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NUMERICAL SOLUTION OF FUZZY INTEGRAL EQUATIONS

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To my brothers Mohammad and Mahyar

for their endless love and support

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Abstract

Starting with the work of Abel in the 1820's, analysts have had a continuing interest in integral equations. The names of many modern mathematicians, notably Cauchy, Fredholm, Hilbert, Volterra, and others, are associated with this topic. There are basically two reasons for this interest. In some cases, as in the work of Abel on tautochrone curves, integral equations are the natural mathematical model for representing a physically interesting situation. The second, and perhaps more common reason, is that integral operators, transforms, and equations, are convenient tools for studying differential equations. Consequently, integral equation techniques are well known to classical analysts and numerical analyst and many elegant and powerful results were developed by them.

In practice, parameters of a mathematical model are uncertain and a good tool for modeling such problems is fuzzy set theory. Fuzzy systems are useful to study a variety of problems ranging from fuzzy metric spaces , population models , the golden mean , particle systems , quantum optics and gravity , synchronize hyperchaotic systems , control chaotic systems, medicine, to bioinformatics and computational biology.

In this work we propose numerical methods for solving nonlinear Volterra fuzzy integral equations. First some basic definitions and results on fuzzy numbers and fuzzy integral, a metric on the space of fuzzy numbers and an existence theorem for fuzzy Volterra integral equations is given. Then we propose a general quadrature method for solving nonlinear fuzzy Volterra integral equations of the second kind and prove convergence of the quadrature method. We also give two other numerical methods with numerical examples in the following

chapters. The first one is explicit Runge-Kutta method and the second one is predictor-corrector method.

Chapter 1

Introduction to Fuzzy Mathematics

1.1 Introduction

Most of our traditional tools for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. By crisp we mean dichotomous, that is, yes-or-no type rather than more-or-less type. In conventional dual logic, for instance, a statement can be true or false and nothing in between. In set theory, an element can either belong to a set or not; and in optimization, a solution is either feasible or not. Precision assumes that the parameters of a model represent exactly either our perception of the phenomenon modeled or the features of the real system that has been modeled. Generally, precision also implies that the model is unequivocal, that is, that it contains no ambiguities.

Certainty eventually indicates that we assume the structures and parameters of the model to be definitely known, and that there are no doubts about their values or their occurrence. If the model under consideration is a formal model [51], that is, if it does not pretend to model reality adequately, then the model assumptions are in a sense arbitrary, that is the model builder can freely decide which model characteristics he chooses. If, however, the model or theory asserts factuality that is, if conclusions drawn from these models have bearing on reality and are supposed to model reality adequately, then the modeling language has to be suited to model the characteristics of the situation under study appropriately.

The usefulness of the mathematical language for modeling purposes is undisputed. For factual models or modeling languages, two major complications arise:

1. Real situations are very often not crisp and deterministic, and they cannot be described precisely.

2. The complete description of a real system often would require far more detailed data than a human being could ever recognize simultaneously, process, and understand.

Fuzzy sets were introduced in 1965 by Lotfi Zadeh with a view to reconcile mathematical modeling and human knowledge in the engineering science. Since then, a considerable body of literature has blossomed around the concept of fuzzy sets in an incredibly wide range of areas, from mathematics and logics to traditional and advanced engineering methodologies (from civil engineering to computational intelligence). Applications are found in many contexts, from medicine to finance, from human factors to consumer products, from vehicle control to computational linguistics; and so on. Fuzzy logic is now currently used in the industrial practice of advanced information technology. L. Zadeh referred to the second point when he wrote, "As the complexity of a system increases, our ability to make precise and yet significant (or relevance) become almost mutually exclusive characteristics". Let us consider characteristic features of real-world systems again: Real situations are very often uncertain or vague in a number of ways. Due to lack of information, the future state of a system might not be known completely.

Sugeno has used the word fuzziness in a radically different context. Fuzziness can be found in many areas of daily life, such as in engineering , medicine , meteorology and others.

It is particularly frequent, however, in all areas of decision-making, reasoning, learning, and so on. Some reasons for this fuzziness have already been mentioned. Others are that most of our daily communication uses "natural languages", and a good part of our thinking is done in it. In these natural languages, the meaning of words are very often vague.

Fuzzy set theory is composed of an organized body of mathematical tools particularly well studied for handling incomplete information, the unsharpness of classes of objects or situations, or the gradualness of preference profiles, in a flexible way. It offers a unifying framework for modeling various types of information ranging from precise numerical, interval-valued data, to symbolic and linguistic knowledge, with a stress on semantic rather than syntax (some misunderstanding with logicians).

The structure of the thesis is organised as follows. The present chapter is meant to provide to the historical emergence of fuzzy sets and the main components of fuzzy set theory, as it stands now. Most basic concepts and formal notations are briefly introduced. In chapter 2 we give an existence theorem from [46] and propose a general quadrature method for solving nonlinear fuzzy Volterra integral equations of the second kind and prove convergence of the proposed method. In chapter 3 some explicit Runge-Kutta method is proposed with some numerical example. In chapter 4 another family of numerical methods for nonlinear fuzzy Volterra integral equations, called predictor-corrector method, is proposed along with convergence proof.

1.2 The Historical Emergence of Fuzzy Sets

About a hundred years ago, the American philosopher Charles Peirce was one of the first scholars in the modern age to point out, and to regret, that "Logicians have too much neglected the study of vagueness, not suspecting the important part it plays in mathematical thought [23]". Bertrand Russel (1923) also expressed this point of view some time later.

Discussion on the links between logic and vagueness are not usual in the philosophical literature in the first half of the century (Copilowish, 1939; Hempel, 1939). Even Wittgenstein (1953) pointed out that concepts in natural language do not possess a clear collection of properties defining them, but have extendable boundaries, and that there are central and less central members in a category. In spite of considerable interest for multiple-valued logics raised in the 1930s by Jan Luckasiewicz (1910a, b; 1920, 1930) and his school who developed logics with intermediary truth value(s) it was the American philosopher.

Max Black (1937) who first proposed so-called "consistency profiles" (the ancestors of fuzzy member ship functions) in order to "characterize vague symbols". H. Weyl (1940), who explicitly replaces it by a continuous characteristic function, has first considered the generalization of the traditional characteristic function. Kaplan and Scott (1951) further proposed the same kind of generalization in 1951. They suggested Caculi for generalized characteristic functions of vague predicates, and the basic fuzzy set connectives already appeared in this thesis. Karl Menger (1951a), who, in 1951, was first to use the term "ensemble flou" (the French counterpart set of "fuzzy set") in the title of a paper of his. The notation of a fuzzy set stems from the observation made by Zadeh (1965a) that "more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership". This observation emphasizes the gap existing between mental representation of reality and usual mathematical representations , which are based on binary logic, precise numbers, differential equations and the like.

Classes of objects referred to in Zadeh's citation exist only through such mental representations through natural language term such as high temperature, young man, big size, etc., and with nouns such as bird, chair, etc. Classical logic is too rigid to account for such categories where that membership is a gradual notation rather than all-or-nothing matter. The power of expressivity of real numbers is far beyond the limited level of precision found in mental representations. The latter are meaningful summaries of perceptive phenomena that account for complexity of the world. Analytical representations of physical phenomena can be faithful as models of reality, but are sometimes difficult to understand because they do not explain much by themselves, and may remain opaque to the non-specialist. Mental representations make more sense but are pervaded with vagueness,which encompasses at the same time the lack of specificity of linguistic term, and the lack of well-defined boundaries of the class of objects they refer to.

In the literature of fuzzy sets, the word fuzzy often stands for the word vague. Some comments on the links between vagueness and fuzziness are useful. In common use, there is a property of objects called "fuzziness"; from the Oxford English Dictionary we read that "fuzzy" means either not firm or sound in substance, or fringed into loose fibers. Fuzzy means also covered by fuzz, i.e., with loose volatile matter. "Something is fuzzy". For example, "a bear is fuzzy". It may sound strange to say, "bald is fuzzy", or that "young is fuzzy". Words (adjectives in this case) bald and young are vague (but not fuzzy in the material sense) because their meanings are not fixed by sharp boundaries. Similarly, objects are not vague. Here however, the word "fuzzy" is applied to words, especially predicates, and is supposed to refer to the gradual nature of some of these words, which causes them to appear as vague. However, the term "vagueness" designates a much larger kind of ill definition for words (including ambiguity), generally.

The specificity of fuzzy sets is to capture the idea of partial membership. The characteristic function of a fuzzy set, often called membership function, is a function whose range is an ordered membership set containing more that two (often a continuum of) values (typically, the unit interval). Therefore, a fuzzy set is often understood as a function. This has been a source of criticism from mathematicians (Arbib, 1977) as functions are already well known, and a theory of functions already exists. However, the novelty of fuzzy set theory, as first proposed by Zadeh, is to treat functions as if they were subsets of their domains, since such functions are used to represent gradual categories. It means that the classical set-theoretic notations like intersection, union, complement, inclusion, etc. are extended to combine functions ranging on an ordered membership set. In elementary fuzzy set theory, the set-union of functions is performed by taking their point wise maximum, their intersection by their point wise minimum, their complementation by means of an order reversing automorphism of the membership scale, and set-inclusion by the point wise inequality between functions. Mathematicians had not envisaged this point of view earlier, if we except some pioneers, mainly logician.

Fuzzy set theory is indeed closely connected to many-value logics that appeared in the thirties, if degrees of membership are understood as degrees of truth, intersection as conjunction, union as disjunction, complementation as negation and set-inclusion as implication. This chapter is meant to account for history of how the notation of fuzzy sets could become known, and it presents a catalogue of basic notations, which are presented in details in the other chapters of this thesis.

1.3 Fuzzy Sets-Basic Definitions

A *classical* (crisp) set is normally defined as a collection of elements or objects $x \in X$ that can be finite, countable, or overcountable. Each single element can either belong to or not belong to a set A, $A \subseteq X$. In the former case, the statement "x belong to A" is true, whereas in the latter case this statement is false. Such a classical set can be described in different ways: one can either enumerate (list) the elements that belong to the set analytically, for instance, by stating conditions for membership; or define the elements by using the characteristic function, in which 1 indicates membership function and 0 non-membership function. For a fuzzy set, the characteristic function allows various degrees of membership for the elements of a given set.

Definition 1.3.1. (Membership function) If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, A(x))|\ x \in X\} \tag{1.3.1}$$

$A(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in A that maps X to the membership space M (when M contains only the two points 0 and 1, A is non-fuzzy and $A(x)$ is identical to the characteristic function of a non-fuzzy set). The range of the membership function is a subset of the non-negative real numbers whose spermium is finite. Elements with a zero degree of membership are normally not listed. It should be noted membership functions enable the notation of a class to be extended to categories that have no clear-cut boundaries, as often encountered in linguistic information.

