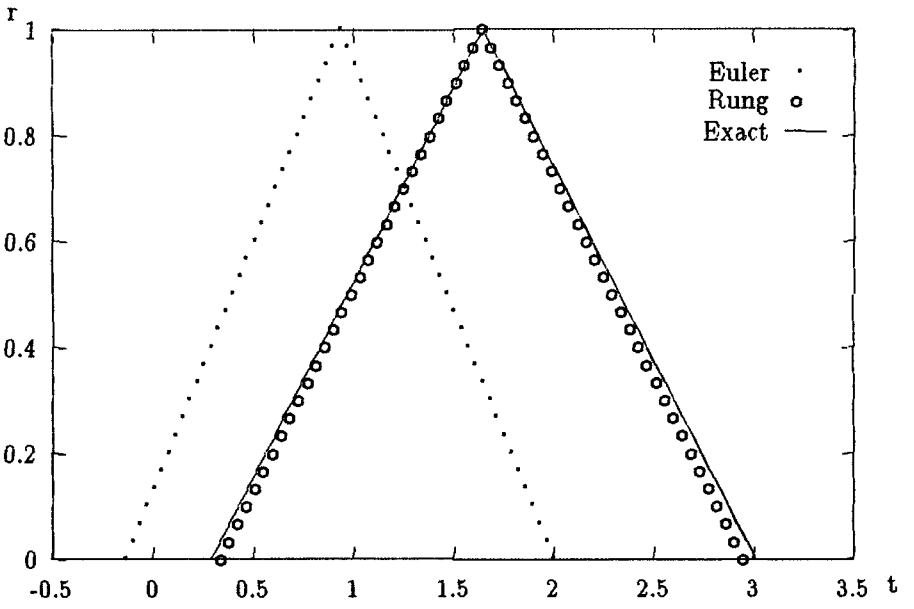


*Figure*2.4.2, $h = .4$

Example 2.4.3. *Consider the example 2.2.3.*

By using Runge-Kutta method at $t_n, 0 \leq n \leq N$

$$k_{1,1}(t_n; r) = (c_{1,1}(r).y_1^2(t_n; r) + c_{2,1}(r)),$$

$$k_{1,2}(t_n; r) = (c_{1,2}(r).y_2^2(t_n; r) + c_{2,2}(r)),$$

$$k_{2,1}(t_n; r) = (c_{1,1}(r).z_{1,1}^2(t_n; r) + c_{2,1}(r)),$$

$$k_{2,2}(t_n; r) = (c_{1,2}(r).z_{1,2}^2(t_n; r) + c_{2,2}(r)),$$

$$k_{3,1}(t_n; r) = (c_{1,1}(r).z_{2,1}^2(t_n; r) + c_{2,1}(r)),$$

$$k_{3,2}(t_n; r) = (c_{1,2}(r).z_{2,2}^2(t_n; r) + c_{2,2}(r)),$$

$$k_{4,1}(t_n; r) = (c_{1,1}(r).z_{3,1}^2(t_n; r) + c_{2,1}(r)),$$

$$k_{4,2}(t_n; r) = (c_{1,2}(r).z_{3,2}^2(t_n; r) + c_{2,2}(r)).$$

Where

$$z_{1,1}(t_n;r) = y_1(t_n;r) + \frac{h}{2}k_{1,1}(t_n;r), \quad z_{1,2}(t_n;r) = y_2(t_n;r) + \frac{h}{2}k_{1,2}(t_n;r),$$

$$z_{2,1}(t_n;r) = y_1(t_n;r) + \frac{h}{2}k_{2,1}(t_n;r), \quad z_{2,2}(t_n;r) = y_2(t_n;r) + \frac{h}{2}k_{2,2}(t_n;r),$$

$$z_{3,1}(t_n;r) = y_1(t_n;r) + hk_{3,1}(t_n;r), \quad z_{3,2}(t_n;r) = y_2(t_n;r) + hk_{3,2}(t_n;r).$$

There are several nonlinear problems which can be solved by GAMS software. Thus the suggested 4th order Runge-Kutta method in this paper can be used. The exact and approximate solutions are shown in Figure 2.4.3 at $t = 1$.

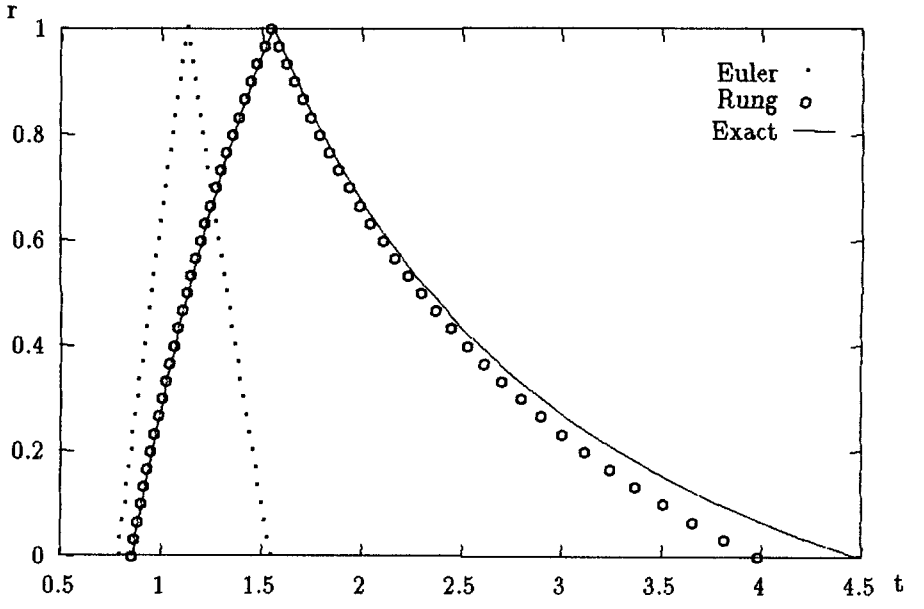


*Figure*2.4.3, $h = .5$

2.5 Adams-Bashforth two-step method

In this method the approximate solution is available at each of the mesh points $t_0, t_1, \ldots, t_i$ before the approximation at t_{i+1} is obtained and because the error $|y(t_j) - Y(t_j)|$ tends to increase with j, it seems reasonable to develop methods that use these more accurate previous data when approximating the solution at t_{i+1}. Methods using the approximation at more than one previous mesh point to determine the approximation at the next point are called *multistep* methods.

Definition 2.5.1. An *m-step multistep method* for solving the initial-value problem is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation, where m is an integer greater than 1:

$$\begin{aligned} y_{i+1} &= a_{m-1}y_i + a_{m-2}y_{i-1} + \ldots + a_0 y_{i+1-m} \\ &+ h\{b_m f(t_{i+1}, y_{i+1}) + b_{m-1} f(t_i, y_i) + \ldots + b_0 f(t_{i+1-m}, y_{i+1-m})\}, \end{aligned} \tag{2.5.1}$$

for $i = m-1, m, \ldots, N-1$, where $a = t_0 \leq t_1 \leq \ldots \leq t_N = b$ *and* $h = \frac{(b-a)}{N} = t_{i+1} - t_i$, the $a_0, a_1, \ldots, a_{m-1}$ and $b_0, b_1, \ldots, b_m$ are constants, and the starting values

$$y_0 = \alpha, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad \ldots, \quad y_{m-1} = \alpha_{m-1}$$

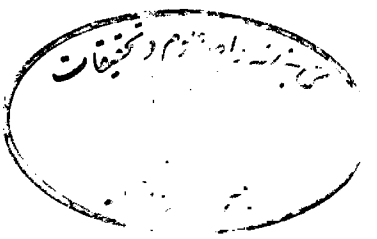
are specified. When $b_m = 0$, the method is called *explicit*, since equation (2.5.1) then gives y_{i+1} explicitly in terms of previously determined values. When $b_m \neq 0$, the method is called *implicit*, since y_{i+1} occurs on both sides of equation (2.5.1) and is specified only implicitly.

Definition 2.5.2. If $y(t)$ is the solution to the initial value problem with the equation (2.5.1) is the $(i+1)$st step in a multistep method, the *local truncation error* at this step is

$$T_{i+1}(h) = \frac{y(t_{i+1}) - a_{m-1}y(t_i) - \ldots - a_0 y(t_{i+1-m})}{h}$$

$$-\{b_m f(t_{i+1}, y_{i+1}) + b_{m-1} f(t_i, y_i) + \ldots + b_0 f(t_{i+1-m}, y_{i+1-m})\};$$

for each $i = m-1, m, \ldots, N-1$.



Adams-Bashforth two-step method is as follows

$$y_0 = \alpha, \quad y_1 = \alpha_1,$$

$$y_{i+1} = y_i + \frac{h}{2}[3f(t_i, y_i) - f(t_{i-1}, y_{i-1})],$$

where $i = 1, 2, \ldots, N-1$. The local truncation error is $T_{i+1}(h) = \frac{5}{12}y^{(3)}(\mu_{i+1})h^2$, for some $\mu_i \in (t_{i-1}, t_{i+1})$. Let the exact solution $Y(t;r) = [Y_1(t;r), Y_2(t;r)]$ is approximated by some $y(t;r) = [y_1(t;r), y_2(t;r)]$. For (1.5.1) and for $r \in (0,1]$ from definition (2.5.1) and Hukuhara difference we define

$$\begin{aligned} &y_0(r) = [\alpha_1(r), \alpha_2(r)] \quad , \quad y_1(r) = [\alpha_{1,1}(r), \alpha_{1,2}(r)] \\ &y_{i+1,1}(r) = y_{i,1}(r) + \tfrac{h}{2}[3f_1(t_i, y_i(r)) - f_1(t_{i-1}, y_{i-1}(r))], \\ &y_{i+1,2}(r) = y_{i,2}(r) + \tfrac{h}{2}[3f_2(t_i, y_i(r)) - f_2(t_{i-1}, y_{i-1}(r))], \end{aligned} \tag{2.5.2}$$

where

$$y_i(r) = [y_{i,1}(r), y_{i,2}(r)],$$

$$f_1(t_i, y_i(r)) = \min\{f(t,u) | u \in [y_{i,1}(r), y_{i,2}(r)]\},$$

$$f_2(t_i, y_i(r)) = \max\{f(t,u) | u \in [y_{i,1}(r), y_{i,2}(r)]\}.$$

Define,

$$\begin{aligned} F[t, y(r)] &= 3f_1(t_i, y_i(r)) - f_1(t_{i-1}, y_{i-1}(r)), \\ G[t, y(r)] &= 3f_2(t_i, y_i(r)) - f_2(t_{i-1}, y_{i-1}(r)). \end{aligned} \tag{2.5.3}$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $[Y(t_n)]_r = [Y_1(t_n; r), Y_2(t_n; r)]$ and $[y(t_n)]_r = [y_1(t_n; r), y_2(t_n; r)]$ respectively. The solution is calculated by points at t_{n+1}. By (2.5.2) and (2.5.3) we have

$$\begin{aligned} Y_1(t_{n+1}; r) &\approx Y_1(t_n; r) + \tfrac{h}{2}F[t_n, Y(t_n; r)], \\ Y_2(t_{n+1}; r) &\approx Y_2(t_n; r) + \tfrac{h}{2}G[t_n, Y(t_n; r)]. \end{aligned} \tag{2.5.4}$$

We define

$$\begin{aligned} y_1(t_{n+1}; r) &= y_1(t_n; r) + \tfrac{h}{2}F[t_n, y(t_n; r)], \\ y_2(t_{n+1}; r) &= y_2(t_n; r) + \tfrac{h}{2}G[t_n, y(t_n; r)]. \end{aligned} \tag{2.5.5}$$

The lemmas (4.5.3) and (4.5.4) will be applied to show convergence of these approximates i.e.,

$$\begin{aligned} \lim_{h \to 0} y_1(t, h; r) &= Y_1(t; r), \\ \lim_{h \to 0} y_2(t, h; r) &= Y_2(t; r). \end{aligned}$$

