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APPROXIMATE ANALYTICAL SOLUTION OF FUZZY PARTIAL DIFFERENTIAL EQUATIONS

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Approximate Analytical Solution of Fuzzy Partial Differential Equations**" by **Hamid Rouhparvar** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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To My Parents

Contents

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Outline of the Thesis

Our discussion begins with Chapter 1 where is offered a summary of fundamentals of fuzzy mathematics. In Chapter 2, it is pointed out that how the strategy Bukley-Feuring is applied to solve the heat-like and wave-like equations by variational iteration method. Application of Hukuhara derivative is studied in a real-world model, and are proved existence and uniqueness theorems in Chapter 3 by Adomian decomposition method. Finally, in Chapter 4, generalized directional derivative concept and its characterizations are presented and also its application is studied to fuzzy partial differential equation.

Articles

1-Existence and uniqueness of solution an uncertain characteristic Cauchy reaction-diffusion equation by Adomian decomposition method, *Mathematical and computational Applications (MCA)*, In press. (ISI)

2-Generalizations of directional derivative of fuzzy-number-valued functions with applications to fuzzy partial differential equations, *Submitted to the Information Sciences.*

3-Existence and uniqueness of fuzzy solution for fuzzy linear parabolic equation by Adomian decomposition method, *Submitted to the Computers and Mathematics with Applications.*

4-The variational iteration method for exact solutions of fuzzy heat-like equations with variable coefficients, *Submitted to the Analele Stiintifice Ale Universitatii Ovidius Constanta-Seria Matematica.*

5-The exact solutions of fuzzy wave-like equations with variable coefficients by a variational iteration method, *Submitted to the Journal of "ASOC".*

6-Analytical approximate solution of fuzzy characteristic Cauchy reaction-diffusion equation with the variational iteration method, 3^{th} *Joint Congress on Fuzzy and Intelligent Systems, Yazd, Iran, July 2009.*

7- A computational method to find an approximate analytical solution for fuzzy differential equations,*Analele Stiintifice Ale Universitatii Ovidius Constanta-Seria Matematica, Vol. 17(1), 2009, 5-14.*(ISI)

8-Solving fuzzy linear systems of equations, *ROMAI Journal, 4, 1(2008), 207-214.*

9-Existence and uniqueness of fuzzy solution for linear Volterra fuzzy integral equations with Adomian decomposition method, *ROMAI Journal, 5, nr. 1 ,2009,In Press.* (Oral presentation in CAIM 2009, Romania)

10-Defuzzification of generalized fuzzy numbers by incentre point and its application, *39th Annual Iranian Mathematics Conference 24-27 August 2008, Shahid Bahonar University, Kerman, Iran.*

11-A method for fully fuzzy linear system of equations, *First Joint Congress on Fuzzy and Intelligent Systems Ferdowsi University of Mashhad, Iran 29-31 Aug 2007.*

12-Solving fuzzy polynomial equation by ranking method, *First Joint Congress on Fuzzy and Intelligent Systems Ferdowsi University of Mashhad, Iran 29-31 Aug 2007.*

13-LU decomposition of a fuzzy matrix, *First Joint Congress on Fuzzy and Intelligent Systems Ferdowsi University of Mashhad, Iran 29-31 Aug 2007.*

14-Approximate analytical solution of fuzzy first order initial value problem, *39th Annual Iranian Mathematics Conference, Kerman, Iran, 2008.*

15-Solving fuzzy differential inclusions using the LU-representation of fuzzy numbers, *J. Sci. I. A. U. (JSIAU)*, In press.

Abstract

The aim of this work is to use different point of views for solving types of *Fuzzy Partial Differential Equations*, (FPDEs) that are applied in various sciences.

In the first point of view, is assumed to be exist fuzzy parameters in the heat-like and wave-like equations, and henceforth this equations are studied by Buckley-Feuring method. In this method, at first the crisp equations are solved by variational iteration method and then in case of possible the crisp solution is translated to the fuzzy solution.

In the second point of view, the characteristic Cauchy problem is considered by fuzzy initial condition and is supposed fuzzy functions are Hukuhara differentiable. Existence, uniqueness and convergence of fuzzy solution are proved by Adomian decomposition method and several examples are solved using parametric representation.