A more convenient notation was proposed by Zadeh [48]. When X is a finite set $\{x_1, ..., x_n\}$ a fuzzy set on X is expressed as

$$A = \frac{A(x_1)}{x_1} + \cdots + \frac{A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{A(x_i)}{x_i}. \tag{1.3.2}$$

When X is not finite, we write

$$A = \int_x A(x).$$

1.3.1 Characteristic of Fuzzy Sets

Fuzzy sets are characterized in more detail by referring to the concept of support, core, normality, convexity, etc. Let $f(X)$ denote the set of all possible fuzzy sets defined on the universe of discourse X. For a fuzzy set $A \in F(X)$ we can give the following definitions.

Definition 1.3.2. (Support) The support of a fuzzy set A is the ordinary subset of X:

$$supp\ A = \{x \in X,\ A(x) > 0\} \tag{1.3.3}$$

Definition 1.3.3. (Core) The core of a fuzzy set A is the set of all points with the membership degree one in A:

$$coreA = \{x \in X |\ A(x) = 1\}. \tag{1.3.4}$$

Definition 1.3.4. (Height of a fuzzy set) The height of A is the least upper bound of $A(x)$ i.e.,

$$\text{hgt}(A) = \sup_{x \in X} A(x). \tag{1.3.5}$$

Definition 1.3.5. (Universal fuzzy set) A is a universal fuzzy set if $A(x) = 1,\ \forall\ x \in X$, that is if $core(A) = X$.

Definition 1.3.6. (Empty fuzzy set) A is an empty fuzzy set if $A(x) = 0,\ \forall\ x \in X$, that is if $supp(A) = \phi$.

Definition 1.3.7. (Fuzzy singleton) Let A be a fuzzy set. If$supp(A) = \{x_0\}$ then A is called a fuzzy point and we use the notation $A = x_0$.

Definition 1.3.8. (Normal fuzzy set) A is said to be normal if and only if $\exists\ x \in X,\ A(x) = 1$; this definition implies $\text{hgt}(A) = 1$.

A more general and even more useful notion is that of an $\alpha-$cut or $\alpha-$level set.

Definition 1.3.9. ($\alpha-$Levels) The set of elements that belong to the fuzzy set A at least to the degree $\alpha \in (0, 1]$ is called the $\alpha-$cut or $\alpha-$level set:

$$[A]^{\alpha} = \{x \in X\ :\ A(x) \geqslant \alpha\}. \tag{1.3.6}$$

If non-equality is hold strictly then $[A]_\alpha$ is called "strong $\alpha-$level set", also for $\alpha = 0$ we have

$$[A]^0 = \bigcup_{\alpha \in (0,1]} [A]^\alpha.$$

Definition 1.3.10. The $m-th$ power of a fuzzy set A is a fuzzy set with membership function

$$A^m(x) = [A(x)]^m \quad , \quad \forall x \in X \quad , \quad \forall m \in \mathbb{R}^+. \tag{1.3.7}$$

Convexity also plays a role in fuzzy set theory. By contrast to classical set theory, however, convexity conditions are defined with reference to the membership function rather than the support of a fuzzy set.

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its $\alpha-$cut according to the formula

$$A(x) = \sup_{\alpha \in (0,1]} \min(\alpha, A_\alpha(x)),$$

where

$$A_\alpha(x) = \begin{cases} 1, & x \in [A]^\alpha, \\ 0, & otherwise. \end{cases}$$

It is easily checked that the following properties hold:

$$[A \cup B]^\alpha = [A]^\alpha \cup [B]^\alpha, \qquad [A \cap B]^\alpha = [A]^\alpha \cap [B]^\alpha.$$

Definition 1.3.11. (Convexity) A fuzzy set A of X is called convex if $[A]^\alpha$ is a convex subset of X for each $\alpha \in [0,1]$. We now state a useful theorem that provides us with an alternative formulation of convexity of fuzzy sets. For the sake of simplicity, we restrict the theorem to fuzzy sets on $\mathbb{R}$, which are of primary interest in this text.

Theorem 1.3.1. *A fuzzy set A on $\mathbb{R}$ is convex if and only if*

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\} \tag{1.3.8}$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0,1]$, where min *denotes the minimum operator.*

Proof. See [20]

1.3.2 Basic Relationships Between Fuzzy Sets

As in set theory, we can define generic relations between two fuzzy sets, such as inclusion and equality. Let $A \in F(X)$ and $B \in F(X)$ be fuzzy sets.

Definition 1.3.12. (Inclusion) We say that A is included in B, denoted by $A \subseteq B$, iff $A(x) \leq B(x),\ \forall x \in X$.

Definition 1.3.13. (Equality) A and B, are said to be equal, denoted by $A = B$, iff $A \subseteq B$, and $B \subseteq A$.

1.3.3 Fuzzy Number

Definition 1.3.14. (Fuzzy number) A fuzzy subset A of the real line $\mathbb{R}$ with membership function $A(x),\ A : \mathbb{R} \to [0,1]$, is called a fuzzy number if

(a) A is normal, i.e., there exists an element x_0 such that $A(x_0) = 1$,

(b) A is convex, i.e., $A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\}$,

(c) $A(x)$ is upper semi-continuous,

(d) $supp(A)$ is bounded, where $supp(A) = cl\{x \in \mathbb{R} : A(x) > 0\}$, and cl is the closure operator.

A fuzzy number A is called *positive* (*negative*) if its membership function is such that $A(x) = 0, \ \forall x < 0 \ (\forall x > 0)$.

Dubois and Prade [20] suggest a special type of representation for fuzzy numbers of the following type: They call L (and R), which map $\mathbb{R}^+ \longrightarrow [0, 1]$, and are decreasing, *shape functions* or *reference functions* if $L(0) = 1, \ L(x) < 1$ for $x > 0$; $L(x) > 0$ for $x < 1$; $L(1) = 0$ or $[L(x) > 0, \forall x$ and $L(+\infty) = 0]$. The most useful type of fuzzy numbers is LR type as follows :

A fuzzy number is a convex fuzzy subset like of the real line is completely defined by its membership function. Let A be a fuzzy number, whose membership function $A(x)$ can generally be defined as [19]

$$A(x) = \begin{cases} L_A(x) & a \leq x \leq b, \\ \omega & b \leq x \leq c, \\ R_A(x) & c \leq x \leq d, \\ 0 & otherwise, \end{cases} \tag{1.3.9}$$

where $0 < \omega \leq 1$ is a constant, $L_A(x) : [a, b] \to [0, \omega]$ and $R_A(x) : [c, d] \to [0, \omega]$ are two strictly monotonic and continuous mapping from $\mathbb{R}$ to closed interval $[0, \omega]$. If $\omega = 1$, then A is a normal fuzzy number; otherwise, it is trapezoidal fuzzy number and is usually denoted by $A = (a, b, c, d; \omega)$ or $A = (a, b, c, d)$ if $\omega = 1$. In particular, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $A = (a, b, d; \omega)$ or $A = (a, b, d)$ if $\omega = 1$. So triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers.

Since $L_A(x)$ and $R_A(x)$ are both strictly monotonic and continuous functions, their inverse functions exist and should also be continuous and strictly monotonic. Let $A_L : [0, \omega] \to [a, b]$ and $A_R : [0, \omega] \to [c, d]$ be the inverse functions of $L_A(x)$ and $R_A(x)$, respectively. Then $A_L(y)$ and $A_R(y)$ should be integrable on the close interval $[0, \omega]$. In other words, both $\int_0^\omega A_L(y)dy$ and $\int_0^\omega A_R(y)dy$ should exist. In the case of trapezoidal fuzzy number, the inverse functions $A_L(y)$ and $A_R(y)dy$ can be analytically expressed as

$$A_L(y) = a + (b-a)y/\omega \qquad 0 \leq y \leq \omega, \tag{1.3.10}$$

$$A_R(y) = d - (d-c)y/\omega \qquad 0 \leq y \leq \omega. \tag{1.3.11}$$

The functions $L_A(x)$ and $R_A(x)$ are also called the left and right side of the fuzzy number A, respectively [19, 20].

In this thesis, we assume that

$$\int_{-\infty}^{+\infty} A(x)dx < +\infty.$$

According to the definition of a fuzzy number, it is seen at once that every $\alpha-$cut of a fuzzy number is a closed interval. Hence, for a fuzzy number A, we have

$$A_\alpha = [A_1(\alpha), A_2(\alpha)], \tag{1.3.12}$$

where

$$A_1(\alpha) = \inf\{x \in \mathbb{R} : A(x) \geq \alpha\},$$

$$A_2(\alpha) = \sup\{x \in \mathbb{R} : A(x) \geq \alpha\}. \tag{1.3.13}$$

If the left and right sides of the fuzzy number A are strictly monotone, obviously, A_1 and A_2 are inverse functions of $L_A(x)$ and $R_A(x)$, respectively.

If $r = 1$, we obtain trapezoidal fuzzy number. When $r = 1,\ b = c$ we obtain triangular fuzzy number, and it is denoted by $A = (a, b, d)$.

In case that $b - a = d - b,\ A$ is a symmetrical triangular fuzzy number.

Since the trapezoidal fuzzy number is completely characterized by $r = 1$ and four real numbers $a \leq b \leq c \leq d$, it is often denoted in brief as $A = (a, b, c, d)$.

The conditions $r = 1,\ a = b$ and $c = d$ imply the close interval and in case $r = 1,\ a = b = c = d = t$ we have the crisp number t.

Note 1: A normal trapezoidal fuzzy number $A = (a, b, c, d)$ is also denoted by (b, c, α, β), where α and β are left and right spread, respectively. Moreover, $\alpha = b - a,\ \beta = d - c$.

Note 2: In this thesis, a family of fuzzy numbers and trapezoidal fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$ and $\mathbb{F}^T(\mathbb{R})$ respectively.

Definition 1.3.15. (Quasi fuzzy number) A quasi fuzzy number A is a fuzzy set of the real line with a normal, convex and continuous membership function satisfying the limit conditions

$$\lim_{t \to +\infty} A(t) = 0, \qquad \lim_{t \to -\infty} A(t) = 0.$$

Remark 1.3.1. Let A be a fuzzy number. Then A_α is a closed convex (compact) subset of $\mathbb{R}$ for all $\alpha \in [0, 1]$.

Definition 1.3.16. We represent an arbitrary fuzzy number by an ordered pair of functions $(u_1(\alpha), u_2(\alpha))$, $0 \leq \alpha \leq 1$, which satisfy the following requirements [**?**]:

1. $u_1(\alpha)$ is a bounded left continuous non-decreasing function over $[0, 1]$,

2. $u_2(\alpha)$ is a bounded left continuous non-increasing function over $[0, 1]$,

3. $u_1(\alpha) \leq u_2(\alpha), 0 \leq \alpha \leq 1$.

A crisp number λ is simply represented by $u_1(\alpha) = u_2(\alpha) = \lambda, 0 \leq \alpha \leq 1$.