Let $F^*(t, u, v)$ and $G^*(t, u, v)$ are obtained by substituting $[y(t)]_r = [u, v]$ in (2.5.3)

$$\begin{aligned} F^*(t, u, v) &= 3f_1(t_i, u, v) - f_1(t_{i-1}, u, v), \\ G^*(t, u, v) &= 3f_1(t_i, u, v) - f_1(t_{i-1}, u, v). \end{aligned}$$

The domain where F^* and G^* are defined is therefore

$$K = \{(t,u,v) | \ 0 \leq t \leq T \ , \ -\infty < v < \infty \ , \ -\infty < u \leq v\}.$$

Theorem 2.5.1. *Let $F^*(t,u,v)$ and $G^*(t,u,v)$ belong to $C^2(K)$ and let the partial derivatives of F^* and G^* be bounded over K. Then, for arbitrary fixed $r, 0 \leq r \leq 1$, the approximately solutions (1.5.1) converge to the exact solutions $Y_1(t;r)$ and $Y_2(t;r)$ uniformly in t.*

Proof. It is sufficient to show

$$\lim_{h\to 0} y_1(t_N;r) = \ Y_1(t_N;r),$$

$$\lim_{h\to 0} y_2(t_N;r) = \ Y_2(t_N;r),$$

where $t_N = T$. For n = 0,1, ...,N-1, by using definition (2.5.2) we get

$$\begin{aligned} Y_1(t_{n+1};r) = \ & Y_1(t_n;r) + \tfrac{h}{2}F^*(t_n, Y_1(t_n;r), Y_2(t_n;r)) + \tfrac{5h^3}{12}Y^{(3)}(\xi_{n,1}), \\ Y_2(t_{n+1};r) = \ & Y_2(t_n;r) + \tfrac{h}{2}G^*(t_n, Y_1(t_n;r), Y_2(t_n;r)) + \tfrac{5h^3}{12}Y^{(3)}(\xi_{n,2}), \end{aligned} \tag{2.5.6}$$

where $\xi_{n,1}, \xi_{n,2} \in (t_n, t_{n+1})$. Denote

$$W_n = Y_1(t_n;r) - y_1(t_n;r),$$

$$V_n = Y_2(t_n;r) - y_2(t_n;r),$$

we have from (2.5.6) and (2.5.5)

$$W_{n+1} = W_n + \frac{h}{2}\{F^*(t_n, Y_1(t_n;r), Y_2(t_n;r)) - F^*(t_n, y_1(t_n;r), y_2(t_n;r))\} + \frac{5h^3}{12}Y^{(3)}(\xi_{n,1}),$$

$$V_{n+1} = V_n + \frac{h}{2}\{G^*(t_n, Y_1(t_n;r), Y_2(t_n;r)) - G^*(t_n, y_1(t_n;r), y_2(t_n;r))\} + \frac{5h^3}{12}Y^{(3)}(\xi_{n,2}).$$

Then

$$|W_{n+1}| \leq |W_n| + Lh.max\{|W_n|, |V_n|\} + \frac{5h^3}{12}M_1,$$

$$|V_{n+1}| \leq |V_n| + Lh.max\{|W_n|, |V_n|\} + \frac{5h^3}{12}M_2,$$

where

$$M_1 = \max|Y_1^{(3)}(t;r)| \quad , \quad M_2 = \max|Y_2^{(3)}(t;r)|$$

for $t \in [0,T]$ and $L > 0$ is a bound for the partial derivatives of F^* and G^*. Thus by lemma 4.5.4

$$|W_n| \leq (1+2Lh)^n|U_0| + \frac{5h^3}{6}M\frac{(1+2Lh)^n - 1}{2Lh},$$

$$|V_n| \leq (1+2Lh)^n|U_0| + \frac{5h^3}{6}M\frac{(1+2Lh)^n - 1}{2Lh},$$

where $|U_0| = |W_0| + |V_0|$, $M = \max\{M_1, M_2\}$. In particular

$$|W_n| \leq (1+2Lh)^N|U_0| + \frac{5h^2}{12}M\frac{(1+2Lh)^{\frac{T}{h}} - 1}{L},$$

$$|V_n| \leq (1+2Lh)^N|U_0| + \frac{5h^2}{12}M\frac{(1+2Lh)^{\frac{T}{h}} - 1}{L},$$

Since $W_0 = V_0 = 0$, we obtain

$$|W_N| \leq \frac{5h^2}{12}M\frac{e^{2LT} - 1}{L},$$

$$|V_N| \leq \frac{5h^2}{12}M\frac{e^{2LT} - 1}{L},$$

and if $h \to 0$ we get $W_N \to 0$ and $V_N \to 0$ which completes the proof. □

2.5.1 Examples

Example 2.5.1. *Consider the example 2.2.1.*

By using Adams-Bashforth Two-Step Method we have

$$y_1(t_{n+1};r) = y_1(t_n;r)[1 + h + \frac{h^2}{2}],$$

$$y_2(t_{n+1};r) = y_2(t_n;r)[1 + h + \frac{h^2}{2}].$$

The distance between the Euler fuzzy number and the Exact fuzzy number is a fuzzy number that we called it μ and also the distance between the Adams fuzzy number and the Exact fuzzy number is a fuzzy number that we called it v. You see that

$$val(\mu) = 55.3993 \quad , \quad val(v) = 33.3503$$

The exact and approximate solutions are compared and plotted at $t = 1$ in Figure 2.5.1.

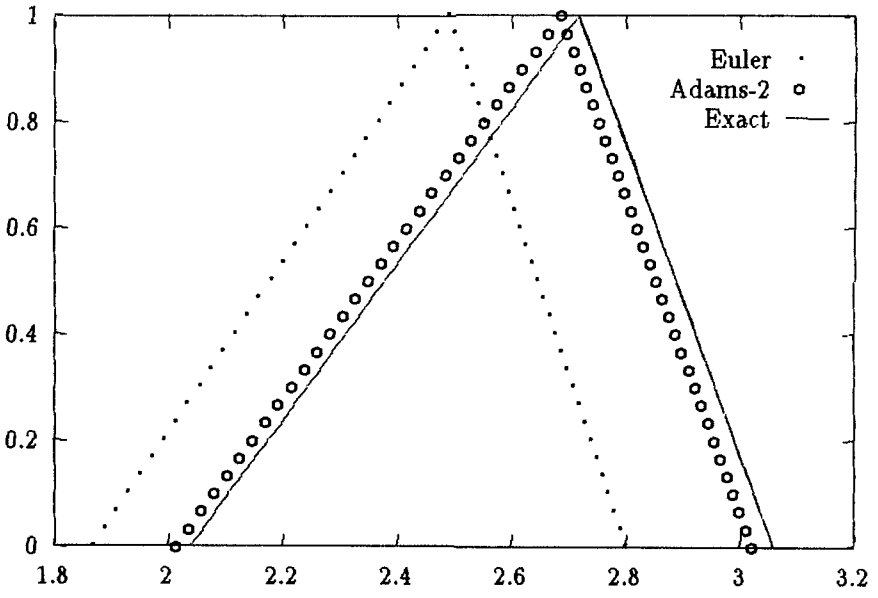


Figure 2.5.1, $h = .2$

Example 2.5.2. *Consider the example 2.2.1.*

By using Adams-Bashforth Two-Step Method at $t_n, 0 \leq n \leq N$ we have

$$f_1(t_n, y_n(r)) = min\{t.u | u \in [y_1(t_n; r), y_2(t_n; r)]\},$$

$$f_2(t_n, y_n(r)) = max\{t.u | u \in [y_1(t_n; r), y_2(t_n; r)]\}.$$

By considering $t > 0$ and $t < 0$,the above minimizing and maximizing problems can be solved by GAMS software.

$$val(\mu) = 121.7972 \ , \ val(v) = 82.2981$$

The exact and approximate solutions are compared and plotted in Figure 2.5.2.

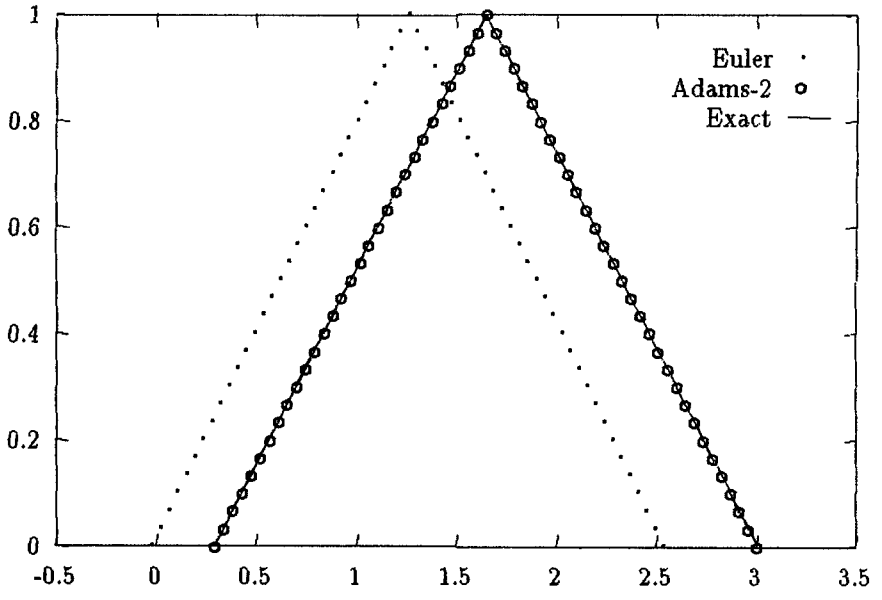


Figure 2.5.2, $h = .2$

Example 2.5.3. *Consider the example 2.2.3.*

By using Adams-Bashforth Two-Step method at $t_n, 0 \leq n \leq N$

$$f_1(t_n, y_n(r)) = (c_{1,1}(r).y_1^2(t_n; r) + c_{2,1}(r)),$$

$$f_2(t_n, y_n(r)) = (c_{1,2}(r).y_2^2(t_n; r) + c_{2,2}(r)).$$

There are several nonlinear problems which can be solved by GAMS software. Thus the suggested Adams-Bashforth Two-Step Method in this paper can be used.

$$val(\mu) = 85.8098 \ , \ val(v) = 82.7309$$

The exact and approximate solutions are shown in Figure 2.5.3 at $t = 1$.

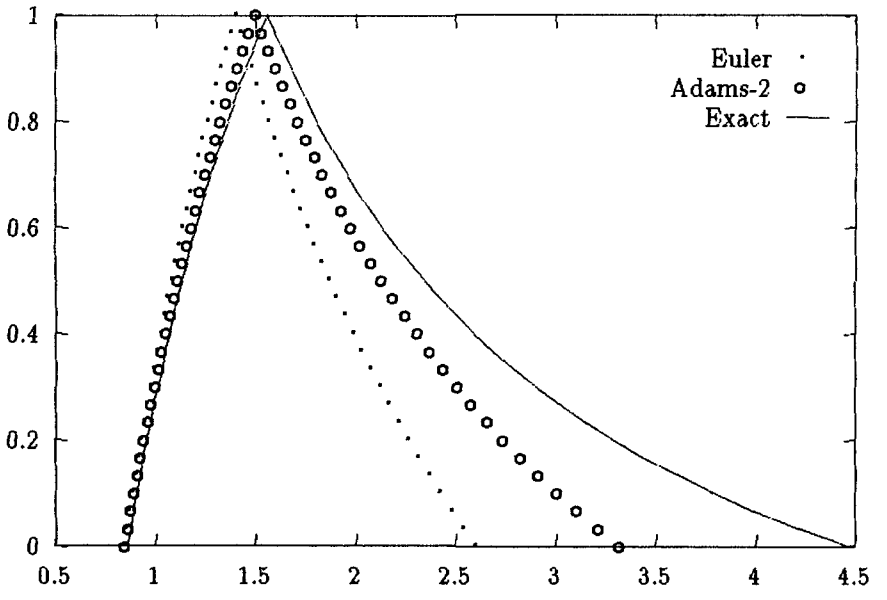


Figure 2.5.3, $h = .1$

2.6 Conclusion

We note that the convergence order of the Euler method in [19] is $O(h)$. It is shown that in Taylor method of order two and Runge-Kutta method of order two and Adams-Bashforth Two-Step method, the convergence order is $O(h^2)$ and in Runge-Kutta method of order four $O(h^4)$ and these solutions have higher accuracy.