In the third point of view, note that Hukuhara derivative is very restrictive. For to void this shortcoming, Bede introduced generalized derivative for fuzzy functions from $\mathbb{R}$ to fuzzy numbers space. In here, the concept of generalized derivative is extended for fuzzy functions from $\mathbb{R}^n$ to fuzzy numbers space and is also studied some its characteristics. As an application, existence and convergence of fuzzy solution is proved using this concept of derivative in a fuzzy partial differential equation.

Chapter 1

Preliminaries

Zadeh is credited with introducing the concept of fuzzy sets in 1965 as a mathematical means of describing vagueness in linguistics. The idea may be considered as a generalization of classical set theory. In the decade since Zadeh's pioneering paper on fuzzy sets [54], many theoretical developments in fuzzy logic took place in the United States, Europe, and Japan. From the mid-1970s to the present, however, Japanese researchers have done an excellent job of advancing the practical implementation of the theory; they have been a primary force in commercializing this technology. Much of the success of the new products associated with the fuzzy technology is due to fuzzy logic, and some is also due to the advanced sensors used in these products. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes, such as thinking and reasoning.

1.1 Introduction

In this chapter, the basic definitions of fuzzy sets and algebraic operations are defined and extension principle are provided which is one of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets. The concepts of this chapter are found in [8, 12, 14, 19, 20, 22, 25, 30, 37, 42, 48, 55].

1.2 Fuzzy sets

The basic idea of fuzzy sets is quite simple. In a conventional (nonfuzzy or crisp) set, an element of the universe either belongs to or does not belong to the set. That is, the membership of an element is crisp-it is either yes (in the set) or no (not in the set).

A fuzzy set is a generalization of an ordinary set in that it allows the degree of membership for each element to range over the unit interval $[0, 1]$. Thus, the membership function of a fuzzy set maps each element of the universe of discourse to its range space, which, in most cases, is assumed to be the unit interval.

One major difference between crisp and fuzzy sets is that crisp sets always have unique membership functions, whereas every fuzzy set has an infinite number of membership functions that may represent it. This enables fuzzy systems to be adjusted for maximum utility in a given situation.

Definition 1.2.1. Let X be a nonempty set. A fuzzy set $\bar{A}$ in X is characterized by its membership function $\mu_{\bar{A}} : X \to [0, 1]$ and $\mu_{\bar{A}}(x)$ is interpreted as the degree of membership of element x in fuzzy set $\bar{A}$ for each $x \in X$.

It is clear that $\bar{A}$ is completely determined by the set of ordered pairs

$$\{(x, \mu_{\bar{A}}(x))|x \in X\}.$$

Definition 1.2.2. Let $\bar{A}$ be a fuzzy subset of X, the support of $\bar{A}$, denoted $supp(\bar{A})$, is the crisp subset of X whose elements all have nonzero membership grades in $\bar{A}$.

$$supp(\bar{A}) = \{x \in X | \mu_{\bar{A}}(x) > 0\}.$$

Definition 1.2.3. A fuzzy subset $\bar{A}$ of a classical set X is called normal if there exists an $x \in X$ such that $\mu_{\bar{A}}(x) = 1$.

Definition 1.2.4. An γ-level set of a fuzzy set $\bar{A}$ of X is a non-fuzzy set denoted by $\bar{A}[\gamma] = [A^-(\gamma), A^+(\gamma)]$ and is defined by

$$\bar{A}[\gamma] = \begin{cases} \{x \in X | \mu_{\bar{A}}(x) \geq \gamma\}, & 0 < \gamma \leq 1, \\ \\ cl(supp\bar{A}), & \gamma = 0, \end{cases}$$

where $cl(supp\bar{A})$ denotes the closure of the support of $\bar{A}$.

Definition 1.2.5. (Extension principle) Assume X and Y are crisp sets and let F be a mapping from X to Y, $F : X \to Y$ such that for each $x \in X$; $F(x) = y \in Y$. Assume $\bar{A}$ is a fuzzy subset of X, using the extension principle, we can define $F(\bar{A})$ as a fuzzy subset of Y such that

$$F(\bar{A})(y) = \begin{cases} \sup_{x \in F^{-1}(y)} \mu_{\bar{A}}(x), & F^{-1}(y) \neq 0, \\ \\ 0, & otherwise, \end{cases}$$

where $F^{-1}(y) = \{x \in X | F(x) = y\}$.