Definition 1.3.17. Let v and w be fuzzy numbers and s be a real number. Then for

$0 \leq \alpha \leq 1$

$$u = v \text{ if and only if } u_1(\alpha) = v_1(\alpha) \text{ and } u_2(\alpha) = v_2(\alpha),$$

$$v + w = (v_1(\alpha) + w_1(\alpha), v_2(\alpha) + w_2(\alpha)),$$

$$v - w = (v_1(\alpha) - w_2(\alpha), v_2(\alpha) - w_1(\alpha)),$$

$$v.w = (\min\{v_1(\alpha).w_1(\alpha), v_1(\alpha).w_2(\alpha), v_1(\alpha).w_2(\alpha), v_2(\alpha).w_2(\alpha)\},$$

$$\max\{v_1(\alpha).w_1(\alpha), v_1(\alpha).w_2(\alpha), v_2(\alpha).w_1(\alpha), v_2(\alpha).w_2(\alpha)\}),$$

Definition 1.3.18. The space $\mathbb{F}^n(\mathbb{R})$ is all of fuzzy subsets U of $\mathbb{R}^n$ which satisfy the following conditions:

1. U is normal,
2. U is convex,
3. U is upper semi-continuous,
4. $[U]^0$ is a bounded subset of $\mathbb{R}^n$,

when $n = 1$, elements of $\mathbb{F}^1(\mathbb{R})$ are Fuzzy numbers.

1.4 The Extension Principle

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. In this elementary form, it was already implied in Zadeh's first contribution (1965). In the meantime, modifications have been suggested [Zadeh 1973a; Zadeh et al. 1975; Jain 1976]. Following Zadeh [1973a] and Dubois and Prade [1980a], we define the extension principle as follows:

Definition 1.4.1. [52] (extension principle) Assume X and Y are crisp sets and let f be a mapping from X to Y,

$$f : X \to Y,$$

such that for each $x \in X$, $f(x) = y \in Y$. Assume A is a fuzzy subset of X, using extension principle, we can define $f(A)$ as a fuzzy subset of Y such that

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Definition 1.4.2. (*sup-min extension n-place functions*) Let X_1, X_2,...,X_n and Y be a family of sets. Assume f is a mapping from the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ into Y. Let A_1, A_2,...,A_n be fuzzy subsets of X_1, X_2,...,X_n, respectively, then we use the extension principle for the evaluation of $f(A_1, A_2, ..., A_n)$. $f(A_1, A_2, ..., A_n)$ is a fuzzy set such that

$$f(A_1, A_2, ..., A_n)(y) = \begin{cases} \sup\{\min\{A_1(x_1), A_2(x_2), \ldots, A_n(x_n)\} \mid x \in f^{-1}(y)\} \\ \qquad\qquad \text{if } f^{-1}(y) \neq \emptyset, \\ 0 \qquad\qquad \text{otherwise,} \end{cases}$$

where $x = (x_1, x_2, ..., x_n)$.

Example 1.4.1. *Let* $f : X \times X \to X$ *be defined as*

$$f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2, \ \lambda_1, \ \lambda_2 \in \mathbb{R}.$$

Suppose A_1 and A_2 are fuzzy subsets of X. Then using the extension principle we get

$$f(A_1, A_2)(y) = \sup_{\lambda_1 x_1 + \lambda_2 x_2 = y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = \lambda_1 A_1 + \lambda_2 A_2$.

Theorem 1.4.1. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then the Zadeh's extension $\hat{f} : \mathbb{F}(\mathbb{R}^n) \to \mathbb{F}(\mathbb{R}^n)$ is well defined,is continuous and*

$$\left[\hat{f}\,(u)\right]^{\alpha} = f\left([u]^{\alpha}\right) \quad \forall \alpha \in [0, 1]\,.$$

Note that this relation is valid if $f : U \to \mathbb{R}^n$,where U is an open sub set in $\mathbb{R}^n$.

Proof. See [44]

By the extension principle we can define fuzzy distance in the following.

Definition 1.4.3. The fuzzy distance function on $\mathbb{F}(\mathbb{R})$, $\delta : \mathbb{F}(\mathbb{R}) \times \mathbb{F}(\mathbb{R}) \to \mathbb{F}(\mathbb{R})$, is defined by

$$\delta(A, B)(z) = \sup_{|x-y|=z} \min(A(x), B(y)). \tag{1.4.1}$$

Definition 1.4.4. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be a function from $\mathbb{F}(X)$ to $\mathbb{F}(Y)$. Then f is called a fuzzy function (or mapping) and we use the notation

$$f : \mathbb{F}(X) \to \mathbb{F}(Y).$$

It should be noted, however, that a fuzzy function is not necessarily defined by Zadeh's extension principle. It can be any function which maps a fuzzy set $A \in \mathbb{F}(X)$ into a fuzzy set $B := f(A) \in \mathbb{F}(Y)$.

Definition 1.4.5. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets . A fuzzy mapping $f : \mathbb{F}(X) \to \mathbb{F}(Y)$ is said to be monotonic increasing if $A, A' \in \mathbb{F}(X)$ and $A \subset A'$ imply that $f(A) \subset f(A')$.

Theorem 1.4.2. *Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets. Then every fuzzy mapping $f : \mathbb{F}(X) \to \mathbb{F}(Y)$ defined by the extension principle is monotonic increasing.*

Proof. Let $A, A' \in \mathbb{F}(X)$ such that $A \subset A'$. Then using the definition of sup-min extension principle we get

$$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \leq \sup_{x \in f^{-1}(y)} A'(x) = f(A')(y)$$

for all $y \in Y$. $\square$

Let A and B be fuzzy numbers with $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$, $0 \leq \alpha \leq 1$. Then it can easily be shown that

$$[A + B]^\alpha = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)],$$

$$[-A]^\alpha = [-a_2(\alpha), -a_1(\alpha)],$$

$$[A - B]^\alpha = [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)],$$

$$[\lambda A]^\alpha = [\lambda a_1(\alpha), \lambda a_2(\alpha)] \quad if \quad \lambda \geq 0,$$

$$[\lambda A]^\alpha = [\lambda a_2(\alpha), \lambda a_1(\alpha)] \quad if \quad \lambda < 0,$$

for all $\alpha \in [0, 1]$, i.e. any α-level set of the extended sum of two fuzzy numbers is equal to the sum of their α-level sets. The following two theorems show that this property is valid for any continuous function.

Theorem 1.4.3. *[41] Let $f : X \to X$ be a continuous function and let A be a fuzzy number. Then,*

$$[f(A)]^\alpha = f([A]^\alpha),$$

where $f(A)$ is defined by the extension principle and

$$f([A]^\alpha) = \{f(x) \mid x \in [A]^\alpha\}.$$

If $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and f is continuous and monotone increasing then from the above theorem we get

$$[f(A)]^\alpha = f([A]^\alpha) = f([a_1(\alpha), a_2(\alpha)]) = [f(a_1(\alpha)), f(a_2(\alpha))].$$

Theorem 1.4.4. *[41] Let $f : X \times X \to X$ be a continuous function and let A and B be fuzzy numbers. Then*

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha)$$

where,

$$f([A]^\alpha, [B]^\alpha) = \{f(x_1, x_2) \mid x_1 \in [A]^\alpha, x_2 \in [B]^\alpha\}.$$

Let $f(x, y) = xy$ and let $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$, $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ be the $\alpha-$level sets of two fuzzy numbers A and$B$. Applying above theorem we get

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) = [A]^\alpha [B]^\alpha.$$

The equation

$$[AB]^\alpha = [A]^\alpha [B]^\alpha = [a_1(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)]$$

holds if and only if A and B are both nonnegative, i.e. $A(x) = B(x) = 0$ for $x \leq 0$.
In general, we obtain a very complicated expression for the α-level sets of the product AB

$$[A]^\alpha [B]^\alpha = [\min\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\},$$

$$\max\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}].$$

If $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, then, according to Zadeh extension principle, we can extend f to $\mathbb{F}^n(\mathbb{R}) \times \mathbb{F}^n(\mathbb{R}) \to \mathbb{F}^n(\mathbb{R})$ by the equation

$$f(u, v)(z) = \sup_{z=f(x,y)} \min\{u(x), v(y)\}. \tag{1.4.2}$$

1.5 Operation on Fuzzy Numbers

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to *add*, *subtract*, *multiply* and *divide* with fuzzy quantities. The process of doing these operations is called $fuzzy$ $arithmetic$.

1.5.1 Addition and Multiplication

Addition: Addition is an increasing operation. Hence, the extended addition ($\oplus$) of fuzzy numbers gives a fuzzy number. Note that $-(A \oplus B) = (-A) \oplus (-B)$. The operation ($\oplus$) is commutative and associative but has no group structure. The identity of ($\oplus$) is the non-fuzzy number 0. But A has no symmetrical element in the sense of a group structure. In particular, $A \oplus (-A) \neq 0, \quad \forall A \in \mathbb{F}(\mathbb{R}) - \mathbb{R}$.

Multiplication: Multiplication is an increasing operation on $\mathbb{R}^+$ and a decreasing operation on $\mathbb{R}^-$. Hence, the product of fuzzy numbers ($\odot$) that are all either positive or negative gives a positive fuzzy number. Note that $-(A) \odot B = -(A \odot B)$, so that the factors can have different signs. The operation ($\odot$) is commutative and associative. The set of positive fuzzy numbers is not a group for ($\odot$): although $\forall A, \quad A \odot 1 = A$, the product $A \odot A^{-1} \neq 1$ as soon as A is not a real number. A has no inverse in the sense of group structure.

1.5.2 Subtraction

Subtraction is neither increasing nor decreasing. However, it is easy to check that $A \ominus B = A \oplus (-B), \quad \forall (A, B) \in \mathbb{F}(\mathbb{R})^2$ so that $A \ominus B$ is a fuzzy number whenever A and B are.

1.5.3 Division

Division is neither increasing nor decreasing. But, since $A \oslash B = A \odot (B^{-1}), \forall A\ \forall B, A \in \mathbb{F}(\mathbb{R}), B \in \mathbb{F}(\mathbb{R} - \{0\})$, $A \oslash B$ is a fuzzy number when A and B are both positive or both negative fuzzy numbers. The division of ordinary fuzzy numbers can be performed similar to multiplication, by decomposition.

1.6 Hausdorff Metric

Denote by κ^n the set of all nonempty compact subsets of $\mathbb{R}^n$ and by κ_c^n the subset of κ^n consisting of nonempty convex compact sets.

In this section, we define a metric space by Hausdorff separation. Recall that

$$\rho(x, A) = \min_{a \in A} \|x - a\| \tag{1.6.1}$$

is the distance of a point $x \in \mathbb{R}^n$ from $A \in \kappa^n$ and that the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \kappa^n$ is defined as

$$\rho(A, B) = \max_{a \in A} \rho(a, B). \tag{1.6.2}$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$.

An open ϵ-neighborhood of $\mathrm{A} \in \kappa^n$ is the set

$$\mathrm{N}(\mathrm{A}, \epsilon) = \{x \in \mathbb{R}^n \ : \ \rho(x, \mathrm{A}) < \epsilon\} = \mathrm{A} + \epsilon \mathrm{B}^n, \tag{1.6.3}$$

where B^n is the open unit ball in $\mathbb{R}^n$, [17].