Chapter 3

Extrapolation Method for Improving Solutions of FIVP

3.1 Introduction

In this chapter we are improving the solutions of the algorithms of chapter 2. First FIVP is solved using Euler method, then midpoint method is applied to improve the solutions.

Suppose (1.5.1) satisfies the hypothesis of theorem 1.5.3. Let the exact solution $y(t;r) = [y_1(t;r), y_2(t;r)]$ of (1.5.1) is approximated by some $w(t;r) = [w_1(t;r), w_2(t;r)]$. From [19],

$$\begin{aligned} w_1(t+h;r) &= w_1(t;r) + hF^*(t, w_1(t;r), w_2(;r)), \\ w_2(t+h;r) &= w_2(t;r) + hG^*(t, w_1(t;r), w_2(t;r)), \end{aligned} \tag{3.1.1}$$

$$\begin{aligned} y_1(t+h;r) &\approx y_1(t;r) + hF^*(t, y_1(t;r), y_2(;r)), \\ y_2(t+h;r) &\approx y_2(t;r) + hG^*(t, y_1(t;r), y_2(t;r)), \end{aligned} \tag{3.1.2}$$

where

$$\begin{aligned} F^*(t, y(t;r)) &= y(t;r), \\ G^*(t, y(t;r)) &= y(t;r). \end{aligned} \tag{3.1.3}$$

The following theorem will be applied to show convergence of these approximates i.e.,

$$\begin{aligned} \lim_{h\to 0} w_1(t,h;r) &= y_1(t;r), \\ \lim_{h\to 0} w_2(t,h;r) &= y_2(t;r). \end{aligned}$$

Let

$$K = \{(t,u,v) |\ 0 \le t \le T \ ,\ -\infty < v < \infty \ ,\ -\infty < u \le v\}.$$

Theorem 3.1.1. *Let $F^*(t,u,v)$ and $G^*(t,u,v)$ are obtained by substituting $[y(t)]_r = [u,v]$ in 3.1.3 be the functions belong to $C^1(K)$, where u and v are constants and $u \le v$. Then, for arbitrary fixed $r : 0 \le r \le 1$, the approximate solutions (1.5.1) converge to the exact solutions $y_1(t;r)$ and $y_2(t;r)$ uniformly in t,*

Proof [19].

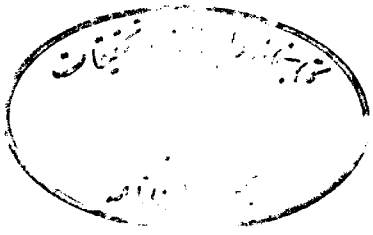
3.2 Extrapolation method

We define

$$y_0(r) = [\alpha_1(r), \alpha_2(r)],\ r \in (0,1], \tag{3.2.1}$$

let us assume that we have a fixed step size h and that we wish to approximate $y_i(t_1;r) = y_i(a+h;r)$, for any $r \in (0,1]$ and $i = 1,2$. For the first extrapolation step we let $h_0 = \frac{h}{2}$ and use (3.1.2) approximation

$$y_i(a+h_0;r) = y_i(a+\frac{h}{2};r)$$



as

$$\begin{aligned} w_{i,1}(r) &= w_{i,0}(r) + h_0 f_i(a, w_0; r), \\ w_{i,0}(r) &= \alpha_i(r), \end{aligned} \tag{3.2.2}$$

for any $r \in (0,1]$ and $i = 1,2$, where $[w_j]_r = [w_{1,j}(r), w_{2,j}(r)]$ for any j, and

$$f_1(t_n, w(t_n); r) = \min\{f(t,u) | u \in [w_1(t_n;r), w_2(t_n;r)]\},$$

$$f_2(t_n, w(t_n); r) = \max\{f(t,u) | u \in [w_1(t_n;r), w_2(t_n;r)]\}.$$

We then apply the Midpoint with $t_0 = a$ and $t_1 = a + h_0 = a + \frac{h}{2}$ to produce a first approximation to $y_i(a+h;r) = y_i(a+2h_0;r)$,

$$w_{i,2}(r) = w_{i,0}(r) + 2h_0 f_i(a+h_0, w_1; r),$$

for any $r \in (0,1]$ and $i = 1,2$. The endpoint correction is applied to obtain the final approximation to $y_i(a+h;r)$ for any $r \in (0,1]$ and $i = 1,2$, for the stepsize h_0. This results in the $O(h_0^2)$ approximation to $y_i(t_1;r)$

$$y^1_{i,1}(r) = \frac{1}{2}[w_{i,2}(r) + w_{i,1}(r) + h_0 f_i(a+2h_0, w_2; r)],$$

for any $r \in (0,1]$ and $i = 1,2$; where $[y^1_1]_r = [y^1_{1,1}(r), y^1_{2,1}(r)]$. We save the approximation $y^1_{i,1}(r)$ and discard the intermediate results $w_{i,1}(r)$ and $w_{i,2}(r)$, for any $r \in (0,1]$ and $i = 1,2$. To obtain the next approximation, $y^1_{i,2}(r)$, to $y(t_1)$, we let $h_1 = \frac{h}{4}$ and use

Euler's method initial values to obtain an approximation to $y_i(a+h_1;r) = y_i(a+\frac{h}{4};r)$ that we will call $w_{i,1}(r)$:

$$\begin{aligned} w_{i,1}(r) &= w_{i,0}(r) + h_1 f_i(a, w_0; r), \\ w_{i,0}(r) &= \alpha_i(r), \end{aligned} \tag{3.2.3}$$

for any $r \in (0,1]$ and $i = 1,2$.

Next we produce approximations $w_{i,2}(r)$ to

$$y_i(a+2h_1;r) = y_i(a+\frac{h}{2};r) \quad and \quad y_i(a+3h_1;r) = y_i(a+\frac{3h}{4};r)$$

given by

$$w_{i,2}(r) = w_{i,0}(r) + 2h_1 f_i(a+h_1, w_1; r),$$

and

$$w_{i,3}(r) = w_{i,1}(r) + 2h_1 f_i(a+2h_1, w_2; r),$$

for any $r \in (0,1]$ and $i = 1,2$. Then we produce the approximation $w_{i,4}(r)$ to $y_i(a+4h_1;r) = y_i(t_1;r)$ given by

$$w_{i,4}(r) = w_{i,2}(r) + 2h_1 f_i(a+3h_1, w_3; r),$$

for any $r \in (0,1]$ and $i = 1,2$. The endpoint correction is now applied to $w_{i,3}(r)$ and $w_{i,4}(r)$ to produce the improved $O(h_1^2)$ approximation to $y_i(t_1;r)$,

$$y_{i,2}^1(r) = \frac{1}{2}[w_{i,4}(r) + w_{i,3}(r) + h_1 f_i(a+4h_1, w_4; r)],$$

for any $r \in (0,1]$ and $i = 1,2$.

The two approximations to $y_i(a+h;r)$ have the property that

$$y_1(a+h;r) = y_{1,1}^1(r) + e_{1,1}(r)(\frac{h}{2})^2 + e_{1,2}(r)(\frac{h}{2})^4 + \ldots = y_{1,1}^1(r) + e_{1,1}(r)\frac{h^2}{4} + e_{1,2}(r)\frac{h^4}{16} + \ldots, \tag{3.2.4}$$

$$y_2(a+h;r) = y^1_{2,1}(r)+e'_{1,1}(r)(\frac{h}{2})^2+e'_{1,2}(r)(\frac{h}{2})^4+\ldots = y^1_{2,1}(r)+e'_{1,1}(r)\frac{h^2}{4}+e'_{1,2}(r)\frac{h^4}{16}+\ldots, \tag{3.2.5}$$

and

$$y_1(a+h;r) = y^1_{1,2}(r)+e_{1,1}(r)(\frac{h}{4})^2+e_{1,2}(r)(\frac{h}{4})^4+\ldots = y^1_{1,2}(r)+e_{1,1}(r)\frac{h^2}{16}+e_{1,2}(r)\frac{h^4}{256}+\ldots, \tag{3.2.6}$$

$$y_2(a+h;r) = y^1_{2,2}(r)+e'_{2,1}(r)(\frac{h}{4})^2+e'_{2,2}(r)(\frac{h}{4})^4+\ldots = y^1_2(t_2;r)+e'_{2,1}(t;r)\frac{h^2}{16}+e'_{2,2}(t;r)\frac{h^4}{256}+\ldots, \tag{3.2.7}$$

for any $r \in (0,1]$, where $[e_j]_r = [e_{1,j}(r), e_{2,j}(r)]$. We can eliminate the $O(h^2)$ portion of this truncation error by averaging these two formulas appropriately, for any $r \in (0,1]$. Specifically, if we subtract (3.2.4) from 4 times (3.2.6) and divide the result by 3 and repeat this process for (3.2.5) and (3.2.7) we will have

$$y_1(a+h;r) = y^1_{1,2}(r) + \frac{1}{3}(y^1_{1,2}(r) - y_{1,1}(r)^1) - e_{2,1}(r)\frac{h^4}{64} + \ldots,$$

$$y_2(a+h;r) = y^1_{2,2}(r) + \frac{1}{3}(y^1_{2,2}(r) - y_{2,1}(r)^1) - e'_{2,2}(r)\frac{h^4}{64} + \ldots.$$

So the approximation

$$y^2_{i,2}(r) = y^1_{i,2}(r) + \frac{1}{3}(y^1_{i,2}(r) - y^1_{i,1}(r))$$

for any $r \in (0,1]$ and $i = 1,2$, has error of order $O(h^4)$. Continuing in this manner, we next let $h_2 = \frac{h}{6}$ and apply Euler method once followed b. Then we use the endpoint correction to determine the h^2 approximation that we denote $y^1_{i,3}(r)$, to $y_i(a+h;r)$, this approximation can be averaged with $y^1_{i,2}(r)$ to produce a second $O(h^4)$ approximation that we denote $y^2_{i,3}(r)$, for $i = 1,2$. Then $y^2_{i,3}(r)$ and $y^2_{i,2}(r)$ are averaged to eliminate the $O(h^4)$ error terms and produce an approximation with error of order $O(h^6)$. Higher-order formulas are generated by continuing the process. The

error is controlled by requiring that the approximations $y^1_{i,1}(r), y^2_{i,2}(r), \ldots$ be computed until $|y^i_{1,i}(r) - y^{i-1}_{1,i-1}(r)|$ and $|y^i_{2,i}(r) - y^{i-1}_{2,i-1}(r)|$ is less than a given ϵ (for example $\epsilon = 0.001$), for $i = 1, 2$. If $y^i_{1,i}(r)$ and $y^i_{2,i}(r)$ are found to be acceptable, then $w_{1,1}(r)$ and $w_{2,1}(r)$ is set to $y^i_{1,i}(r)$ and $y^i_{2,i}(r)$ respectively, and computations begin again to determine $[w_{1,2}(r), w_{2,2}(r)]$ which will approximate $y_{i,2}(r) = y_i(a + 2h; r)$, for any $r \in (0, 1]$, and $i = 1, 2$. The process is repeated until the approximation $w_{1,N}(r)$ and $w_{2,N}(r)$ to $y_1(b; r)$ and $y_2(b; r)$, for any $r \in (0, 1]$.