Definition 1.2.6. A fuzzy set $\bar{A}$ of X is called convex if $\bar{A}[\gamma]$ is a convex subset of X for all $\gamma \in [0,1]$. That is, $\mu_{\bar{A}}(\gamma t + (1-\gamma)x) \geq \min(\mu_{\bar{A}}(t), \mu_{\bar{A}}(x))$ for all $\gamma \in [0,1]$ and $t, x \in X$.

1.3 Fuzzy numbers

In this section, is denoted the definition of fuzzy number and its properties.

Definition 1.3.1. Let $E = \{\bar{A} | \bar{A} : \mathbb{R} \to [0,1], \, has\, the\, following\, properties\, (1){-}(4)\}$:

(1) $\forall \bar{A} \in E$, $\bar{A}$ is normal;

(2) $\forall \bar{A} \in E$, $\bar{A}$ is a convex fuzzy set;

(3) $\forall \bar{A} \in E$, $\bar{A}$ is upper semi-continuous on $\mathbb{R}$; and

(4) $\bar{A}[0]$ is a compact set.

Then E is called fuzzy number space and $\forall \bar{A} \in E$, $\bar{A}$ is called a fuzzy number.

Definition 1.3.2. Let $\bar{A}$ be a fuzzy number. If $supp(\bar{A}) = \{x_0\}, x_0 \in \mathbb{R}$, then $\bar{A}$ is called a fuzzy point and we use the notation $\bar{A} = \tilde{x}_0$. It is easy to see that $\bar{A}[\gamma] = [x_0, x_0] = \{x_0\}$, $\forall \gamma \in [0,1]$. Obviously, $\mathbb{R} \subset E$.

The following lemma is called the Negoita-Ralescu Stacking Theorem.

Lemma 1.3.1. *Let* $[A^-(\gamma^1), A^+(\gamma^1)]$, $0 < \gamma^1 \leq 1$, *be a given family of non-empty intervals. If*

(i) $[A^-(\gamma^1), A^+(\gamma^1)] \supset [A^-(\gamma^2), A^+(\gamma^2)]$ *for* $0 < \gamma^1 \leq \gamma^2$, *and*

(ii) $[\lim_{k\to\infty} A^-(\gamma^1_k), \lim_{k\to\infty} A^+(\gamma^1_k)] = [A^-(\gamma^1), A^+(\gamma^1)]$,

whenever γ^1_k *is a nondecreasing sequence converging to* $\gamma^1 \in (0,1]$, *then the family* $[A^-(\gamma^1), A^+(\gamma^1)]$, $0 < \gamma^1 \leq 1$, *represents the* γ^1*-level sets of a fuzzy number* $\bar{A}$ *in* E. *conversely, if* $[A^-(\gamma^1), A^+(\gamma^1)]$, $0 < \gamma^1 \leq 1$, *are the* γ^1*-level sets of a fuzzy number* $\bar{A} \in E$, *then the conditions (i) and (ii) hold true.*

For $\bar{A}, \bar{B} \in E$ and $\tau \in \mathbb{R}$ we can define sum and scalar multiplication on E, respectively, by

$$(\bar{A} \oplus \bar{B})[\gamma] = \bar{A}[\gamma] + \bar{B}[\gamma],$$

$$(\tau \odot \bar{A})[\gamma] = \tau\bar{A}[\gamma], \ \ \forall \gamma \in [0,1],$$

where $\bar{A}[\gamma] + \bar{B}[\gamma]$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\tau\bar{A}[\gamma]$ means the usual product between a scalar and a subset of $\mathbb{R}$ and

$$(A+B)^-(\gamma) = A^-(\gamma) + B^-(\gamma), \quad (A+B)^+(\gamma) = A^+(\gamma) + B^+(\gamma),$$

$$(\tau A)^-(\gamma) = \begin{cases} \tau A^-(\gamma) & \tau \geq 0, \\ \tau A^+(\gamma) & \tau < 0, \end{cases}$$

and

$$(\tau A)^+(\gamma) = \begin{cases} \tau A^+(\gamma) & \tau \geq 0, \\ \tau A^-(\gamma) & \tau < 0, \end{cases} \qquad for\, any\, \gamma \in [0,1].$$

Definition 1.3.3. Let $\bar{A}, \bar{B} \in E$. If there exists $\bar{C} \in E$ such that $\bar{A} = \bar{B} \oplus \bar{C}$, then $\bar{C}$ is called H-difference of $\bar{A}$ and $\bar{B}$ and it is denoted by $\bar{A} -_H \bar{B} = \bar{C}$.