Definition 1.6.1. A mapping $F : \mathbb{R}^n \to \kappa^n$ is *upper semi-continuous* (usc) at x_0 if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0)$ such that

$$F(x) \subset N(F(x_0), \epsilon) = F(x_0) + \epsilon B^n, \tag{1.6.4}$$

for all $x \in N(x_0, \delta)$.

Definition 1.6.2. The *Hausdorff metric* d_H on κ^n is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}. \tag{1.6.5}$$

The space (κ^n, d_H) is a complete metric space. Let D^n denote the set of usc normal fuzzy sets on $\mathbb{R}^n$ with compact support. That is, $u \in D^n$, then $u : \mathbb{R}^n \to [0, 1]$ is usc, $supp(u)$ is compact and there exists at least one $\xi \in supp(u)$ for which $u(\xi) = 1$. The β-level set of u, $0 < \beta \leq 1$ is

$$[u]^\beta = \{x \in \ \mathbb{R}^n \ : \ u(x) \geqslant \beta\}. \tag{1.6.6}$$

Clearly, for $\alpha \leqslant \beta, [u]_\alpha \supseteq [u]_\beta$. The level sets are nonempty from normality and compact by usc and compact support. The metric d_∞ is defined on D^n as

$$d_\infty(u, v) = \sup\{d_H([u]^\alpha, [v]^\alpha) \ : \ 0 \leqslant \alpha \leqslant 1\}, \ \ u, v \in D^n \tag{1.6.7}$$

and (D^n, d_∞) is a complete metric space. $\mathbb{F}^n(\mathbb{R})$ is the subset of fuzzy convex elements of D^n. The metric space $(\mathbb{F}^n(\mathbb{R}), d_\infty)$ is also complete, [18].

Chapter 2

Fuzzy Volterra Integral Equations

Fuzzy systems were found useful in some problems ranging from population models [32], particle systems [24, 25], quantum optics and gravity [26], synchronize hyperchaotic systems [50], control chaotic systems [27, 36], medicine [5, 11, 33, 42], to bioinformatics and computational biology [10, 14].

In recent years the theory of fuzzy differential equations is well developed. The study of initial value problems for fuzzy differential equations was initiated by Goetschel and Voxman [31] and Kaleva [37], where the existence and uniqueness of the solution was proved using the Banach fixed point principle, and in [38], where the fuzzy variant of Peanos theorem was obtained. Numerical methods for fuzzy differential equations are built in [1, 2, 28] using the fuzzy variant of the Euler and Runge-Kutta method. In [34], uncertain dynamical systems are associated to fuzzy initial value problems and in [35], numerical methods to approximate the fuzzy reachable sets of these initial value problems are developed. as in the crisp case, fuzzy differential equations and fuzzy integral equations are closely related and methods for solving them are similar. Existence and uniqueness of the solution of fuzzy Volterra integral equations was proved in [43] and numerical methods for linear fuzzy integral equations and nonlinear fuzzy Fredhohlm integral equations can be found in [3, 12, 29, 30]. In this thesis we propose quadrature method for nonlinear Volterra fuzzy integral equations. In order to do this, first we state some existence and uniqueness theorems.

2.1 Existence Theorem

We consider the following nonlinear fuzzy Volterra integral equation

$$f(x) = g(x) \oplus \int_0^x K(x,s,f(s))\mathrm{d}s, \qquad 0 \le x \le a, \tag{2.1.1}$$

where $g : [0,a] \to \mathbb{F}(\mathbb{R})$, $K : U \to \mathbb{F}(\mathbb{R})$ and $U = \{(x,s,y)|0 \le s \le x \le a, y \in \mathbb{F}(\mathbb{R}) \text{ and } D(y,g(x)) \le b\}$. We assume that conditions of the following theorem is satisfied to guarantee existence of unique solution.

Theorem 2.1.1. *[46]Let* a, b *and* L *be positive numbers and for some fixed* $\alpha \in (0,1)$ *define* $c = \alpha/L$. *Suppose:*

(i) $g : [0,a] \to \mathbb{F}(\mathbb{R})$ *is continuous with respect to metric* D,

(ii) $K : U \to \mathbb{F}(\mathbb{R})$ *is continuous where* $U = \{(x,s,y)|0 \le s \le x \le a, y \in \mathbb{F}(\mathbb{R}) \text{ and } D(y,g(x)) \le b\}$.

(iii) K *satisfies Lipschitz condition with respect to* y *on* U, *i.e. there exist some positive real number* L *such that* $D(K(x,s,y),K(x,s,z)) \le LD(y,z)$, *if* $(x,s,y),(x,s,z) \in U$.

If $M = \max_U D(K(x,s,y),0)$, *then there is a unique solution of (2.1.1) on* $[0,T]$ *where* $T = \min\{a, b/M, c\}$.

Writing equations (2.1.1) in terms of level sets we have the following system of two integral equations with two unknowns

$$\begin{aligned} \underline{f}(x;r) &= \underline{g}(x;r) + \int_0^x \underline{K}(x,s,f(s);r)\mathrm{d}s, \qquad 0 \le x \le a, \\ \overline{f}(x;r) &= \overline{g}(x;r) + \int_0^x \overline{K}(x,s,f(s);r)\mathrm{d}s, \qquad 0 \le x \le a, \end{aligned} \tag{2.1.2}$$

for $0 \leq r \leq 1$ where $\underline{K}$ and $\overline{K}$ are functions from U into $\mathbb{R}$ such that

$$\underline{K}(x,s,u;r) = \min\{z|z \in [K(x,s,u)]^r\},$$
$$\overline{K}(x,s,u;r) = \max\{z|z \in [K(x,s,u)]^r\}. \quad (2.1.3)$$

2.2 Quadrature Methods for Nonlinear Fuzzy Volterra Integral Equation

Let $f : [a,b] \to E$ be an integrable fuzzy function with r-level sets $[f(t)]^r = [\underline{f}(x;r), \overline{f}(x;r)]$. In [6, 8] some quadrature methods for $\int_a^b f(x)\mathrm{d}x$ were given as follows

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{j=0}^{n} w_j \odot f(x_j) \quad (2.2.1)$$

where $x_0 = a < x_1 < \ldots < x_n = b$ is a partition of $[a,b]$ and the weights w_j are chosen so that to obtain highest possible accuracy. In this thesis we only use quadrature method with $w_j > 0$, $0 \leq j \leq n$. If we use a method of order p then

$$D(\int_a^b f(x)\mathrm{d}x, \sum_{j=0}^{n} w_j \odot f(x_j)) \leq Kh^p, \quad (2.2.2)$$

for some fixed K. i.e.

$$D(\int_a^b f(x)\mathrm{d}x, \sum_{j=0}^{n} w_j \odot f(x_j)) = O(h^p). \quad (2.2.3)$$

Therefore every quadrature method with nonnegative weights for crisp functions can be used to approximate fuzzy integrals with the same order of approximation.

Suppose for a given stepsize $h > 0$ we know the solution of equation (2.1.1) at points $x_i = ih, i = 0, 1, \ldots, n-1$. An approximation to $f(x_n)$ can then be computed by replacing the integral on the right hand side of (2.1.1) by a numerical integration rule using values of the integrand at $x_i, i = 0, 1, \ldots, n$ and solving the resulting equation for $f(x_n)$. Denoting by f_n an approximation to $f(x_n)$ and letting $g_n = g(x_n)$, we have

$$f_n = g_n \oplus h \odot \sum_{k=0}^{n} w_{nk} \odot K(x_n, x_k, f_k), \quad n = 1, 2, \ldots, N, \tag{2.2.4}$$

$$f_0 = g_0, \tag{2.2.5}$$

where the numbers w_{nk} are the weights associated with the quadrature rule. If $w_{nn} \neq 0$ then the unknown f_n is implicitly defined by (2.2.4) and for sufficiently small h the equation has a unique solution. These equations can be solved by an iterative method for solving fuzzy equations [4]. For example if we use trapezoidal quadrature rule then

$$w_{n0} = w_{nn} = \frac{1}{2}, \quad w_{nk} = 1, \quad k = 1, 2, \ldots, n-1, \tag{2.2.6}$$

so (2.2.4) reduces to the simple form

$$\begin{aligned} f_n &= g_n \oplus \frac{h}{2} \odot (K(x_n, x_0, f_0) \oplus 2 \odot \sum_{k=1}^{n-1} K(x_n, x_k, f_k) \oplus K(x_n, x_n, f_n)), \\ & \qquad n = 1, 2, 3, \ldots, N, \\ f_0 &= g_0. \end{aligned} \tag{2.2.7}$$

However the trapezoidal rule is of low order. So obtaining accurate solution requires the stepsize h to be very small. In order to increase accuracy without choosing h too small we can use higher order Newton-Cotes or Gregory rule. For example fifth order Gregory formula has weights

$$w_{n0} = w_{nn} = \frac{251}{720}, \tag{2.2.8}$$

$$w_{n1} = w_{n,n-1} = \frac{299}{240}, \tag{2.2.9}$$

$$w_{n2} = w_{n,n-2} = \frac{211}{240}, \tag{2.2.10}$$

$$w_{n3} = w_{n,n-3} = \frac{739}{720}, \tag{2.2.11}$$

$$w_{ni} = 1, \qquad i = 4, 5, \ldots, n-4. \tag{2.2.12}$$

Thus if we use (2.2.4) with these weights the values of $f_1, f_2, \dots, f_6$ can not be calculated and must be computed some other way. This reflects the fact that higher order integration rules require a minimum number of points. f_0 is of course always taken as $f(0) = g(0)$. The use of the Newton-Cotes type formulas introduces a slight complication, since these rules involve some restriction on the number of points. For example, compound Simpson's rule can be applied only when n is even. For odd n some adjustment has to be made. One way is to apply the so-called three-eighths rule over points $x_{n-3}, x_{n-2}, x_{n-1}, x_n$ and the Simpson's rule over the rest of the interval,

$$\int_{x_0}^{x_n} f(x)\mathrm{d}x = \int_{x_0}^{x_{n-3}} f(x)\mathrm{d}x \oplus \int_{x_{n-3}}^{x_n} f(x)\mathrm{d}x. \tag{2.2.13}$$

The first integral in the right hand side is approximated by compound Simpson's rule and the second by three-eighths rule. i.e.,

$$\int_{x_{n-3}}^{x_n} f(x)\mathrm{d}x \approx \frac{3h}{8} \odot (f_{n-3} \oplus 3f_{n-2} \oplus 3f_{n-1} \oplus f_n) \tag{2.2.14}$$

Thus the weights will be

$$\begin{array}{lll} n \text{ is even:} & w_{n0} = w_{nn} = \frac{1}{3}, & \\ & w_{n,2i} = \frac{2}{3}, & i = 1, 2, \dots, \frac{n}{2} - 1, \\ & w_{n,2i+1} = \frac{4}{3}, & i = 0, 1, \dots, \frac{n}{2} - 1, \\ n \text{ is odd:} & w_{n0} = w_{nn} = \frac{1}{3}, & n \geq 5, \\ & w_{n,2i} = \frac{2}{3}, & i = 1, 2, \dots, \frac{n-5}{2}, \\ & w_{n,2i+1} = \frac{4}{3}, & i = 0, 1, \dots, \frac{n-5}{2}, \\ & w_{n,n-3} = \frac{17}{24} - \frac{1}{3}\delta_{n3}, & \\ & w_{n,n-1} = w_{n,n-2} = \frac{9}{8}, & \\ & w_{n,n} = \frac{3}{8}, & \end{array} \tag{2.2.15}$$

where δ_{ij} denotes the Kronecker delta.