3.3 Examples

Example 3.3.1. *Consider the fuzzy initial value problem, [23].*

$$\begin{cases} y'(t) &= [y(t)]^2, \quad t \geq 0, \\ y(0) &= (r, 1 - ln(r)), \quad 0 < r \leq 1. \end{cases}$$

The exact fuzzy solution $y(t)$ is defined on $[0, t_\beta]$ with $t_\beta = \frac{1}{1-ln(\beta)}$ by

$$[y(t)]_r = [y_1(t; r), y_2(t; r)], \quad \beta \leq r \leq 1,$$

where

$$y_1(t; r) = \frac{r}{1 - rt} \quad , \quad y_2(t; r) = \frac{1 - ln(r)}{1 - (1 - ln(r))}$$

and $t_\beta \to 0$ as $\beta \to 0^+$, [23]. The exact, Euler and Extrapolation solutions at $t = 0.2$ are compared in Table 3.3.1.

r	Euler	Extra	Exact
0.0	(0.000000,Inf)	(0.00000000,Inf)	(0.000000,Inf)
0.1	(0.102020,6.323393)	(0.102041,9.728272)	(0.102041,9.728279)
0.2	(0.208162,4.372998)	(0.208333,5.457783)	(0.208333,5.457792)
0.3	(0.318548,3.413183)	(0.319149,3.941250)	(0.319149,3.941258)
0.4	(0.433306,2.804949)	(0.434783,3.107118)	(0.434783,3.107120)
0.5	(0.552563,2.371791)	(0.555556,2.560058)	(0.555556,2.560058)
0.6	(0.676450,2.041527)	(0.681818,2.165019)	(0.681818,2.165019)
0.7	(0.805100,1.778117)	(0.813953,1.861859)	(0.813953,1.861864)
0.8	(0.938650,1.561196)	(0.952381,1.619256)	(0.952381,1.619261)
0.9	(1.077236,1.378229)	(1.097560,1.419077)	(1.097561,1.419079)
1	(1.221000,1.221000)	(1.249999,1.249999)	(1.250000,1.250000)

Table 3.3.1, $t_\beta = 0.2, h = .1$

Example 3.3.2. *Consider the example 2.2.1.*

Using the Euler approximation with $N = 10$ we obtain

$$w_1(1;r) = (.75 + .25r)(1 + \frac{1}{10})^{10},$$

$$w_2(1;r) = (1.125 - .125r)(1 + \frac{1}{10})^{10}.$$

The distance between the Euler fuzzy number and the Exact fuzzy number is a fuzzy number that we called it μ and also the distance between the Extrapolation method fuzzy number and the Exact fuzzy number is a fuzzy number that we called it v. You see that

$$val(\mu) = 2.7841 \ , \ \ val(v) = 5.6646e - 04$$

The exact and Euler and Extrapolation solutions at the end point are compared in Table 3.3.2.

r	Euler	Extra	Exact
0.0	(1.687500,2.531250)	(2.038710,3.058064)	(2.03871,3.058067)
0.1	(1.743750,2.503125)	(2.106667,3.024086)	(2.106668,3.024089)
0.2	(1.800000,2.475000)	(2.174623,2.990107)	(2.174625,2.990110)
0.3	(1.856250,2.446875)	(2.242580,2.956129)	(2.242583,2.956131)
0.4	(1.912500,2.418750)	(2.310537,2.922150)	(2.310540,2.922153)
0.5	(1.968750,2.390625)	(2.378494,2.888172)	(2.378497,2.888174)
0.6	(2.025000,2.362500)	(2.446451,2.854193)	(2.446454,2.854196)
0.7	(2.081250,2.334375)	(2.514408,2.820215)	(2.514411,2.820217)
0.8	(2.137500,2.306250)	(2.582365,2.786236)	(2.582368,2.786239)
0.9	(2.193750,2.278125)	(2.650322,2.752258)	(2.650325,2.752260)
1	(2.250000,2.250000)	(2.718279,2.718279)	(2.718282,2.718282)

Table 3.3.2, $h = .5$

Example 3.3.3. *Consider the example 2.2.3.*

The exact and Euler's and Extrapolation solutions are compared in table 3.3.3.

$$val(\mu) = 4.8290 \quad , \quad val(v) = 4.5673e - 015$$

r	Euler	Extra	Exact
0.0	(0.839847,2.597979)	(0.860330,4.469118)	(0.860329,4.469125)
0.1	(0.882277,2.419006)	(0.907805,3.788140)	(0.907805,3.788162)
0.2	(0.926897,2.258159)	(0.958504,3.285733)	(0.958504,3.285743)
0.3	(0.973930,2.112983)	(1.012873,2.899134)	(1.012873,2.899146)
0.4	(1.023628,1.981418)	(1.071439,2.591938)	(1.071439,2.591944)
0.5	(1.076273,1.861721)	(1.134832,2.341526)	(1.134832,2.341533)
0.6	(1.132181,1.752413)	(1.203806,2.133136)	(1.203806,2.133143)
0.7	(1.191712,1.652232)	(1.279281,1.956710)	(1.279281,1.956714)
0.8	(1.255272,1.560100)	(1.362381,1.805153)	(1.362381,1.805155)
0.9	(1.323323,1.475088)	(1.454505,1.673324)	(1.454505,1.673325)
1	(1.396394,1.396394)	(1.557407,1.557407)	(1.557408,1.557408)

Table 3.3.3, $h = 0.1$

3.4 Conclusion

The order of convergence of Euler's Method is $O(h)$ that we can see in [11]. Now in this method we can get the higher-order of convergence, so that $|y_1^i(t_i;r) - y_1^{i-1}(t_{i-1};r)|$ and $|y_2^i(t_i;r) - y_2^{i-1}(t_{i-1};r)|$ are less than a given tolerance. Therefore, we can improve the solution of FIVP arbitrary.

Chapter 4

Numerical Solutions of Fuzzy Differential Inclusions

Differential equations in a fuzzy environment have been suggested as a way of modeling uncertain and incompletely specified systems. Formulation of the concept usually interprets the solution as a flow on some appropriate space of fuzzy sets and has been largely concerned with existence and uniqueness problems.

4.1 Introduction

In this chapter we are going to *operationalize* our approach, i.e. we are going to propose a method for computing the set of approximations to all solutions to a FIVP using numerical methods. Also since finding this set of solutions analytically does only work with trivial examples, a numerical approach seems to be the only way of

solving such problems.

4.2 Euler method

Consider (1.6.3) and let

$$F(t, x_r(t); r) = [f(t, x(t))]_r \in \kappa_c^n, \quad x_r(t) \in [X(t)]_r, \quad r \in [0,1].$$

and

$$f = (f^1, \ldots, f^n)^t \quad , \quad f_1 = (f_1^1, \ldots, f_1^n)^t \quad , \quad f_2 = (f_2^1, \ldots, f_2^n)^t.$$

We construct n families of r-parameterized interval-valued mappings

$$f^k : I \times \mathrm{E}^n \to [f^k(t, x_r(t); r), f^k(t, x_r(t); r)]$$

in the following way:

$$\begin{cases} f_1^k(t, x_r(t); r) = f^k(t, U_{min}^k; r) = \min\{f^k(t, U) : U \in [X(t)]_r\}, \\ f_2^k(t, x_r(t); r) = f^k(t, U_{max}^k; r) = \max\{f^k(t, U) : U \in [X(t)]_r\}, \quad k = 1, \ldots, n, \quad r \in [0,1]. \end{cases} \tag{4.2.1}$$

Obviously the *fuzzy set valued function* $F : \Omega \times [0,1] \to \kappa_c^n$ is as follows;

$$\begin{aligned} F(t, x_r(t); r) &= \textstyle\prod_{k=1}^n [f_1^k(t, x_r(t); r), f_2^k(t, x_r(t); r)] \\ &= [f_1(t, x_r(t); r), f_2(t, x_r(t); r)] \in \kappa_c^n, \quad x_r(t) \in [X(t)]_r, \quad 0 \le r \le 1, \end{aligned} \tag{4.2.2}$$

this means

$$\begin{aligned} &\{(f^1(t, x_r(t); r), \ldots, f^n(t, x_r(t); r)) : f^i(t, x_r(t); r) \in \\ &[f_1^i(t, x_r(t); r), f_2^i(t, x_r(t); r)], \quad i = 1, \ldots, n\} = F(t, x_r(t); r), \quad r \in [0,1], \end{aligned} \tag{4.2.3}$$

and also

$$[Y_0]_r = \prod_{k=1}^n [y_1^k(0; r), y_2^k(0; r)]$$

is the surface of n-dimensional rectangles for any $r \in [0,1]$ where

$$Y_1(0;r) = (y_1^1(0;r), \ldots, y_1^n(0;r)) \quad , \quad Y_2(0;r) = (y_2^1(0;r), \ldots, y_2^n(0;r)).$$

Now the problem (1.6.3) is transformed to

$$x_r'(t) \in [f_1(t, x_r(t);r), f_2(t, x_r(t);r)], \quad x_r(0) \in [Y_1(0;r), Y_2(0;r)] = [Y_0]_r, \quad \forall\, r \in [0,1]. \tag{4.2.4}$$

If $x_r(0)$ chosen randomly then the selection of $x_r'(t)$ is random too that the set of all selections are formed the reachable set hence (4.2.4) converted to a crisp differential inclusions for any $r \in [0,1]$. The (4.2.4) might be stiff differential equation, then by considering the fact that R^n be equipped with the scalar product $\langle .,. \rangle$ and the corresponding induced norm $\|.\|$, we have the following theorem.

Let $x_r(t_i) \approx y_i(r)$ for all $r \in [0,1]$, then the Euler method for approximating the reachable set of problem (4.2.4) is proposed as follows:

$$\begin{aligned} &y_0(r) \in [Y_0]_r \subset R^n, \\ &y_{i+1}(r) = \bigcup_{s_r \in [Y(t_i)]_r} (s_r + hf(t_i, s_r; r)), \quad i = 0, \ldots, N-1, \quad r \in [0,1], \end{aligned} \tag{4.2.5}$$

where

$$[Y(t_i)]_r = \prod_{k=1}^{n} [y_1^k(t_i;r), y_2^k(t_i;r)], \quad i = 0, \ldots, N-1, \quad r \in [0,1].$$

Theorem 4.2.1. *Let $F \in C^1(\Omega)$ in (1.6.3) be a compact convex valued mapping such that satisfies Lipschitz condition in x with Lipschitz constant $L > 0$ and x_r be a solution of (1.6.3) then $lim_{h \to 0} y_N(r) = x_r(T)$, for any $r \in [0,1]$.*

Proof. Let, a.e.

$$x_r(t_{i+1}) = \bigcup_{\bar{x}_r \in [X(t_i)]_r} (\bar{x}_r + hf(t_i, \bar{x}_r; r)),$$

and

$$y_{i+1}(r) = \bigcup_{\bar{y}_r \in [Y(t_i)]_r} (\bar{y}_r + hf(t_i, \bar{y}_r; r) + O(h^2)).$$

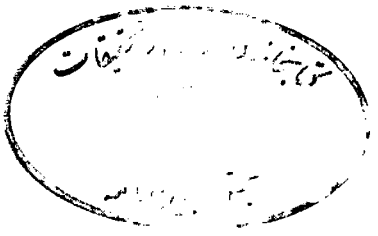
It is enough to prove $lim_{h\to 0} \parallel \bar{y}_{i+1}(r) - \bar{x}_r(t_{i+1}) \parallel = 0, i = 0, \ldots, N-1, r \in [0,1]$.