It is obvious that if the H-difference $\bar{A} -_H \bar{B}$ exists, then $(A-B)^-(\gamma) = A^-(\gamma) - B^-(\gamma)$ and $(A-B)^+(\gamma) = A^+(\gamma) - B^+(\gamma)$.

Theorem 1.3.2. *(1) If we show* $\tilde{0} = \chi_{\{0\}}$, *the characteristic function of zero, then* $\tilde{0} \in E$ *is neutral element with respect to* $\oplus$, *i.e.* $\bar{A} \oplus \tilde{0} = \tilde{0} \oplus \bar{A} = \bar{A}$, *for all* $\bar{A} \in E$.

(2) With respect to $\tilde{0}$*, none of* $\bar{A} \in E \setminus \mathbb{R}$*, has inverse in* E *(with respect to* $\oplus$*).*

(3) For each $\tau, \kappa \in \mathbb{R}$ *with* $\tau, \kappa \geq 0$ *or* $\tau, \kappa \leq 0$ *and each* $\bar{A} \in E$*, we have* $(\tau+\kappa) \odot \bar{A} = \tau \odot \bar{A} \oplus \kappa \odot \bar{A}$*. For general* $\tau, \kappa \in \mathbb{R}$*, the above property dose not hold.*

(4) For each $\tau \in \mathbb{R}$ *and each* $\bar{A}, \bar{B} \in E$*, we have* $\tau \odot (\bar{A} \oplus \bar{B}) = \tau \odot \bar{A} \oplus \tau \odot \bar{B}$*.*

(5) For each $\tau, \kappa \in \mathbb{R}$ *and each* $\bar{A} \in E$*, we have* $\tau \odot (\kappa \odot \bar{A}) = (\tau\kappa) \odot \bar{A}$*.*

The metric structure is given by the Hausdorff distance

$$D : E \times E \to \mathbb{R}^{>0} \cup \{0\}$$

by

$$D(\bar{A}, \bar{B}) = \sup_{0 \leq \gamma \leq 1} \max\{|A^-(\gamma) - B^-(\gamma)|, |A^+(\gamma) - B^+(\gamma)|\}.$$

(E, D) is a complete metric space and the following properties are well-known:

- $D(\bar{A} \oplus \bar{C}, \bar{B} \oplus \bar{C}) = D(\bar{A}, \bar{B})$, $\forall \bar{A}, \bar{B}, \bar{C} \in E$,
- $D(\theta\bar{A}, \theta\bar{B}) = |\theta| D(\bar{A}, \bar{B})$, $\forall \bar{A}, \bar{B} \in E$ and $\theta \in \mathbb{R}$,
- It can be proved straightaway that $D(\bar{A} \oplus \bar{B}, \bar{C} \oplus \bar{D}) \leq D(\bar{A}, \bar{C}) + D(\bar{B}, \bar{D})$ for $\bar{A}, \bar{B}, \bar{C}$ and $\bar{D} \in E$, also as a result we will have
  - $D(\bar{A} \oplus \bar{B}, \bar{C}) \leq D(\bar{A}, \bar{C}) + D(\bar{B}, \tilde{0})$,
  - $D(\bar{A} \oplus \bar{B}, \tilde{0}) \leq D(\bar{A}, \tilde{0}) + D(\bar{B}, \tilde{0})$.

The parametric representation of fuzzy number $\bar{A}$ is any pair $(A^-(\gamma), A^+(\gamma))$ of functions $A^\pm : [0, 1] \to \mathbb{R}$ that satisfy the following theorem.

Theorem 1.3.3. *(Representation theorem) If* $\bar{A} \in E$*, then* $A^-(\gamma)$*,* $A^+(\gamma)$ *are functions on* $[0, 1]$ *satisfying the following conditions (1)-(4):*

(1) $A^-(\gamma)$ *is a nondecreasing function on* $[0,1]$*,*

(2) $A^+(\gamma)$ *is a nonincreasing function on* $[0,1]$*,*

(3) $A^-(\gamma)$ *and* $A^+(\gamma)$ *are bounded and left continuous on* $(0,1]$*, and right continuous at* $\gamma = 0$*, and*

(4) $A^-(\gamma) \leq A^+(\gamma)$*, for each* $\gamma \in [0,1]$*.*

Conversely, if functions $A^-(\gamma)$ *and* $A^+(\gamma)$ *on* $[0,1]$ *satisfy conditions (1)-(4), then there exists a unique fuzzy number* $\bar{A}(\gamma) = (A^-(\gamma), A^+(\gamma)) \in E$*.*

1.4 Fuzzy functions

Let T be a real interval, the function $F : T \to E$ is called a fuzzy function (or mapping).