Later in chapter (3) a starting method will be given but first in the next section we discuss convergence of the method.

2.3 Convergence of the approximate solution

In order to prove the convergence of the method we consider the set of values $e_j = D(f_j, f(x_j))$ which we will call the discretization error. We are interested in the behavior of this discretization error as a function of the stepsize h. The following three definitions and lemma are taken from [40].

Definition 2.3.1. A method of the form (2.2.4) is said to be a convergent approximation method (for some equation or class of equations) if

$$\max_{0\leq j\leq N} e_j \to 0$$

as $h \to 0$, $N \to \infty$ such that $Nh = a$.

Definition 2.3.2. If, for all admissible h, there exists a number $M < \infty$, independent of h, such that

$$\max_{0\leq j\leq N} e_j \leq Mh^q \tag{2.3.1}$$

and if q is the largest number for which such an inequality holds, then q is called the order of convergence of the method.

We will see in the following that the order of convergence is closely connected with the accuracy of the numerical integration employed.

Definition 2.3.3. Let f be the solution of (2.1.1). Then the function

$$\delta(h, x_n) = D\left(\int_0^{x_n} K(x_n, s, f(s))\mathrm{d}s, h \odot \sum_{j=o}^{n} w_{nj} \odot K(x_n, x_j, f(x_j))\right) \tag{2.3.2}$$

is the local consistency error for (2.2.4).

Lemma 2.3.1. *If ξ_i is a sequence of real numbers satisfying*

$$|\xi_n| \le A\sum_{i=0}^{n-1}\xi_i + B_n \qquad n = r, r+1, \dots, \tag{2.3.3}$$

where

$$A > 0, \qquad |B_n| \le B, \qquad \sum_{i=0}^{r-1}\xi_i \le \eta. \tag{2.3.4}$$

Then

$$|\xi_n| \le (1+A)^{n-r}(B + A\eta), \quad n = r, r+1, \dots. \tag{2.3.5}$$

In particular, if $A = hM$, $M > 0$, and $x = nh$, then

$$|\xi_n| \le (B + A\eta)^{Mx} \tag{2.3.6}$$

Now we employ this lemma to prove the convergence theorem and the fact that order of accuracy of the method is the same as the order of quadrature method.

Theorem 2.3.2. *Consider the approximate solution of (2.1.1) by (2.2.4) and assume that*

(i) the quadrature formula is such that there exist a constant C (independent of h) such that

$$\max_{0\le n\le N}\delta(h, x_n) \le Ch^q, \tag{2.3.7}$$

(ii) the weights satisfy

$$\max_{0\le i\le n\le N} w_{ni} \le W < \infty,$$

(iii) the starting errors satisfy $e_j \le Mh^{q-1}$, $j = 0, 1, \dots, r-1$, where $M > 0$ is a constant.

Then the method is a convergent approximation method and the order of convergence is at least q.

Proof. Putting $x = x_n$ in (2.1.1) and considering (2.2.4), we get for $n = r, r+1, \ldots$

$$\begin{aligned}
e_n &= D(f_n, f(x_n)) \\
&= D\left(g_n \oplus h \odot \sum_{j=o}^{n} w_{nj} \odot K(x_n, x_j, f_j), g_n \oplus \int_0^{x_n} K(x_n, s, f(s))\mathrm{d}s\right) \\
&= D\left(h \odot \sum_{j=o}^{n} w_{nj} \odot K(x_n, x_j, f_j), \int_0^{x_n} K(x_n, s, f(s))\mathrm{d}s\right) \\
&\leq D\left(h \odot \sum_{j=o}^{n} w_{nj} \odot K(x_n, x_j, f_j), h \odot \sum_{j=o}^{n} w_{nj} \odot K(x_n, x_j, f(x_j))\right) \\
&\quad + D\left(\int_0^{x_n} K(x_n, s, f(s))\mathrm{d}s, h \odot \sum_{j=o}^{n} w_{nj} \odot K(x_n, x_j, f(x_j))\right) \\
&\leq h \sum_{j=o}^{n} w_{nj} D\left(K(x_n, x_j, f_j), K(x_n, x_j, f(x_j))\right) + \delta(h, x_n).
\end{aligned}$$

Using the Lipschitz conditions and assumption (ii) we have

$$e_n \leq hWL \sum_{j=o}^{n} e_j + \delta(h, x_n). \tag{2.3.8}$$

Hence choosing $h < 1/LW$,

$$(1 - hWL)e_n \leq hWL \sum_{j=o}^{n-1} e_j + \delta(h, x_n), \tag{2.3.9}$$

$$e_n \leq \frac{hWL}{1 - hWL} \sum_{j=o}^{n-1} e_j + \frac{\delta(h, x_n)}{1 - hWL}. \tag{2.3.10}$$

Applying lemma (2.3.1) and (2.3.6) it follows that

$$e_n \leq \frac{1}{1 - hWL} \left(\max_{0 \leq j \leq n} \delta(h, x_j) + hWL \sum_{j=0}^{r-1} e_j \right) e^{WLx_n/(1-hWL)}, \tag{2.3.11}$$

assumptions (i) and (iii) gives

$$e_n \leq \frac{1}{1 - hWL}(C + rWLM)h^q e^{WLx_n/(1-hWL)}. \tag{2.3.12}$$

This proves that the method is convergent and the order of convergence is at least q. □

Chapter 3

Runge-Kutta method

In the solution of fuzzy (or crisp) ordinary differential equations, a class of algorithms called explicit Runge-Kutta methods is popular in practice and has been studied in great depth [1, 13, 40]. These methods are self-starting and they can also be used to produce starting values for the methods of section (2.3). In this chapter we develop explicit Runge-Kutta Method for nonlinear fuzzy Volterra integral equations and discuss their convergence and accuracy as a special case of quadrature methods.

3.1 Explicit Runge-Kutta method

The advantage of explicit methods is that they doesn't lead to nonlinear fuzzy equations. To develop such methods, one first writes down a general form for computing the solution at $x_n + h$ in terms of the solution at previous points. A similar idea can be used for integral equations. To present the discretization of (2.1.1) by explicit Runge-Kutta method we define

$$f_{nq} = \varphi_n(x_n + \theta_q h) \oplus h \sum_{j=0}^{q-1} A_{qj} \odot K(x_n + d_{qj}h, x_n + c_j h, f_{nj}), \quad q = 1, 2, \ldots, p, \qquad (3.1.1)$$

where

$$\varphi_n(x) = g(x) \oplus \sum_{k=0}^{n-1} h \odot \sum_{j=0}^{p-1} A_{pj} \odot K(x_k + d_{pj}h, x_n + c_j h, f_{nj}). \tag{3.1.2}$$

These equations present a p-stage explicit Runge-Kutta method for (2.1.1). Using $\varphi_n(0) = f_{00} = g(0)$, these equations can be used to determine the sequence of values $f_{00}, f_{01}, \ldots, f_{0p} = f_{10}, f_{11}, \ldots, f_{1p}, \ldots$. We consider f_{nq} as an approximation to $f(x_n + \theta_q h)$ and the value f_{np} is taken as an approximation to $f(x_n + h)$; also $f_{n+1,0} = f_{np}$. The quantity $\varphi(x)$ is a numerical approximation to the lag term

$$F_n(x) = g(x) \oplus \int_0^{x_n} K(x, s, f(s))\mathrm{d}s, \quad x \geq x_n, (n = 0, 1, \ldots, N-1), \tag{3.1.3}$$

So the second term in (3.1.1) is an approximation to

$$\int_{x_n}^{x_n+\theta_q h} K(x_n + \theta_q h, s, f(s))\mathrm{d}s. \tag{3.1.4}$$

Actually, (3.1.1) is so general and we can add some restriction on the parameters. For example, by setting $\theta_q = d_{qj} = c_q$ we obtain a set of formulas sometimes called Pouzet-type method. The parameters of a Pouzet-type formulae can be displayed conveniently in tabular form as [9]

$$\begin{array}{c|ccccc}
\theta_0 & A_{00} & & & & \\
\theta_1 & A_{10} & & & & \\
\theta_2 & A_{20} & A_{21} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
\theta_{p-1} & A_{p-1,0} & A_{p-1,1} & \ldots & A_{p-1,p-2} & \\
\hline
\theta_p = 1 & A_{p,0} & A_{p,1} & \ldots & A_{p,p-2} & A_{p,p-1}
\end{array} \tag{3.1.5}$$

Once we have chosen a set of θ_i with $0 \leq \theta_i \leq 1$, we can use standard quadrature rules to get the parameters A_{qj}. In this way, Runge-Kutta method can be considered as a quadrature method of the form (2.2.4) and according to theorem (2.3.2) it is convergent of the same order of the quadrature rules. For example one possible choice for θ_i with $p = 3$

is $\theta_i = \frac{i}{3}$, $i = 0, 1, 2, 3$ with the quadrature rules

$$\int_0^{\frac{h}{3}} f(s)\mathrm{d}s \approx \frac{h}{3} \odot f(0), \tag{3.1.6}$$

$$\int_0^{\frac{2h}{3}} f(s)\mathrm{d}s \approx \frac{2h}{3} \odot f(\frac{h}{3}), \tag{3.1.7}$$

$$\int_0^{h} f(s)\mathrm{d}s \approx \frac{h}{4} \odot f(0) \oplus \frac{3h}{4} \odot f(\frac{2h}{3}). \tag{3.1.8}$$

The above formulae are (3.1.6) a rectangle (Euler) rule, (3.1.7) the mid-point rule, and (3.1.8) a Radau rule. These methods are of order one, two and three respectively. Using these formulae to replace the integrals in (3.1.1), we are led to the formulas

$$\begin{aligned} f_{n1} &= g(x_n + \frac{h}{3}) \oplus \frac{h}{4}\sum_{i=0}^{n-1}\left(K(x_n + \frac{h}{3}, x_i, f_{i,0}) \oplus 3K(x_n + \frac{h}{3}, x_i + \frac{2h}{3}, f_{i2})\right) \\ &\quad \oplus \frac{h}{3}K(x_n + \frac{h}{3}, , x_n, f_{n,0}), \end{aligned} \tag{3.1.9}$$

$$\begin{aligned} f_{n2} &= g(x_n + \frac{2h}{3}) \oplus \frac{h}{4}\sum_{i=0}^{n-1}\left(K(x_n + \frac{2h}{3}, x_i, f_{i,0}) \oplus 3K(x_n + \frac{2h}{3}, x_i + \frac{2h}{3}, f_{i2})\right) \\ &\quad \oplus \frac{2h}{3}K(x_n + \frac{2h}{3}, x_n + \frac{h}{3}, f_{n1}), \end{aligned} \tag{3.1.10}$$

$$f_{n3} = g(x_n + h) \oplus \frac{h}{4}\sum_{i=0}^{n}\left(K(x_n + h, x_i, f_{i,0}) \oplus 3K(x_n + h, x_i + \frac{2h}{3}, f_{i2})\right), \tag{3.1.11}$$

with $f_{0,0} = g(0)$ and $f_{i,0} = f_{i-1,3}$. The associated table for this method is

$$\begin{array}{c|ccc} 0 & 0 & & \\ \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & 0 & \frac{1}{3} & \\ \hline 1 & \frac{1}{4} & 0 & \frac{3}{4} \end{array} \tag{3.1.12}$$

Order of the quadrature rules (3.1.6)-(3.1.8) and theorem (2.3.2) show that

$$e_{n3} = D(f_{n3}, f(nh)) = O(h^3), \qquad n = 1, 2, \ldots \tag{3.1.13}$$

i.e. this method is of order three.