Since

$$\bar{y}_{i+1}(r) = \bar{y}_i(r) + hf(t_i, \bar{y}_i(r); r) + O(h^2),$$

and

$$\bar{x}_r(t_{i+1}) \approx \bar{x}_r(t_i) + hf(t_i, \bar{x}_r(t_i); r),$$



then

$$\parallel \bar{y}_{i+1}(r) - \bar{x}_r(t_{i+1}) \parallel \leq \parallel \bar{y}_i(r) - \bar{x}_r(t_i) \parallel (1 + Lh) + O(h^2).$$

By using the lemma (4.5.3) for all t_i in particular at T,

$$\parallel \bar{y}_N(r) - \bar{x}_r(T) \parallel \leq \frac{1}{L} O(h)[e^{LT} - 1],$$

hence proof is completed. □

4.2.1 Examples

Example 4.2.1. *Consider a fuzzy differential inclusions with constant coefficients*

$$\begin{cases} \frac{dx_1(t;r)}{dt} \in 3x_1(t;r) - 2x_2(t;r), & 0 \leq t \leq 0.01, \\ \frac{dx_2(t;r)}{dt} \in 2x_1(t;r) - x_2(t;r), \end{cases} \tag{4.2.6}$$

as an initial value for the fuzzy initial-value problem of (4.2.6) we take a number $Y_0 \in E^2$ *such that for* $r \in [0,1]$

$$[Y_0]_r = \{(x_1(0;r), x_2(0;r)) \in R^2 : x_1(0;r) \in [r-1, 1-r]\ ,\ x_2(0;r) \in [.5+.5r, 1.5-.5r]\},$$

where

$$\sum_r([Y_0]_r, t) = \begin{pmatrix} x_1(t;r) \\ x_2(t;r) \end{pmatrix} = \begin{pmatrix} e^t[x_1(0;r) + 2t(x_1(0;r) - x_2(0;r))] \\ e^t[x_2(0;r) + 2t(x_1(0;r) - x_2(0;r))] \end{pmatrix}, \quad (4.2.7)$$

and $(x_1(0;r), x_2(0;r)) \in [Y_0]_r$. Figure 4.2.1 and 4.2.2 show the $r-$ level sets of $\bigcup_r \sum_r([Y_0]_r, T)$ and the approximation of it with $h = .01$ for $r \in \{0, 0.1, \ldots, 1\}$, respectively.

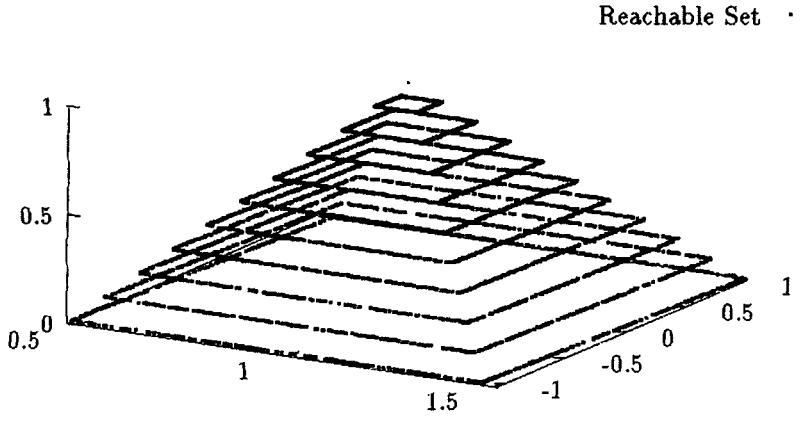


Figure 4.2.1.

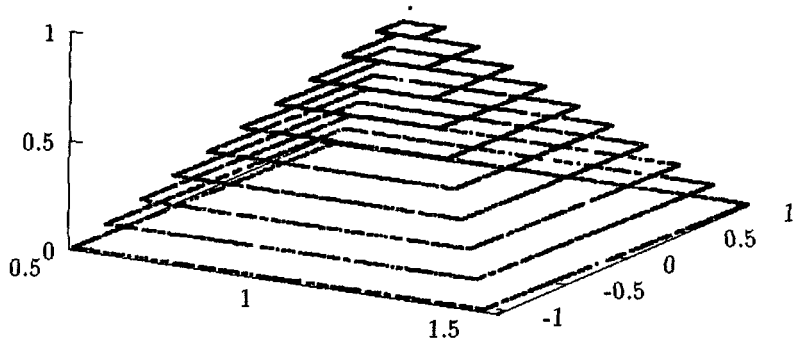


Figure 4.2.2, $h = 0.01$.

Let $C = \bigcup_r \sum_r([Y_0]_r, T)$ and B be the approximation of C which obtained by numerical methods. In Table 4.2.1 we are

h	$d_\infty(C, B)$
0.01	0.0789
0.005	0.0716
0.0025	0.0710
0.00125	0.0694
0.000625	0.0611

Table 4.2.1: The distance between reachable set and its approximation

Example 4.2.2. *Consider the following fuzzy differential inclusions*

$$x_1'(t;r) \in \ -x_2(t;r) + 0.1x_1(t;r)(9 - x_1(t;r)^2 - x_2(t;r)^2) + w(r),$$

$$x_2'(t;r) \in \ -x_1(t;r) + 0.1x_2(t;r)(9 - x_1(t;r)^2 - x_2(t;r)^2) + w(r),$$

we take a number $Y_0 \in E^2$ *such that*

$$[Y_0]_r = \{(x_1(0;r), x_2(0;r)) \in R^2 : x_1(0;r) \in [r-1, 1-r]\ ,\ x_2(0;r) \in [.5+.5r, 1.5-.5r]\}$$

and $w(r) \in [r-1, 1-r], r \in [0,1]$.

Then

$$f_1(t, x_r(t); r) = \begin{pmatrix} -x_2(t;r) + 0.1x_1(t;r)(9 - x_1(t;r)^2 - x_2(t;r)^2) + r - 1 \\ -x_1(t;r) + 0.1x_2(t;r)(9 - x_1(t;r)^2 - x_2(t;r)^2) + r - 1 \end{pmatrix},$$

and

$$f_2(t, x_r(t); r) = \begin{pmatrix} -x_2(t;r) + 0.1x_1(t;r)(9 - x_1(t;r)^2 - x_2(t;r)^2) + 1 - r \\ -x_1(t;r) + 0.1x_2(t;r)(9 - x_1(t;r)^2 - x_2(t;r)^2) + 1 - r \end{pmatrix}.$$

The Figure 4.2.3 shows the $r-$ level set $(r = 0.1)$ of approximations of reachable set by using Euler method with $h = 0.3$ and $h = 0.15$ is as follows

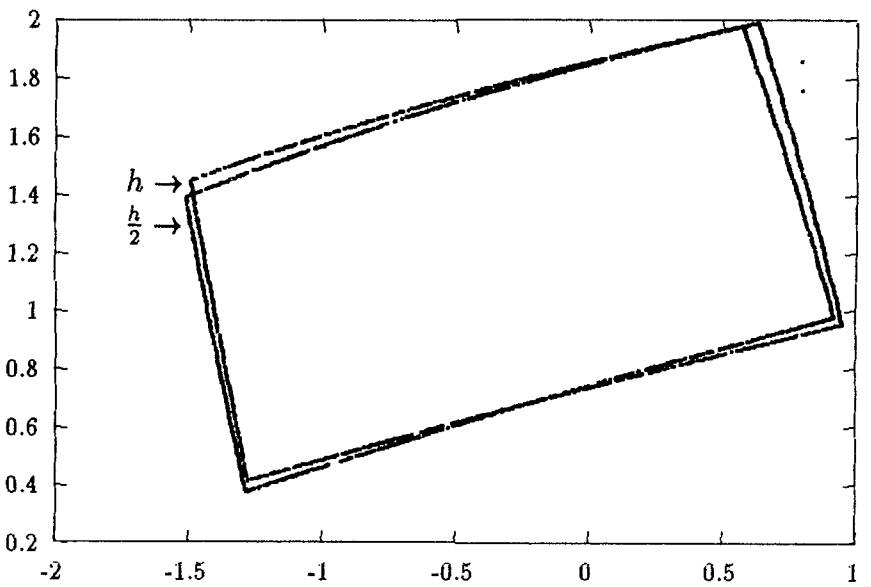


Figure 4.2.3, $t = 0.3, r = 0.1$

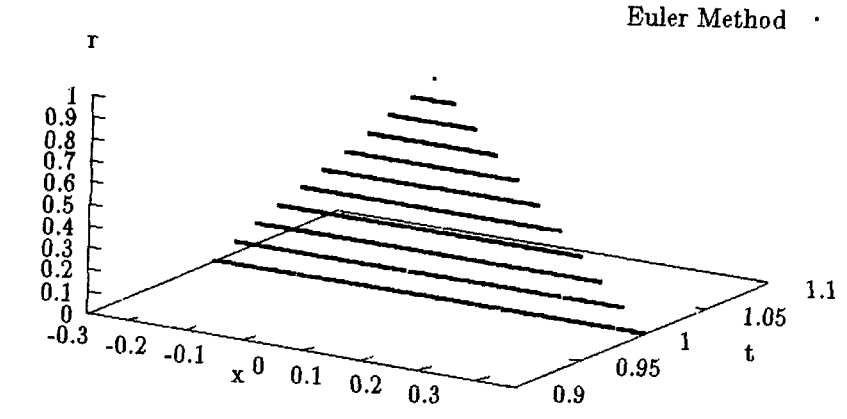


Figure 4.2.5, $t = 1$.

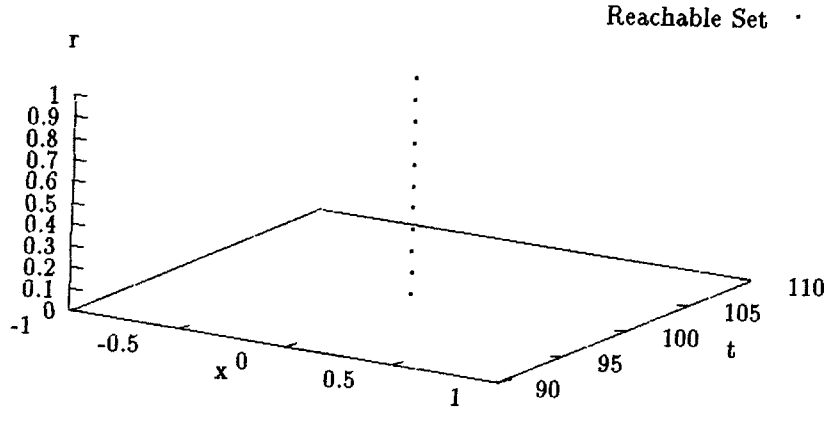


Figure 4.2.6, $t = 100$.

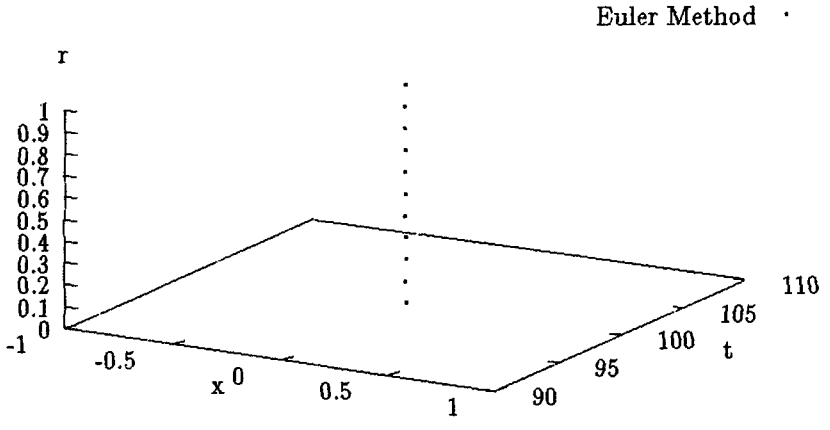


Figure 4.2.7, $t = 100$.

h	$d_\infty(C, B)$
$\frac{1}{10}$	0.0209
$\frac{1}{20}$	0.0097
$\frac{1}{40}$	0.0042
$\frac{1}{80}$	0.0030
$\frac{1}{160}$	0.0024

Table 4.2.2: The distance between reachable set and its approximation, $t = 1$.