Definition 1.4.1. A function $F : T \to E$ is called levelwise continuous at $t_0 \in T$ if the set-valued function $F(t)[\gamma] = [F^-(t,\gamma), F^+(t,\gamma)]$ is continuous at $t = t_0$ with respect to the metric D for all $\gamma \in [0,1]$.

Definition 1.4.2. (Seikkala derivative) The Seikkala derivative of $F(t)$, written $SDF(t)$, was defined in [43]. This definition was as follows: if $[F'^-(t,\gamma), F'^+(t,\gamma)]$ are the γ-cuts of a fuzzy number for each $t \in T$, then $SDF(t)$ exists and $SDF(t)[\gamma] = [F'^-(t,\gamma), F'^+(t,\gamma)]$. Also, $SDF(t)$ is a fuzzy number for all $t \in T$.

Definition 1.4.3. (Hukuhara derivative) A function $F : T \to E$ is differentiable at $t_0 \in T$ if there exists a $F'(t_0) \in E$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) -_H F(t_0)}{h} \quad \textit{and} \quad \lim_{h \to 0^+} \frac{F(t_0) -_H F(t_0 - h)}{h},$$

exist and equal to $F'(t_0)$. Here the limit is taken in the metric space (E, D).

Theorem 1.4.1. *Let $F : T \to E$ be H-differentiable. Denote*

$$F(t)[\gamma] = [F^-(t,\gamma), F^+(t,\gamma)], \quad \gamma \in [0,1].$$

Then $F^-(t,\gamma)$ and $F^+(t,\gamma)$ are differentiable and

$$F'(t)[\gamma] = [F'^-(t,\gamma), F'^+(t,\gamma)].$$

Corollary 1.4.2. *If $F : T \to E$ is H-differentiable on T then for each $\gamma \in [0,1]$ the real function $t \to diam F(t)[\gamma]$ is nondecreasing on T.*

Theorem 1.4.3. *If $F : T \to E$ is H-differentiable then it is continuous.*

Definition 1.4.4. (Generalized derivative) Let $F : T \to E$ and $t_0 \in T$. We say that F is strongly generalized differentiable on t_0, if there exists an element $F'(t_0) \in E$ such that

(i) for all $h > 0$ sufficiently small, there exist $F(t_0+h) -_H F(t_0)$ and $F(t_0) -_H F(t_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0^+} \frac{F(t_0+h) -_H F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0-h)}{h} = F'(t_0),$$

or

(ii) for all $h > 0$ sufficiently small, there exist $F(t_0) -_H F(t_0+h)$ and $F(t_0-h) -_H F(t_0)$ and the limits

$$\lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0+h)}{(-h)} = \lim_{h\to 0^+} \frac{F(t_0-h) -_H F(t_0)}{(-h)} = F'(t_0),$$

or

(iii) for all $h > 0$ sufficiently small, there exist $F(t_0+h) -_H F(t_0)$ and $F(t_0-h) -_H F(t_0)$ and the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) -_H F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0-h) -_H F(t_0)}{(-h)} = F'(t_0),$$

or

(iv) for all $h > 0$ sufficiently small, there exist $F(t_0) -_H F(t_0+h)$ and $F(t_0) -_H F(t_0-h)$ and the limits

$$\lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0+h)}{(-h)} = \lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0-h)}{h} = F'(t_0).$$

(h and $(-h)$ at denominators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$, respectively.)

Theorem 1.4.4. *If F is differentiable in the form (ii) of Definition 1.4.4, then it is continuous.*

Theorem 1.4.5. *Let $F : T \to E$ be a fuzzy function and denotes*

$$F(t)[\gamma] = [F^-(t,\gamma), F^+(t,\gamma)], \quad \gamma \in [0,1].$$

(1) If F is (i)-differentiable in the sense of Definition 1.4.4, then $F^-(t,\gamma)$ and $F^+(t,\gamma)$ are differentiable functions and $F'(t)[\gamma] = [F'^-(t,\gamma), F'^+(t,\gamma)]$.