We can write Eqs. (3.1.9-3.1.11) in terms of r-level sets as follows

$$\begin{aligned}
\underline{f_{n1}}(r) &= \underline{g}(x_n+\frac{h}{3};r)+\frac{h}{4}\sum_{i=0}^{n-1}\Big(\underline{K}(x_n+\frac{h}{3},x_i,f_{i,0};r)+3\underline{K}(x_n+\frac{h}{3},x_i+\frac{2h}{3},f_{i2};r)\Big)\\
&\quad+\frac{h}{3}\underline{K}(x_n+\frac{h}{3},x_n,f_{n,0};r), \qquad (3.1.14)\\
\overline{f_{n1}}(r) &= \overline{g}(x_n+\frac{h}{3};r)+\frac{h}{4}\sum_{i=0}^{n-1}\Big(\overline{K}(x_n+\frac{h}{3},x_i,f_{i,0};r)+3\overline{K}(x_n+\frac{h}{3},x_i+\frac{2h}{3},f_{i2};r)\Big)\\
&\quad+\frac{h}{3}\overline{K}(x_n+\frac{h}{3},x_n,f_{n,0};r), \qquad (3.1.15)\\
\underline{f_{n2}}(r) &= \underline{g}(x_n+\frac{2h}{3};r)+\frac{h}{4}\sum_{i=0}^{n-1}\Big(\underline{K}(x_n+\frac{2h}{3},x_i,f_{i,0};r)+3\underline{K}(x_n+\frac{2h}{3},x_i+\frac{2h}{3},f_{i2};r)\Big)\\
&\quad+\frac{2h}{3}\underline{K}(x_n+\frac{2h}{3},x_n+\frac{h}{3},f_{n1};r), \qquad (3.1.16)\\
\overline{f_{n2}}(r) &= \overline{g}(x_n+\frac{2h}{3};r)+\frac{h}{4}\sum_{i=0}^{n-1}\Big(\overline{K}(x_n+\frac{2h}{3},x_i,f_{i,0};r)+3\overline{K}(x_n+\frac{2h}{3},x_i+\frac{2h}{3},f_{i2};r)\Big)\\
&\quad+\frac{2h}{3}\overline{K}(x_n+\frac{2h}{3},x_n+\frac{h}{3},f_{n1};r), \qquad (3.1.17)\\
\underline{f_{n3}}(r) &= \underline{g}(x_n+h;r)+\frac{h}{4}\sum_{i=0}^{n}\Big(\underline{K}(x_n+h,x_i,f_{i,0};r)+3\underline{K}(x_n+h,x_i+\frac{2h}{3},f_{i2};r)\Big), \qquad (3.1.18)\\
\overline{f_{n3}}(r) &= \overline{g}(x_n+h;r)+\frac{h}{4}\sum_{i=0}^{n}\Big(\overline{K}(x_n+h,x_i,f_{i,0};r)+3\overline{K}(x_n+h,x_i+\frac{2h}{3},f_{i2};r)\Big), \qquad (3.1.19)
\end{aligned}$$

for $0\le r\le 1$ with $\underline{f_{0,0}}(r)=\underline{g}(0;r)$, $\overline{f_{0,0}}(r)=\overline{g}(0;r)$, $\underline{f_{i,0}}(r)=\underline{f_{i-1,3}}(r)$ and $\overline{f_{i,0}}(r)=\overline{f_{i-1,3}}(r)$.

3.2 Numerical Examples

Example 3.2.1. *Consider fuzzy integral equation*

$$f(x)=g(x)\oplus\int_0^x\frac{s}{1+(f(s))^2}\mathrm{d}s, \qquad 0\le x\le 1$$

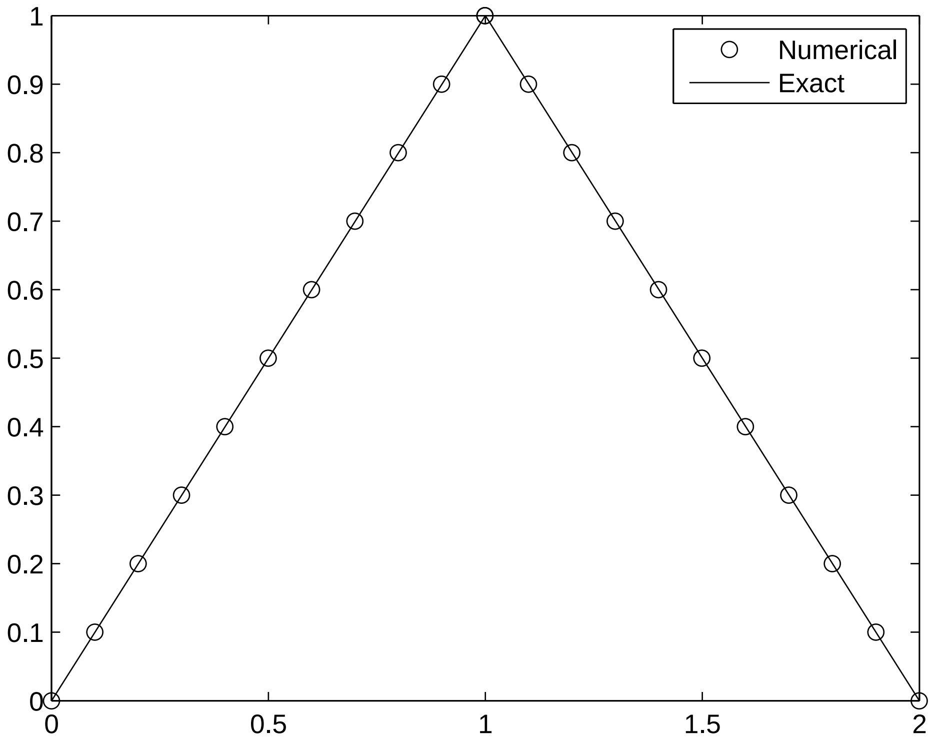

Figure 3.1: Exact vs. Numerical solution of example 3.2.1

where $g(x)$ is a fuzzy function with level sets

$$[g(x)]^r = [\underline{g}(x;r), \overline{g}(x;r)] = [r - \frac{x^2}{2r^2 - 8r + 10}, 2 - r - \frac{x^2}{2r^2 + 2}].$$

The exact solution is $f(t) = A$ where A is triangle fuzzy number $(0, 1, 2)$. The approximation of $f(1)$ using formulae (3.1.14-3.1.19) with $h = 0.5$ is compared with the exact solution in Fig.3.1

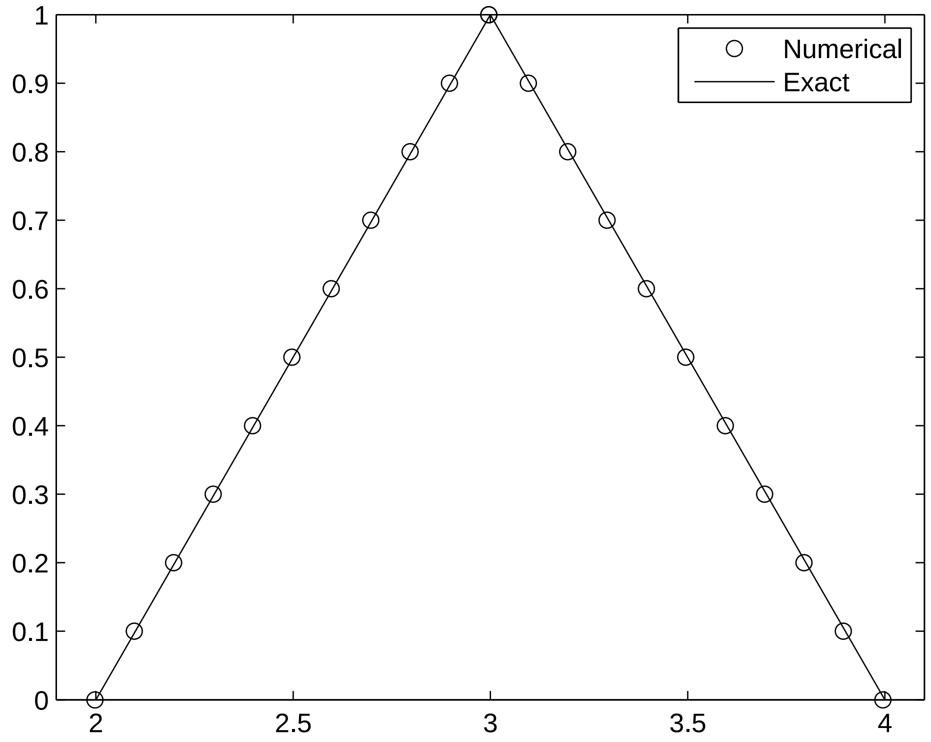

Figure 3.2: Exact vs. Numerical solution of example 3.2.2

Example 3.2.2. *Consider fuzzy integral equation*

$$f(x) = g(x) \oplus \int_0^x \frac{s\sqrt{1+f(s)}}{1+t^2}\mathrm{d}s, \qquad 0 \leq x \leq 5$$

with

$$[g(x)]^r = [2 + r - \frac{x^2\sqrt{5+4r+r^2}}{2+2x^2}, 4 - r - \frac{x^2\sqrt{17-8r+r^2}}{2+2x^2}].$$

The exact solution is $f(t) = B$ *where* B *is triangle fuzzy number* $(2, 3, 4)$. *The approximation of* $f(5)$ *using formulae (3.1.14-3.1.19) with* $h = 1$ *is compared with the exact solution in Fig.3.2*

Chapter 4

Predictor-Corrector method for fuzzy Volterra integral equations

In this chapter we consider predictor-corrector methods using Newton-Cotes quadrature formulas along with a discussion on the discretization error.