4.3 Taylor method of order two

Let $x_r(t_i) \approx y_i(r)$ for all $r \in [0, 1]$, then the Taylor method of order two for approximating the reachable set of (4.2.4) is proposed as follows:

$y_0(r) \in [Y_0]_r \subset R^n$,

$$y_{i+1}(r) \in \bigcup_{s_r \in [Y(t_i)]_r} (s_r + h[F(t_i, s_r; r) + \tfrac{h}{2} F'(t_i, s_r; r)]), \quad i = 0, \ldots, N-1, \forall r \in [0, 1], \tag{4.3.1}$$

where

$$[Y(t_i)]_r = \prod_{k=1}^{n} [y_1^k(t_i;r), y_2^k(t_i;r)], \quad i = 0, \ldots, N-1, \quad r \in [0,1].$$

Theorem 4.3.1. *Let $F \in C^2(\Omega)$ in (1.6.3) be a compact convex valued mapping such that satisfies Lipschitz condition in x with Lipschitz constant $L > 0$ and x_r be a solution of (1.6.3) then $lim_{h\to 0} y_N(r) = x_r(T)$, for any $r \in [0,1]$.*

proof. Let, a.e.

$$x_r(t_{i+1}) = \bigcup_{\bar{x}_r \in [X(t_i)]_r} (\bar{x}_r + h[F(t_i, \bar{x}_r; r) + \frac{h}{2} F'(t_i, \bar{x}_r; r)]), \ a.e.$$

and

$$y_{i+1}(r) = \bigcup_{\bar{y}_r \in [Y(t_i)]_r} (\bar{y}_r + h[F(t_i, \bar{y}_r; r) + \frac{h}{2} F'(t_i, \bar{y}_r; r)] + O(h^3)).$$

It is enough to prove $lim_{h\to 0} \parallel \bar{y}_{i+1}(r) - \bar{x}_r(t_{i+1}) \parallel = 0, i = 0, \ldots, N-1, r \in [0,1]$. Since

$$\bar{y}_{i+1}(r) = \bar{y}_i(r) + h\phi^*(t_i, \bar{y}_i(r); r) + O(h^3),$$

and

$$\bar{x}_r(t_{i+1}) \approx \bar{x}_r(t_i) + h\phi^*(t_i, \bar{x}_r(t_i); r),$$

where

$$\phi^*(t,u;r) = F(t,u;r) + \frac{h}{2} F'(t,u;r),$$

then

$$\parallel \bar{y}_{i+1}(r) - \bar{x}_r(t_{i+1}) \parallel \leq \parallel \bar{y}_i(r) - \bar{x}_r(t_i) \parallel (1 + Lh) + O(h^3).$$

By using the lemma (4.5.3) for all t_i in particular at T proof is completed

$$\parallel \bar{y}_N(r) - \bar{x}_r(T) \parallel \leq \frac{1}{L} O(h^2)[e^{LT} - 1]. \quad \square$$

4.3.1 Examples

Example 4.3.1. *Consider the 4.2.1.*

Let $C = \bigcup_r \sum_r([Y_0]_r, T)$ and B be the approximation of C which obtained by numerical methods. In Table 4.3.1 we compare $d_\infty(C, B)$ for Taylor method of order two and Euler method.

h	*Taylor*	*Euler*
0.01	3.3661e-006	0.0789
0.005	8.571e-007	0.0716
0.0025	2.1886e-007	0.0710
0.00125	5.3138e-008	0.0694
0.000625	1.3425e-008	0.0611

Table 4.3.1: The Hausdorff distance between reachable set and its approximations

Example 4.3.2. *Consider the example 4.2.2.*

The Figure 4.3.1 shows the $r-$ level sets of approximations of reachable set by using Taylor and Euler methods at $t = 0.3$ with $h = 0.3$ for for $r = 0.1$ is as follows

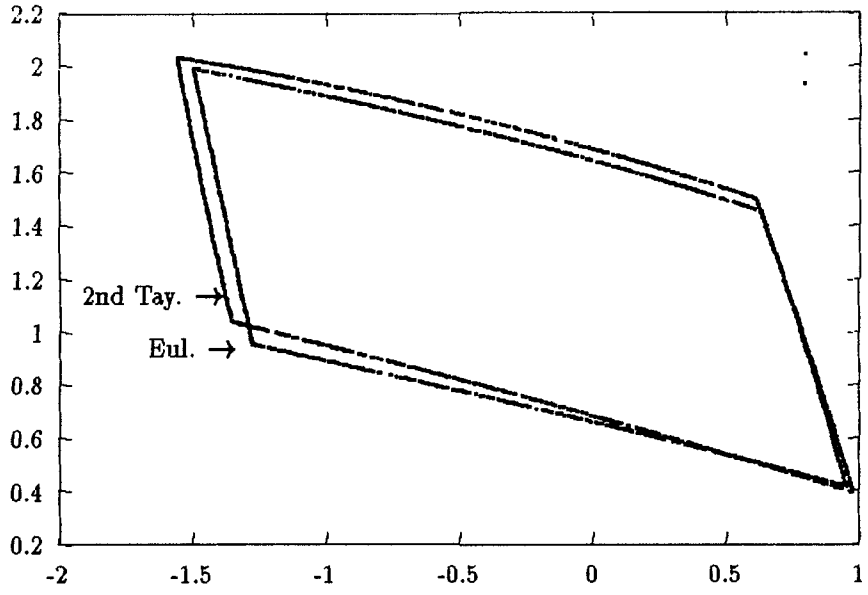


Figure 4.3.1, $t = 0.3, r = 0.1$

Example 4.3.3. *Consider the example 4.2.1.*

By using current method it can be seen that $diam(x(t)) \to 0$ as $t \to \infty$ in this example.

h	*Taylor*	*Euler*
$\frac{1}{10}$	6.6133e-004	0.0209
$\frac{1}{20}$	1.5902e-004	0.0097
$\frac{1}{40}$	3.8995e-005	0.0042
$\frac{1}{80}$	9.6605e-006	0.0030
$\frac{1}{160}$	2.4051e-006	0.0024

Table 4.3.2: The Hausdorff distance between reachable set and its approximations

4.4 Runge-Kutta method of order two

Let $x_r(t_i) \approx y_i(r)$ for all $r \in [0,1]$, then the Runge-Kutta method of order two for approximating the reachable set of problem (4.2.4) is proposed as follows:

$$\begin{aligned} &y_0(r) \in [Y_0]_r \subset R^n, \\ &k_1(r) \in F(t_i, y_i(r); r), \\ &k_2(r) \in F(t_i + h, y_i(r) + hk_1(r); r), \end{aligned} \tag{4.4.1}$$

$$y_{i+1}(r) = \bigcup_{s_r \in [Y(t_i)]_r} (s_r + \frac{h}{2}(k_1(r) + k_2(r))), \quad i = 0, \ldots, N-1, \quad r \in [0,1],$$

where $k_1(r) = F(t_i, s_r; r)$ and $k_2(r) = F(t_i + h, s_r + hk_1(r); r)$, and

$$[Y(t_i)]_r = \prod_{k=1}^{n} [y_1^k(t_i; r), y_2^k(t_i; r)], \quad i = 0, \ldots, N-1, \quad r \in [0,1].$$

Theorem 4.4.1. *Let $F \in C^2(\Omega)$ in (1.6.3) be a compact convex valued mapping such that satisfies Lipschitz condition in x with Lipschitz constant $L > 0$ and x_r be a solution of (1.6.3) then $lim_{h \to 0} y_N(r) = x_r(T)$, for any $r \in [0,1]$.*

proof. Let, a.e.

$$x_r(t_{i+1}) = \bigcup_{\bar{x}_r \in [X(t_i)]_r} (\bar{x}_r + \frac{h}{2}(k_1(r) + k_2(r))), \ a.e.$$

and

$$y_{i+1}(r) = \bigcup_{\bar{y}_r \in [Y(t_i)]_r} (\bar{y}_r + \frac{h}{2}(k_1(r) + k_2(r)) + O(h^3)).$$

It is enough to prove $lim_{h \to 0} \parallel \bar{y}_{i+1}(r) - \bar{x}_r(t_{i+1}) \parallel = 0, i = 0, \ldots, N-1, r \in [0,1]$. Since

$$\bar{y}_{i+1}(r) = \bar{y}_i(r) + \frac{h}{2}\phi^*(t_i, \bar{y}_i(r); r) + O(h^3)$$

and

$$\bar{x}_r(t_{i+1}) \approx \bar{x}_r(t_i) + \frac{h}{2}\phi^*(t_i, \bar{x}_r(t_i); r)$$

then

$$\| \bar{y}_{i+1}(r) - \bar{x}_r(t_{i+1}) \| \leq \| \bar{y}_i(r) - \bar{x}_r(t_i) \| (1 + L\frac{h}{2}) + O(h^3).$$

By using lemmas (4.5.3) and (4.5.4) for all t_i in particular at T proof is completed

$$\| \bar{y}_N(r) - \bar{x}_r(T) \| \leq \frac{2}{L} O(h^2)[e^{\frac{LT}{2}} - 1]. \ \square$$

4.4.1 Examples

Example 4.4.1. *Consider the example 4.2.1.*

Figure 4.4.1 and 4.4.2 show the $r-$ level sets of $\bigcup_r \sum_r([Y_0]_r, T)$ and it's approximation at the end points of interval with $h = .01$ for $r \in \{0, 0.1, \ldots, 1\}$.

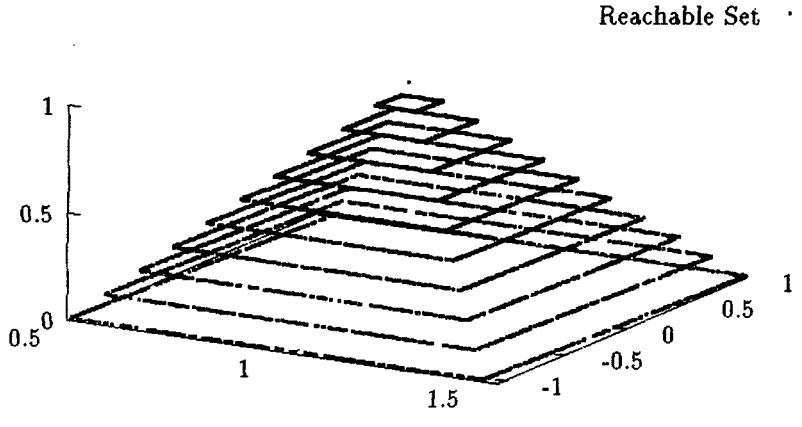


Figure 4.4.1

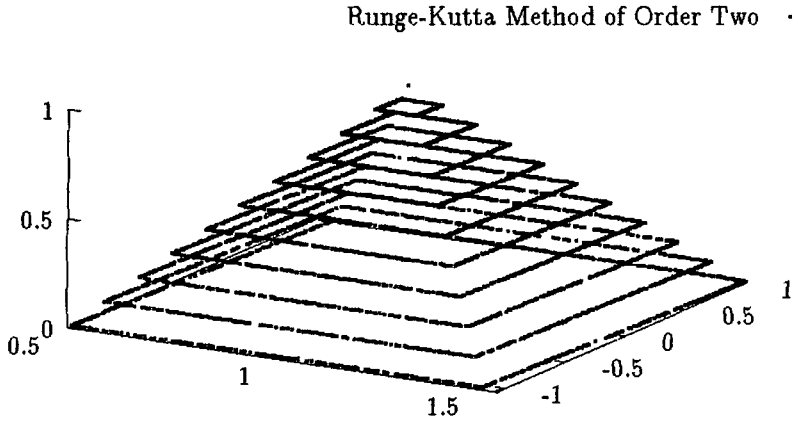


Figure 4.4.2, $h = 0.01$.