(2) If F is (ii)-differentiable in the sense of Definition 1.4.4, then $F^-(t,\gamma)$ and $F^+(t,\gamma)$ are differentiable functions and $F'(t)[\gamma] = [F'^+(t,\gamma), F'^-(t,\gamma)]$.

Definition 1.4.5. Let $F : M(\subset \mathbb{R}^n) \to E$ be a fuzzy function and $x_0 \in M$. If for $y \in \mathbb{R}^n$, there exists $\delta > 0$ such that $x_0 + hy, x_0 - hy \in M$ and the H-differences

$F(x_0 + hy) -_H F(x_0)$ and $F(x_0) -_H F(x_0 - hy)$ exist for any real number $h \in (0, \delta)$, and there exists $D_y F(x_0) \in E$ such that

$$\lim_{h \to 0^+} \frac{F(x_0 + hy) -_H F(x_0)}{h} = \lim_{h \to 0^+} \frac{F(x_0) -_H F(x_0 - hy)}{h} = D_y F(x_0).$$

Then we say F to be H-differentiable in the direction y at x_0 and call $D_y F(x_0)$ the H-derivative of F at x_0 in the direction y. (Here h at denominator means $\frac{1}{h}\odot$.)

Definition 1.4.6. Let $F : T \to E$. The integral of F over T, denoted $\int_T F(t)dt$ is defined levelwise by the equation

$$(\int_T F(t)dt)[\gamma] = \int_T F(t)[\gamma]dt =$$

$$\{\int_T f(t)dt | f : T \to \mathbb{R} \; is\, a\, measurable\, selection\, for\, F(t)[\gamma]\},$$

for all $0 < \gamma \leq 1$.

Also we have

$$(\int_T F(t)dt)[\gamma] = \int_T F(t)[\gamma]dt = [\int_T F^-(t,\gamma)dt, \int_T F^+(t,\gamma)dt].$$

Corollary 1.4.6. *If $F : T \to E$ is continuous then it is integrable.*

Theorem 1.4.7. *Let $F : T \to E$ be integrable and $a, b, c \in T$. then*

$$\int_a^b F = \int_a^c F \oplus \int_c^b F.$$

Theorem 1.4.8. *Let $F, G : T \to E$ be integrable and $\tau \in \mathbb{R}$. then*

(i) $\int(F \oplus G) = \int F \oplus \int G$,

(ii) $\int(\tau \odot F) = \tau \odot \int F$,

(iii) $D(F, G)$ *is integrable,*

(iv) $D(\int F, \int G) \leq \int D(F, G)$.

Definition 1.4.7. A function $F : M(\subset \mathbb{R}^n) \to E$ is bounded if there exists a constant $N > 0$ such that $D(F(x), \tilde{0}) \leq N$ for all $x \in M$.

Chapter 2

Fuzzy Partial Differential Equations under Buckley-Feuring Method

In various subjects of science and engineering, analytical and numerical solutions of nonlinear evaluation equations are essentially important, for instance, heat-like models and wave-like models. Heat-like models can exactly describe some nonlinear phenomena, for example, the most celebrated Navier-Stokes equations can be converted into various heat-like equation in some special cases [39]. Wave-like models can exactly describe some nonlinear phenomena, for example, wave-like equations can describe earthquake stresses [26] and non-homogeneous elastic waves in soils [35]. It is possible to exist imprecise parameters in heat-like and wave-like equations, therefore, it is necessary to be solved this equations to existing status. Since fuzzy sets theory is a powerful tool for modeling imprecise and processing vague in mathematical models

[16, 17, 36, 38], so will be studied this equations in the fuzzy setting.

2.1 Introduction

In this chapter, the Variational Iteration Method (VIM) and the same strategy as Buckley and Feuring are used for find exact fuzzy solution of the fuzzy heat-like equations and fuzzy wave-like equations in one and two dimensions. Several examples are given to show the Buckley-Feuring solution and Seikkala solution.