4.1 Predictor-corrector method using Newton-Cotes formulas

In previous chapters we obtained some explicit and implicit methods for approximate solutions of fuzzy integral equations. Implicit methods have higher order of accuracy but as we mentioned before these methods require solving a nonlinear algebraic fuzzy equation at each step. An implicit method is as follows

$$f_n = g_n \oplus h \odot \sum_{k=0}^{n} w_{nk} \odot K(x_n, x_k, f_k), \quad n = 1, 2, \ldots, N, \tag{4.1.1}$$

$$f_0 = g_0, \tag{4.1.2}$$

where $w_{nn} \neq 0$.

The following iterative method for determining f_n suggests itself naturally:

$$f_n^{i+1} = g_n \oplus h\Big(\sum_{k=0}^{n-1} w_{nk} K(x_n, x_k, f_k) \oplus w_{nn} K(x_n, x_k, f_n^i)\Big), \quad i = 0, 1, \ldots . \tag{4.1.3}$$

For given $f_0, f_1, f_2, \ldots, f_{n-1}$ a good initial value f_n^0 for the iteration (4.1.3) can be found with the aid of an explicit method. For this reason, one also calls explicit methods predictor methods, and implicit methods corrector methods [through the iteration (4.1.3) one corrects f_n^i].

The iterative method (4.1.3) converges for sufficiently small h. In order to obtain a criteria for convergence, from equations (4.1.1) and (4.1.3) and Lipschitz condition we have

$$\begin{aligned} D(f_n^{i+1}, f_n) &= D\Big(g_n \oplus h \sum_{k=0}^{n} w_{nk} K(x_n, x_k, f_k) \\ &\qquad , g_n \oplus h \sum_{k=0}^{n-1} w_{nk} K(x_n, x_k, f_k) \oplus w_{nn} K(x_n, x_k, f_n^i)\Big) \\ &= h w_{nn} D(K(x_n, x_k, f_k), K(x_n, x_k, f_n^i) \\ &\leq hWLD(f_n, f_n^i) \end{aligned} \tag{4.1.4}$$

and hence

$$D(f_n^{i+1}, f_n) \leq (hWL)^{i+1} D(f_n, f_n^0). \tag{4.1.5}$$

Thus if $h < \frac{1}{WL}$, then $f_n^i \to f_n$ as $i \to \infty$.

A predictor-corrector method using Newton-Cotes formulas in the predictor and corrector formulas takes the following form:

$$p_n = g_n \oplus h\Big(\sum_{k=0}^{s} w_{kj} K(x_n, x_k, f_k) \oplus \sum_{k=s}^{n-1} w_{nk}^* K(x_n, x_k, f_k)\Big) \tag{4.1.6}$$

$$c_n = g_n \oplus h\Big(\sum_{k=0}^{n-1} w_{nk} K(x_n, x_k, f_k) \oplus w_{nn} K(x_n, x_k, p_n)\Big), \qquad w_{nn} \neq 0 \tag{4.1.7}$$

where p_n is a predicted value for f_n, c_n is a corrected value for f_n and the numbers W_{nk} are a set of weights associated with a quadrature rule. Corrections can be made more than once and the value that is finally accepted is denoted by f_n.

In order to obtain an example of such a method using Simpson's rule we should consider two cases separately.

First suppose n is even. Using Simpson's rule over the interval $[x_0, x_{n-2}]$ and midpoint rule over the interval $[x_{n-2}, x_n]$ we have

$$\int_{x_0}^{x_{n-2}} f(x)\mathrm{d}x \approx \frac{h}{3}(f_0 \oplus 4f_1 \oplus 2f_2 \oplus 4f_2 \oplus \ldots \oplus 4f_{n-3} \oplus f_{n-2}), \tag{4.1.8}$$

$$\int_{x_{n-2}}^{x_n} f(x)\mathrm{d}x \approx 2hf_{n-1}. \tag{4.1.9}$$

Hence

$$\begin{aligned} p_n = \; & g_n \oplus \frac{h}{3}\Big(K(x_n, x_0, f_0) \oplus 4K(x_n, x_1, f_1) \oplus 2K(x_n, x_2, f_2) \\ & \oplus 4K(x_n, x_3, f_3) \oplus \ldots \oplus 4K(x_n, x_{n-3}, f_{n-3}) \\ & \oplus K(x_n, x_{n-2}, f_{n-2})\Big) \oplus 2hK(x_n, x_{n-1}, f_{n-1}). \end{aligned} \tag{4.1.10}$$

This predicted value p_n can be corrected using Simpson's rule over the interval $[x_0, x_n]$ as follows

$$\begin{aligned} c_n = \; & g_n \oplus \frac{h}{3}\Big(K(x_n, x_0, f_0) \oplus 4K(x_n, x_1, f_1) \oplus 2K(x_n, x_2, f_2) \\ & \oplus 4K(x_n, x_3, f_3) \oplus \ldots \oplus 4K(x_n, x_{n-3}, f_{n-3}) \\ & \oplus 2K(x_n, x_{n-2}, f_{n-2}) \oplus 4K(x_n, x_{n-1}, f_{n-1}) \oplus K(x_n, x_n, p_n)\Big). \end{aligned} \tag{4.1.11}$$

Now suppose n is odd. Using Simpson's rule over the interval $[x_0, x_{n-3}]$ and open Newton-Cotes formula over the interval $[x_{n-3}, x_n]$ we have

$$\int_{x_0}^{x_{n-3}} f(x)\mathrm{d}x \approx \frac{h}{3}(f_0 \oplus 4f_1 \oplus 2f_2 \oplus 4f_2 \oplus \ldots \oplus 4f_{n-4} \oplus f_{n-3}), \tag{4.1.12}$$

$$\int_{x_{n-3}}^{x_n} f(x)\mathrm{d}x \approx \frac{3h}{2}(f_{n-2} \oplus f_{n-1}). \tag{4.1.13}$$

Hence

$$\begin{aligned} p_n &= \frac{h}{3}\Big(K(x_n,x_0,f_0)\oplus 4K(x_n,x1,f_1)\oplus 2K(x_n,x_2,f_2) \\ &\oplus 4K(x_n,x_3,f_3)\oplus\ldots\oplus 4K(x_n,x_{n-4},f_{n-4})\oplus K(x_n,x_{n-3},f_{n-3})\Big) \\ &\oplus\frac{3h}{2}(K(x_n,x_{n-2},f_{n-2})\oplus K(x_n,x_{n-1},f_{n-1})). \end{aligned} \tag{4.1.14}$$

This predicted value p_n can be corrected using Simpson's rule over the interval $[x_0, x_{n-3}]$ and three-eighths rule over the interval $[x_{n-3}, x_n]$

$$\begin{aligned} c_n &= \frac{h}{3}\Big(K(x_n,x_0,f_0)\oplus 4K(x_n,x_1,f_1)\oplus 2K(x_n,x_2,f_2) \\ &\oplus 4K(x_n,x_3,f_3)\oplus\ldots\oplus 4K(x_n,x_{n-4},f_{n-4})\oplus K(x_n,x_{n-3},f_{n-3})\Big) \\ &\oplus\frac{3h}{8}\Big(K(x_n,x_{n-3},f_{n-3})\oplus 3K(x_n,x_{n-2},f_{n-2}) \\ &\oplus 3K(x_n,x_{n-1},f_{n-1})\oplus K(x_n,x_n,f_n)\Big). \end{aligned} \tag{4.1.15}$$

Thus weights for this method is as follows

$$\begin{array}{lll} n \text{ is even:} & s = n-2, & \\ & w_{n0} = w_{nn} = \frac{1}{3}, & \\ & w_{n,2i} = \frac{2}{3}, & i = 1,2,\ldots,\frac{n}{2}-1, \\ & w_{n,2i+1} = \frac{4}{3}, & i = 0,1,\ldots,\frac{n}{2}-1, \\ & w^*_{n,n-1} = 2, & \\ & w^*_{ns} = w^*_{nn} = 0. & \end{array} \tag{4.1.16}$$

$$\begin{array}{lll} n \text{ is odd:} & s = n-3, & \\ & w_{n0} = w_{nn} = \frac{1}{3}, & n \geq 5, \\ & w_{n,2i} = \frac{2}{3}, & i = 1,2,\ldots,\frac{n-5}{2}, \\ & w_{n,2i+1} = \frac{4}{3}, & i = 0,1,\ldots,\frac{n-5}{2}, \\ & w_{n,n-3} = \frac{17}{24} - \frac{1}{3}\delta_{n3}, & \\ & w_{n,n-1} = w_{n,n-2} = \frac{9}{8}, & \\ & w_{n,n} = \frac{3}{8}, & \\ & w^*_{n,n-2} = w^*_{n,n-1} = \frac{3}{2}, & \\ & w^*_{n,s} = w^*_{n,n} = 0. & \end{array} \tag{4.1.17}$$

We should know f_1 as starting value which could be obtained by some other method of the same order, say Runge-Kutta method.

4.2 Convergence of Predictor-Corrector Method

In previous section we saw that in the case of correcting until we obtain the exact value f_n for equation (4.1.1) the predictor-corrector method is convergent. In practise only a finite number of iterations are performed. The accuracy of the result is then dependent on the accuracy of the predicted value. In particular, we shall discuss the discretization error of the method when only one correction is made.

Definition 4.2.1. Let z_n denote the predicted value for $f(x_n)$ which would have been obtained if the exact values $f(x_n)$, $i = 0, 1, \ldots, n-1$ were used in an approximating method of the form (4.1.6). Then the local discretization error d_n at x_n is defined by $d_n = D(z_n, f(x_n))$.

Assuming the conditions required for a unique solution of equation (2.1.1) hold, we shall state the following theorem dealing with error bounds.

Theorem 4.2.1. *Assume*

(i) $d_n \leq Ch^{q-1}$ *for all* n*, where* $q > 1$ *and* $C \neq 0$ *is a constant (independent of* h*),*

(ii) the starting errors satisfy $e_k < Mh^{q-l}$, $k = 0, 1, \ldots, m-1$*, where* $M > 0$ *is a constant,*

(iii) equation (4.1.7) with $W_{nn} \neq 0$ *satisfies*

$$\delta(h, x_n) = D\Big(\int_0^{x_n} K(x_n, s, f(s))\mathrm{d}s, h\sum_{k=0}^{n} w_{nk}K(x_n, x_k, f(x_k))\Big) < C_n h^q$$

$$n = m, m+1, m+2, \ldots \qquad (4.2.1)$$

where $C_n > 0$ is a constant.

Using (4.1.7) as a corrector only once there exists a constant $M_1 > 0$ such that $e_n < M_1 h^q$ for $n = m+1, m+2, \ldots$.