Let $C = \bigcup_r \sum_r([Y_0]_r, T)$ and B be the approximation of C which obtained by numerical methods. In Table 4.4.1 we compare $d_\infty(C, B)$ for Runge-Kutta method of order two and Euler method.

h	*Runge.*	*Euler*
0.01	3.3664e-006	0.0789
0.005	8.5709e-007	0.0716
0.0025	2.1885e-007	0.0710
0.00125	5.3140e-008	0.0694
0.000625	1.3428e-008	0.0611

Table 4.4.1: The distance between reachable set and its approximations

Example 4.4.2. *Consider the example 4.2.2.*

Figure 4.4.3 shows the $r-$ level sets of approximations of reachable set by using Runge-Kutta and Euler methods at $t = 0.3$ with $h = 0.3$ for for $r = 0.1$ is as follows

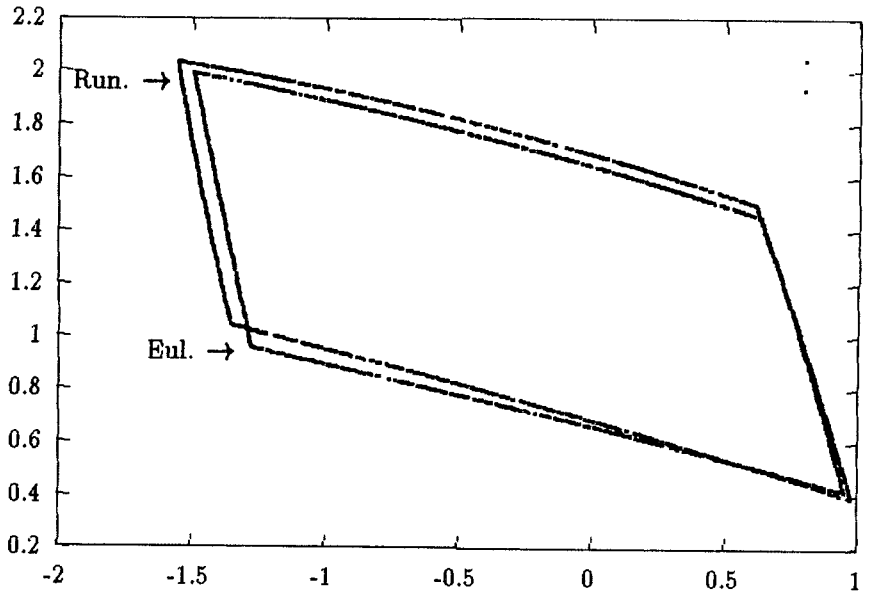


Figure 4.4.3, $t = 0.3, r = 0.1$

The method presented in this section for approximation reachable set, is based on pyramidal fuzzy numbers. The $r-$level sets of this fuzzy numbers are $n-$dimensional rectangles that the convex hull of the corners of rectangles or the set of all points on them (in $n-$dimensional space) forms the reachable set. In this section each point ($n-$dimensional vectors) which is belonged to reachable set is approximated by using the Runge-Kutta method of order two and Taylor method of order two which in contrast with the Euler method has a higher order of convergence.

4.5 Extrapolation Method

In this section a numerical method for solving *fuzzy differential inclusions* is considered and the extrapolation method to increase the accuracy of approximations to the solution is applied. *Fuzzy reachable* set can be approximated by proposed method with complete error analysis which is discussed in detail. The method is illustrated by solving some linear and nonlinear fuzzy initial value problems.

Let $\sum_r([Y_0]_r, t)$ be the Reachable set formed on the exact solutions and $\sum'_r([Y_0]_r, t)$ be the Reachable set formed on the approximate solutions at t. Let us assume that we have a fixed step size h and that we wish to approximate $x(t_1; r) = x(h; r) \in \sum_r([Y_0]_r, t_1)$, with $t_0 = 0$ for $\forall r \in [0, 1]$. For the first extrapolation step we let $h_0 = \frac{h}{2}$ and use Euler's method with randomly $y_0 \in [Y_0]_r$ to approximate $x(h_0; r) = x(\frac{h}{2}; r)$ as

$$y_1(r) \in y_0(r) + h_0 f(a, y_0; r), \quad \forall r \in [0, 1]. \tag{4.5.1}$$

We then apply the Midpoint with $t_0 = a$ and $t_1 = a + h_0 = a + \frac{h}{2}$ to produce a first approximation to $x(h; r) = x(2h_0; r) \in \sum_r([Y_0]_r, t_2)$, as $y_2(r) \in y_0(r) + 2h_0 f(h_0, y_1; r)$. The endpoint correction is applied to obtain the final approximation to $x(h; r)$ for $\forall r \in [0, 1]$ for the stepsize h_0. This results in the $O(h_0^2)$ approximation to $x(t_1; r)$

$$x_1^1(r) \in \frac{1}{2}[y_2(r) + y_1(r) + h_0 f(2h_0, y_2; r)], \quad \forall r \in [0, 1].$$

We save the approximation $x_1^1(r) \in \sum_r([Y_0]_r, t_1)$ and discard the intermediate results $y_1(r) \in \sum'_r([Y_0]_r, t_1)$ and $y_2(r) \in \sum'_r([Y_0]_r, t_2)$ for $\forall r \in [0, 1]$. To obtain the next approximation, $x_2^1(r)$ to $x(t_1; r)$ we let $h_1 = \frac{h}{4}$ and use Euler's method initial values to obtain an approximation to $x(h_1; r) = x(\frac{h}{4}; r)$ that we will call $y_1(r) \in \sum'_r([Y_0]_r, t_1)$

$$y_1(r) \in y_0(r) + h_1 f(0, y_0; r), \quad \forall r \in [0, 1]. \tag{4.5.2}$$

Next we produce approximations $y_2(r) \in \sum_r'([Y_0]_r, t_2)$ to $x(2h_1; r) = x(\frac{h}{2}; r)$ and $y_3(r) \in \sum_r'([Y_0]_r, t_3)$ to $x(3h_1; r) = x(\frac{3h}{4}; r)$ given by

$$y_2(r) \in y_0(r) + 2h_1 f(h_1, y_1(r); r), \quad \forall r \in [0,1],$$

and

$$y_3(r) \in y_1(r) + 2h_1 f(2h_1, y_2(r); r), \quad \forall r \in [0,1].$$

Then we produce the approximation $y_4(r) \in \sum_r'([Y_0]_r, t_4)$ to $x(4h_1; r) = x(t_1; r)$ given by

$$y_4(r) \in y_2(r) + 2h_1 f(3h_1, y_3(r); r), \quad \forall r \in [0,1].$$

The endpoint correction is now applied to $y_3(r) \in \sum_r'([Y_0]_r, t_3)$ and $y_4(r) \in \sum_r'([Y_0]_r, t_4)$ to produce the improved $O(h_1^2)$ approximation to $x(t_1; r)$

$$x_2^1(r) \in \frac{1}{2}[y_4(r) + y_3(r) + h_1 f(4h_1, y_4(r); r)], \quad \forall r \in [0,1].$$

The approximation to $x(h; r)$ has the property that

$$x(h;r) \in x_1^1(r) + e_1(r)(\frac{h}{2})^2 + e_2(r)(\frac{h}{2})^4 + \ldots = x_1^1(r) + e_1(r)\frac{h^2}{4} + e_2(r)\frac{h^4}{16} + \ldots, \quad (4.5.3)$$

and

$$x(h;r) \in x_2^1(r) + e_1(r)(\frac{h}{4})^2 + e_2(r)(\frac{h}{4})^4 + \ldots = x_2^1(r) + e_1(r)\frac{h^2}{16} + e_2(r)\frac{h^4}{256} + \ldots, \quad (4.5.4)$$

for $\forall r \in [0,1]$, where $e : [0,1] \to R^n$ and $x : [0,T] \times [0,1] \to R^n$. We can eliminate the $O(h^2)$ portion of this truncation error by averaging these two formulas appropriately, for any $r \in [0,1]$. Specifically, if we subtract (4.5.3) from 4 times (4.5.4) and divide the result by 3 and we have

$$x(h;r) = x_2^2(r) - e_1(r)\frac{h^4}{64} + \ldots.$$

So the approximation

$$x_2^2(r) \in x_2^1(r) + \frac{1}{3}(x_2^1(r) - x_1^1(r))$$

for $\forall r \in [0,1]$, has error of order $O(h^4)$. Continuing in this manner, we next let $h_2 = \frac{h}{6}$ and apply Euler method once followed b. Then we use the endpoint correction to determine the h^2 approximation, $x_3^1(r)$, to $x(h;r)$, this approximation can be averaged with $x_2^1(r)$ to produce a second $O(h^4)$ approximation that we denote $x_3^2(r)$. Then $x_3^2(r)$ and $x_2^2(r)$ are averaged to eliminate the $O(h^4)$ error terms and produce an approximation with error of order $O(h^6)$. Higher-order formulas are generated by continuing the process. The error is controlled by requiring that the approximations $x_1^1(r), x_2^2(r), \ldots$ be computed until $|x_i^i(r) - x_{i-1}^{i-1}(r)|$ is less than a given tolerance. If $x_i^i(r)$ is found to be acceptable, then $y_1(r)$ is set to $x_i^i(r)$ and computations begin again to determine $y_2(r)$, which will approximate $x(t_2;r) = x(2h;r)$, for any $r \in [0,1]$. The process is repeated until the approximation $y_N(r) \in A_r'([Y_0]_r, t_N)$ to $x(b;r) \in A_r([Y_0]_r, t_N)$, for $\forall r \in [0,1]$, is determined.

4.5.1 Examples

Example 4.5.1. *Consider the example 4.2.1.*

Figure 4.5.1 and 4.5.2 and 4.5.3 show the plan of $r-$ level sets of $\bigcup_r \sum_r([Y_0]_r, T)$ (pyramidal) and the approximation of it with Euler method with $h = 0.3$ and Extrapolation method respectively for $r \in \{0, 0.1, \ldots, 1\}$.

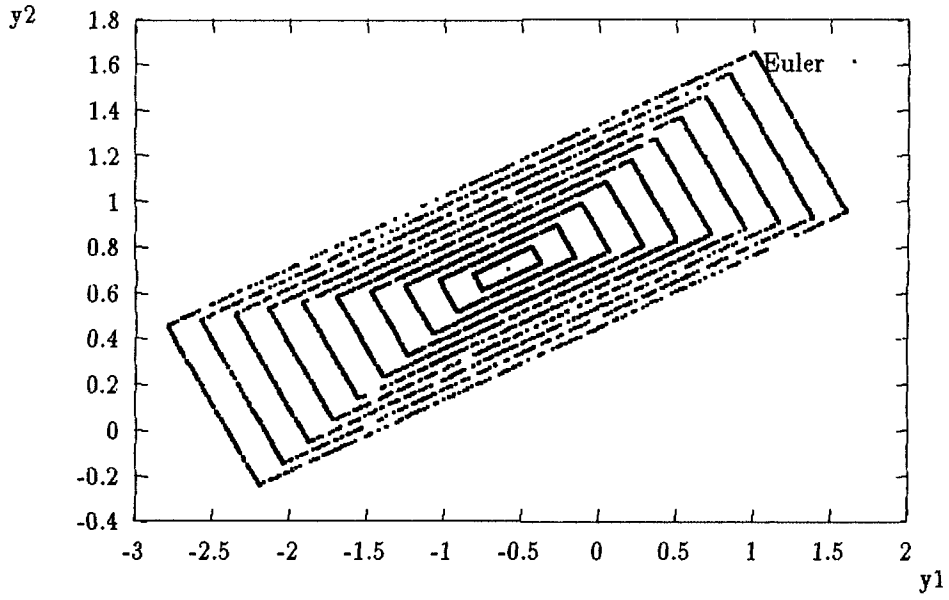


Figure 4.5.1.