2.2 Fuzzy heat-like equations

In this section, we consider the heat-like equations in one and two dimensions which can be written in the forms

(a) One-dimensional:

$$u_t(t,x) + p(x)u_{xx}(t,x) = F(t,x,k), \tag{2.2.1}$$

(b) Two-dimensional:

$$u_t(t,x,y) + p(x)u_{xx}(t,x,y) + q(y)u_{yy}(t,x,y) = F(t,x,y,k), \tag{2.2.2}$$

or

$$u_t(t,x,y) + q(y)u_{xx}(t,x,y) + p(x)u_{yy}(t,x,y) = F(t,x,y,k), \tag{2.2.3}$$

subject to certain initial and boundary conditions.

These initial and boundary conditions, in state of two-dimensional, can come in a variety of forms such as $u(0,x,y) = c_1$ or $u(0,x,y) = g_1(x,y,c_2)$ or $u(M_1,x,y) = g_2(x,y,c_3,c_4)$,. . . .

In this section and sections 2.3 and 2.4 the method is applied for the heat-like equation (2.2.2). For Eqs. (2.2.1) and (2.2.3), it is similar to (2.2.2), so we will omit them. In the following lines, components of Eq. (2.2.2) are enumerated:

- $I_j = [0, M_j]$ are three intervals, which $M_j > 0\,(j = 1, 2, 3)$.
- $F(t, x, y, k)$, $u(t, x, y)$, $p(x)$ and $q(y)$ will be continuous functions for $(t, x, y) \in \prod_{j=1}^{3} I_j$.
- $p(x)$ and $q(y)$ have a finite number of roots for each $(x, y) \in I_2 \times I_3$.
- $k = (k_1, \ldots, k_n)$ and $c = (c_1, \ldots, c_m)$ are vectors of constants with k_j in interval J_j and c_r in interval L_r.

Assume the Eq. (2.2.2) has a solution

$$u(t, x, y) = g(t, x, y, k, c), \tag{2.2.4}$$

for continuous g[1] with $(t, x, y) \in \prod_{j=1}^{3} I_j$, $k \in J = \prod_{j=1}^{n} J_j$ and $c \in L = \prod_{r=1}^{m} L_r$.

Now suppose the value of the k_j and c_r are imprecise. We will model this uncertainty by substituting triangular fuzzy numbers for the k_j and c_r. If we fuzzify Eq. (2.2.2), then we obtain the fuzzy heat-like equation. Using the extension principle we compute $F(t, x, y, \bar{K})$ from $F(t, x, y, k)$, where has $\bar{K} = (\bar{K}_1, \ldots, \bar{K}_n)$ for $\bar{K}_j$ a triangular fuzzy number in J_j, $1 \leq j \leq n$. In result, the function $u : \prod_{j=1}^{3} I_j \to E$ becomes a fuzzy function. So the fuzzy heat-like equation is

$$u_t \oplus p(x) \odot u_{xx} \oplus q(y) \odot u_{yy} = F(t, x, y, \bar{K}), \tag{2.2.5}$$

[1] $g_t(t, x, y, k, c) + p(x) g_{xx}(t, x, y, k, c) + q(y) g_{yy}(t, x, y, k, c)$ is continuous for $(t, x, y) \in \prod_{j=1}^{3} I_j, k \in J, c \in L$.

subject to certain initial and boundary conditions. The initial and boundary conditions can be of the form $u(0,x,y)=\bar{C}_1$ or $u(0,x,y)=g_1(x,y,\bar{C}_2)$ or $u(M_1,x,y)=g_2(x,y,\bar{C}_3,\bar{C}_4)$,.... The functions g_j are obtained using the extension principle. We wish to solve the problem given in Eq. (2.2.5). Finally, we fuzzify g in Eq. (2.2.4). Let $z(t,x,y)=g(t,x,y,\bar{K},\bar{C})$, where z is computed using the extension principle and is a fuzzy solution. In Section 2.4, we will discuss solution with the same strategy as Buckley-Feuring for fuzzy heat-like equation. Let $\bar{K}[\gamma]=\prod_{j=1}^{n}\bar{K}_j[\gamma]$ and $\bar{C}[\gamma]=\prod_{r=1}^{m}\bar{C}_r[\gamma]$.

Of course, we suppose there are no Fourier series, Bessel functions and Legendre functions used to define g. Since we will need to fuzzify g we do not wish to fuzzify Fourier series, Bessel functions or Legendre functions in this chapter.

2.3 He's variational iteration method

The VIM was proposed by He in [27, 28] and it has been shown to solve effectively, easily and accurately, a large class of linear or nonlinear problems [49, 50, 44]. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise a few approximations can be used for numerical purposes.