Proof. The proof is by induction. From (ii) $e_k \leq Mh^{q-l},\ k = 0, 1, \ldots, m-1$. Let $n = m$. Using the predictor, there exists a constant $M^* > 0$ such that $D(f(x_m), p_m) \leq M^* h^{q-1}$. Equations (4.1.7) and (4.2.1) and Lipschitz condition yield

$$\begin{aligned}
e_m &= D(f(x_m), f_m) \\
&= D\Big(g(x_m) \oplus \int_0^{x_m} K(x_m, s, f(s))\mathrm{d}s \\
&\qquad , g(x_m) \oplus h\big(\sum_{k=0}^{m-1} w_{mk} K(x_m, x_k, f_k) \oplus w_{mm} K(x_m, x_k, p_m)\big)\Big) \\
&= D\Big(\int_0^{x_m} K(x_m, s, f(s))\mathrm{d}s, h\big(\sum_{k=0}^{m-1} w_{mk} K(x_m, x_k, f_k) \oplus w_{mm} K(x_m, x_k, p_m)\big)\Big) \\
&\leq D\Big(\int_0^{x_m} K(x_m, s, f(s))\mathrm{d}s, h\sum_{k=0}^{m} w_{mk} K(x_m, x_k, f(x_k))\Big) \\
&\quad + D\Big(h\sum_{k=0}^{m-1} w_{mk} K(x_m, x_k, f(x_k)), h\sum_{k=0}^{m-1} w_{mk} K(x_m, x_k, f_k)\Big) \\
&\quad + D\Big(hw_{mm} K(x_m, x_m, f(x_m)), hw_{mm} K(x_m, x_m, p_m)\Big) \\
&\leq \delta(h, x_m) + h\sum_{k=0}^{m-1} w_{mk} D\Big(K(x_m, x_k, f(x_k)), K(x_m, x_k, f_k)\Big) \\
&\quad + hw_{mm} D(K(x_m, x, f(x_m)), K(x_m, x_m, p_m)) \\
&\leq C_m h^q + hWL\big(\sum_{k=0}^{m-1} e_k + D(f(x_m), p_m)\big) \\
&\leq C_m h^q + hWL\big(mCh^{q-1} + M_1 h^{q-1}\big) \\
&\leq (C_m + aWLC + M_1)h^q.
\end{aligned}$$

Hence there exist M_2 such that

$$e_m \leq M_2 h^q.$$

Assume there exists a constant $M_3 \geq 0$ such that $e_n \leq M_3 h^q,\ n = m+1, m+2, \ldots, r-1,\ (r \leq N)$. Let $n = r$. By (i), we have $D(z_r, f(x_r)) \leq Ch^{q-1}$. Using the definitions of z_r and p_r, we may write

$$\begin{aligned}
D(p_r, z_r) &= D\Big(g_r \oplus h \sum_{k=0}^{r-1} w^*_{rk} K(x_r, x_k, f_k), g_r \oplus h \sum_{k=0}^{r-1} w^*_{rk} K(x_r, x_k, f(x_k))\Big) \\
&= D\Big(h \sum_{k=0}^{r-1} w^*_{rk} K(x_r, x_k, f_k), h \sum_{k=0}^{r-1} w^*_{rk} K(x_r, x_k, f(x_k))\Big) \\
&\leq h\Big(\sum_{k=0}^{r-1} w^*_{rk} D(K(x_r, x_k, f_k), K(x_r, x_k, f(x_k)))\Big) \\
&\leq hWL \sum_{k=0}^{r-1} D(f_k, f(x_k)) \\
&= hWL \sum_{k=0}^{r-1} e_k \\
&\leq rWLM_2 h^{q+1} \\
&\leq WLM_2 N h^{q+1} \\
&= aWLM_2 h^q
\end{aligned}$$

Thus there exists a constant $B \geq 0$ such that $D(p_r, z_r) \leq Bh^q$. Moreover

$$\begin{aligned}
D(p_r, f(x_r)) &\leq D(p_r, z_r) + D(z_r, f(x_r)) \\
&\leq Bh^q + Ch^{q-1}
\end{aligned}$$

Therefore there exists a constant B_2 such that $D(p_r, f(x_r)) < B_2 h^{q-1}$. To complete the proof

$$
\begin{aligned}
e_r &= D(f(x_r), f_r) \\
&= D\Big(g(x_r) \oplus \int_0^{x_r} K(x_r, s, f(s))\mathrm{d}s \\
&\qquad , g(x_r) \oplus h\big(\sum_{k=0}^{r-1} w_{rk} K(x_r, x_k, f_k) \oplus w_{rr} K(x_r, x_k, p_r)\big)\Big) \\
&= D\Big(\int_0^{x_r} K(x_r, s, f(s))\mathrm{d}s, h\big(\sum_{k=0}^{r-1} w_{rk} K(x_r, x_k, f_k) \oplus w_{rr} K(x_r, x_k, p_r)\big)\Big) \\
&\leq D\Big(\int_0^{x_r} K(x_r, s, f(s))\mathrm{d}s, h\sum_{k=0}^{r} w_{rk} K(x_r, x_k, f(x_k))\Big) \\
&\quad + D\Big(h\sum_{k=0}^{r-1} w_{rk} K(x_r, x_k, f(x_k)), h\sum_{k=0}^{r-1} w_{rk} K(x_r, x_k, f_k)\Big) \\
&\quad + D\Big(h w_{rr} K(x_r, x_r, f(x_r)), h w_{rr} K(x_r, x_r, p_r)\Big) \\
&\leq \delta(h, x_r) + h\sum_{k=0}^{r-1} w_{rk} D\Big(K(x_r, x_k, f(x_k)), K(x_r, x_k, f_k)\Big) \\
&\quad + h w_{rr} D(K(x_r, x_, f(x_r)), K(x_r, x_r, p_r)) \\
&\leq C_r h^q + hWL\big(\sum_{k=0}^{r-1} e_k + D(f(x_r), p_r)\big) \\
&\leq C_r h^q + hWL\big(rM_3 h^{q-1} + B_2 h^{q-1}\big) \\
&\leq (C_r + aWLM_3 + B_2)h^q.
\end{aligned}
$$

Hence there exist M^* such that

$$e_r \leq M^* h^q.$$

□

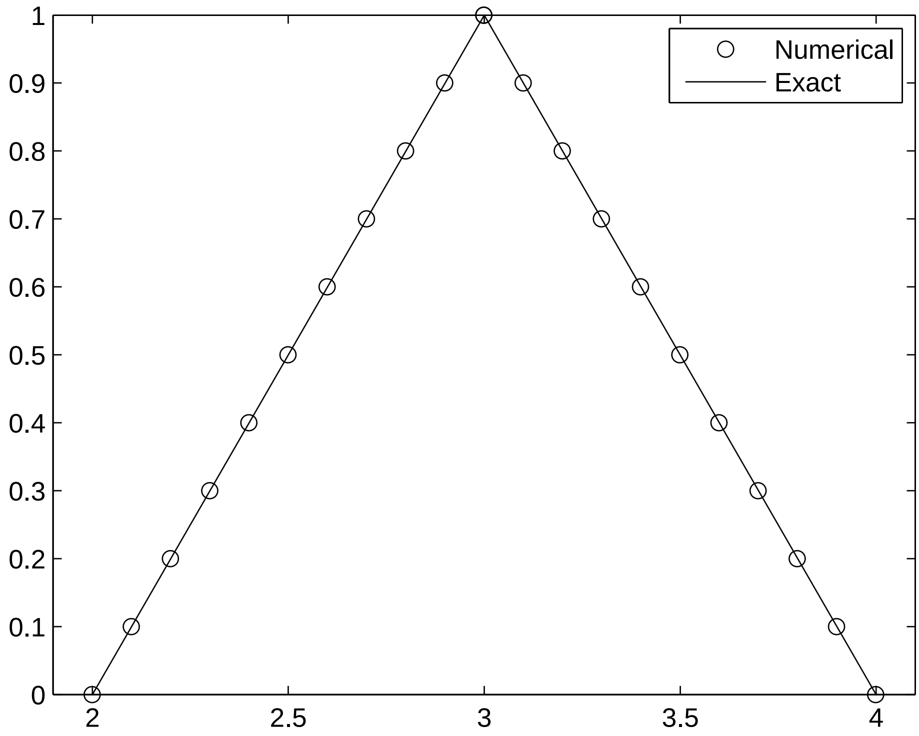

Figure 4.1: Exact vs. Numerical solution of example 4.3.1

4.3 Numerical Example

Example 4.3.1. *We consider again example 3.2.2 of chapter 3 with exact solution* $f(t) = B$ *where* B *is triangle fuzzy number* $(2, 3, 4)$. *The approximation of* $f(5)$ *using formulae (4.1.10-4.1.14) with* $h = .5$ *is compared with the exact solution in Fig.4.1*

Conclusion

In this thesis we developed quadrature methods for solving nonlinear fuzzy Volterra integral equations. Convergence of the proposed method was proved and some explicit Runge-Kutta method developed as an especial case of quadrature method. We also developed a predictor-corrector method for solving nonlinear fuzzy Volterra integral equations numerically. The convergence of the numerical methods was illustrated with some examples.

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حل عددی معادلات انتگرال فازی

چکیده

پس از کار آبل در سال 1820 آنالیزدانان همواره به معادلات انتگرالی علاقمند بوده اند .نامهای بسیاری از ریاضی دانان مدرن، شامل کشی، هیلبرت، فردهلم، ولترا و دیگران با این موضوع عجین شده است .دو دلیل برای این علاقه وجود دارد .در بعضی مواقع، مانند کار آبل روی منحنی همزمانی، معادلات انتگرالی مدل ریاضی طبیعی برای یک مساله ی جالب فیزیکی هستند .دلیل دوم و شاید رایجتر این است که عملگر ها، تبدیلها و معادلات انتگرالی ابزار مناسبی برای مطالعه ی معادلات دیفرانسیلی هستند .بنابراین تکنیکهای معادلات انتگرالی برای آنالیز دانان و آنالیز عددی دانان شناخته شده بود و نتایج بسیار زیبا و قدرتمندی توسط آنان توسعه داده شد.

در عمل پارامترهای یک مدل ریاضی دقیق نیستند و یک ابزار خوب برای چنین مسائلی نظریه ی مجموعه های فازی است .سیستمهای فازی برای مطالعه ی دسته ی وسیعی از مسائل شامل فضاهای متریک فازی، مدل های جمعیتی، میانگین طلایی، سیستمهای ذرات، بینایی کوانتمی و جاذبه، سیستمهای همزمانی آشوبناک، سیستمهای کنترل آشوبناک، پزشکی، بیوانفورماتیک و زیست شناسی محاسباتی مفید هستند.

ما در این تحقیق روشی برای حل عددی معادلات انتگرالی فازی ولترای غیرخطی پیشنهاد می کنیم .در ابتدا تعاریف پایه از اعداد فازی و انتگرال فازی، یک متریک روی فضای اعداد فازی و قضیه ی وجود جواب برای معادلات انتگرالی فازی آورده می شود .سپس روش کلی کوادراتور برای حل عددی معادلات انتگرالی فازی ولترای نوع دوم ارئه می شود و همگرایی آن اثبات می شود .در ادامه دو روش عددی دیگر نیز همراه با مثالهای عددی ارئه می شود که روش اول، رونگه-کوتای صریح و دومی روش پیشگو-اصلاحگر است.