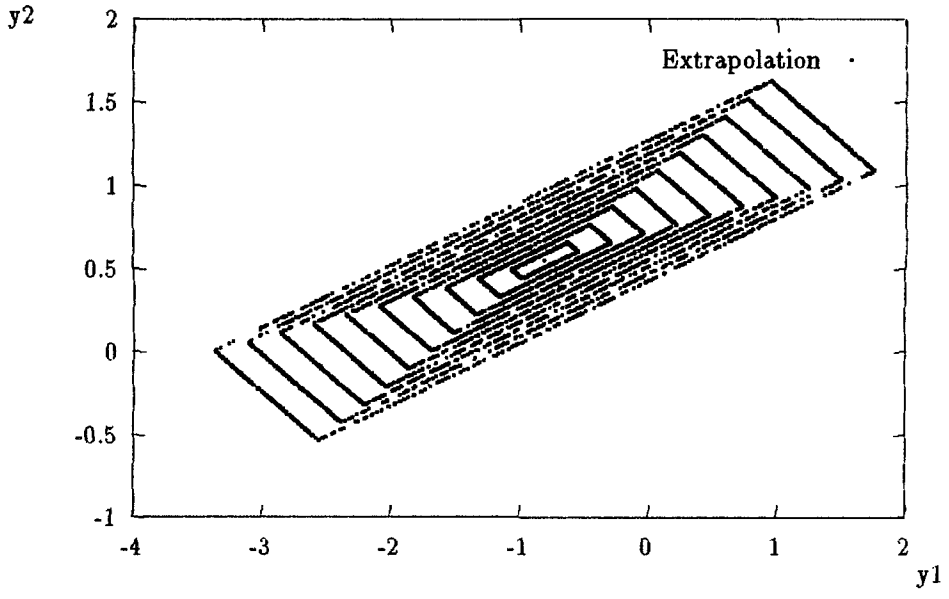


Figure 4.5.2.

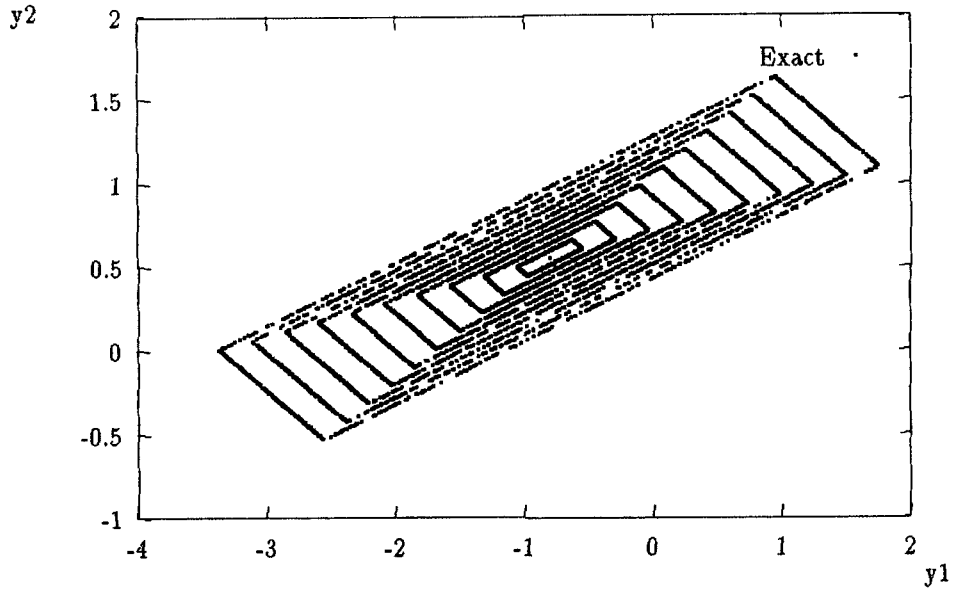


Figure 4.5.3.

Let $C = \bigcup_r \sum_r([Y_0]_r, T)$ and B be the approximation of C which obtained by numerical methods. In Table 4.5.1 we compare $d_\infty(C, B)$ for Extrapolation method with $\epsilon = 0.001$ and Euler method.

Euler	*Extra.*
0.7226	1.7251e-005

Table 4.5.1: The distance between reachable set and its approximations

Example 4.5.2. *Consider the example 4.2.2.*

The Hausdorff distance between Extrapolation method for $h = 0.5$ and $h = 0.25$ with $\epsilon = e - 5$ is $7.1801e - 008$ in figures 4.5.4,4.5.5 and Euler method is 0.3202 in Figure 4.5.6.

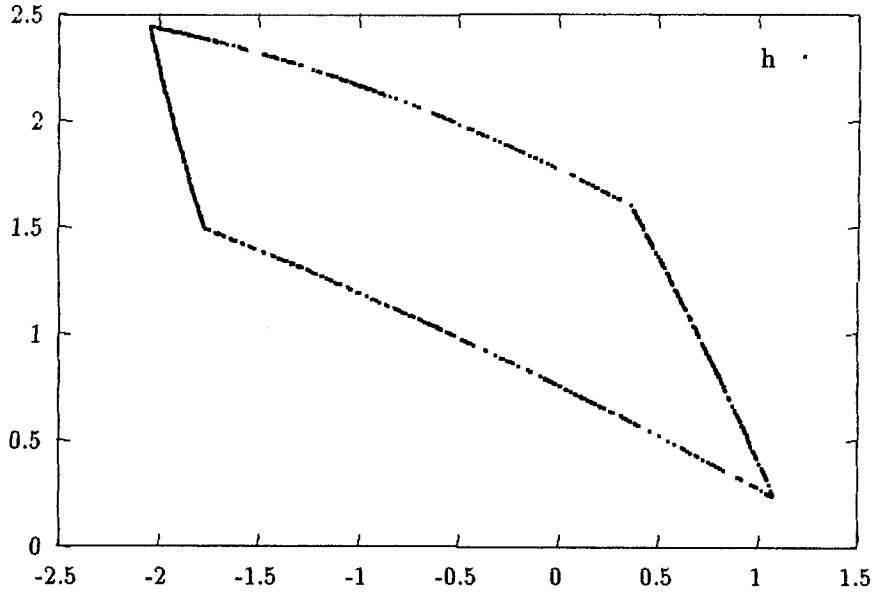


Figure 4.5.4, Extrapolation method with h.

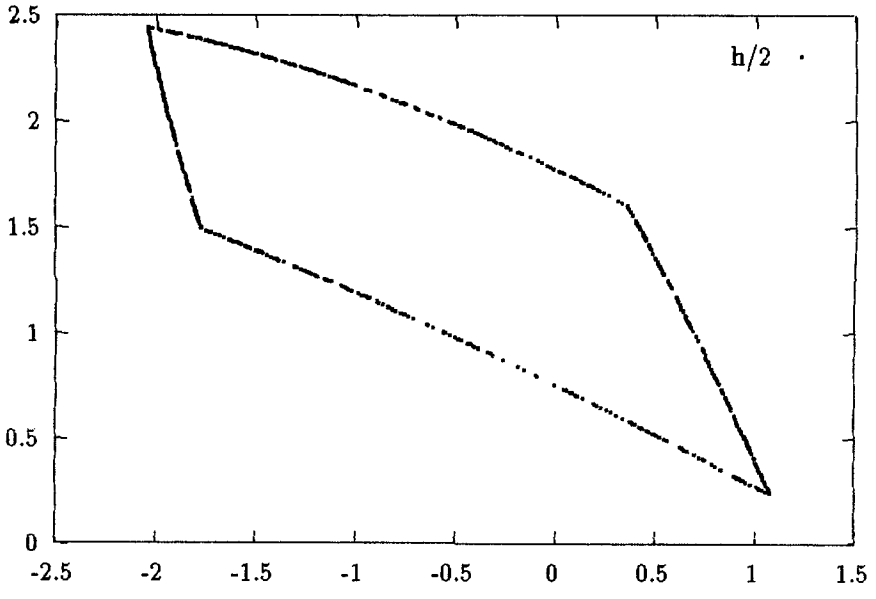


Figure 4.5.5, Extrapolation method with $\frac{h}{2}$.

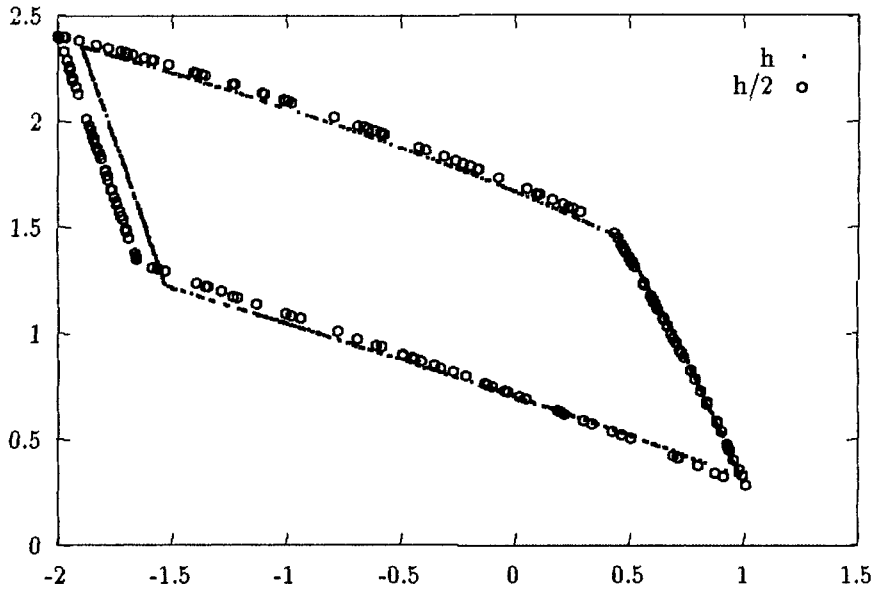


Figure 4.5.6, Euler method with h and $\frac{h}{2}$.

Example 4.5.3. *Consider the example 4.2.1, [6].*

Now we obtain the reachable set and its approximation at $t = 1$ with $h = 0.1$ and $\epsilon = e - 5$ in Table 4.5.1. It can be seen that in this example $diam(x(t)) \to 0$ as $t \to \infty$.

Euler	*Extra.*
0.0209	4.0449e-004

Table 4.5.1: The distance between reachable set and its approximations

The method presented in this section for approximation reachable set, is based on pyramidal fuzzy numbers. The $r-$level sets of these fuzzy numbers are $n-$dimensional rectangles that the convex hull of the corners of rectangles or the set of all points on them (in $n-$dimensional space) forms the reachable set. In this section each point

($n-$dimensional vectors) which is belonged to reachable set is approximated by using the Euler method and then improved by Extrapolation method.

Open problems

1. Using Adomian decomposition for solving fuzzy differential equations.
2. Linear and nonlinear shooting methods for solving fuzzy differential equations.
3. Numerical solution of fuzzy integral equations by using different methods
4. Using Adomian decomposition for solving fuzzy integral equations.
5. Improving of the solutions of fuzzy integral equations by using Ramberg method.
6. Numerical solution of fuzzy partial differential equations
7. Stability in fuzzy partial differential equations.

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Appendix

This appendix reviews notion, theorems and lemmas which are used throughout the text.

Corollary 4.5.1. *If $x : I \to E^n$ is differentiable on I for each $r \in [0,1]$ the real function $t \to diam[x(t)]_r$ is nondecreasing on I, [17].*

Theorem 4.5.2. *If $x : I \to E^n$ is differentiable then it is continuous, [17].*

Lemma 4.5.3. *Let a sequence of numbers $\{W_n\}_{n=0}^N$ satisfy*

$$|W_{n+1}| \leq A|W_n| + B, \quad 0 \leq n \leq N-1,$$

for some given positive constants A and B. Then

$$|W_n| \leq A^n|W_0| + B\frac{A^n - 1}{A-1}, \quad 0 \leq n \leq N, \ [19].$$

Lemma 4.5.4. *Let a sequence of numbers* $\{W_n\}_{n=0}^N$, $\{V_n\}_{n=0}^N$ *satisfy*

$$|W_{n+1}| \leq |W_n| + A.max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \leq |V_n| + A.max\{|W_n|, |V_n|\} + B,$$

for some given positive constants A *and* B, *and denote*

$$U_n = |W_n| + |V_n|, \quad 0 \leq n \leq N.$$

Then

$$U_n \leq \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A} - 1}, \quad 0 \leq n \leq N,$$

where $\overline{A} = 1 + 2A$ *and* $\overline{B} = 2B$, *[19].*