Consider the differential equation

$$L_t u + L_x u + L_y u + Nu = F(t,x,y,k), \tag{2.3.1}$$

where L_t, L_x and L_y are linear operators of t, x and y, respectively, and N is a nonlinear operator, also $F(t,x,y,k)$ is the source non-homogeneous term. In VIM, a correction functional for Eq. (2.3.1) can be written as

$$u_{n+1}(t,x,y) = u_n(t,x,y) + \int_0^t \lambda\{L_s u_n + (L_x + L_y + N)\tilde{u}_n - \tilde{F}\}\, ds. \tag{2.3.2}$$

It is obvious that successive approximations u_j, $j \geq 0$, can be established by determining λ, a general Lagrange multiplier which can be identified optimally via the variational theory [27].

The functions $\tilde{u}_n$ and $\tilde{F}$ are restricted variations, which mean $\delta\tilde{u}_n = 0$ and $\delta\tilde{F} = 0$. It is required first to determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations u_{n+1}, $n \geq 0$, of the solution $u(t,x,y)$ will follow immediately upon using any selective function u_0. The initial values $u(0,x,y)$ and $u_t(0,x,y)$ are usually used for selecting the zeroth approximation u_0. Therefore, the solution is given as

$$u(t,x,y) = \lim_{n\to\infty} u_n(t,x,y). \tag{2.3.3}$$

According to the VIM, we construct a correction functional for Eq. (2.2.2) in the form

$$u_{n+1}(t,x,y) = u_n(t,x,y) + \int_0^t \lambda(s)\{(u_n)_s + p(x)(\tilde{u}_n)_{xx} + q(y)(\tilde{u}_n)_{yy} - \tilde{F}\}\, ds, \tag{2.3.4}$$

where $n \geq 0$ and λ is a lagrange multiplier and $\delta\tilde{u}_n$ is the restricted variation, i.e. $\delta\tilde{u}_n = 0$ [27, 28]. To find the optimal value of λ, we have

$$\delta u_{n+1}(t,x,y) = \delta u_n(t,x,y) + \delta \int_0^t \lambda(s)\{(u_n)_s + p(x)(\tilde{u}_n)_{xx} + q(y)(\tilde{u}_n)_{yy} - \tilde{F}\}\, ds,$$

or

$$\delta u_{n+1}(t,x,y) = \delta u_n(t,x,y) + \delta \int_0^t \lambda(s)(u_n)_s\, ds.$$

Therefore we have

$$\delta u_{n+1}(t,x,y) = (1+\lambda)\delta u_n(t,x,y) - \int_0^t \lambda'(s)\delta u_n(s,x,y)\,ds,$$

which yields

$$\lambda'(s) = 0,$$

$$1+\lambda(s)|_{s=t} = 0.$$

Hence,the Lagrange multiplier is $\lambda = -1$. Submitting the results into Eq. (2.3.4) leads to the following iteration formula

$$u_{n+1}(t,x,y) = u_n(t,x,y) - \int_0^t \{(u_n)_s + p(x)(u_n)_{xx} + q(y)(u_n)_{yy} - F\}\,ds. \qquad (2.3.5)$$

Iteration formula starts with an initial approximation, for example $u_0(t,x,y) = u(0,x,y)$. Also the VIM is used for system of linear and nonlinear partial differential equations [49] which handled in obtaining Seikkala solution.

2.4 Buckley-Feuring Solution (BFS) and Seikkala Solution (SS)

In [16], Buckley and Feuring presented the BFS . For all t, x, y and γ,

$$z(t,x,y)[\gamma] = [z^-(t,x,y,\gamma), z^+(t,x,y,\gamma)], \qquad (2.4.1)$$

and

$$F(t,x,y,\bar{K})[\gamma] = [F^-(t,x,y,\gamma), F^+(t,x,y,\gamma)], \qquad (2.4.2)$$

that by definition

$$z^-(t,x,y,\gamma) = \min\{g(t,x,y,k,c)|\, k \in \bar{K}[\gamma], c \in \bar{C}[\gamma]\}, \qquad (2.4.3)$$

$$z^+(t,x,y,\gamma) = \max\{g(t,x,y,k,c)|\, k \in \bar{K}[\gamma], c \in \bar{C}[\gamma]\}, \tag{2.4.4}$$

and

$$F^-(t,x,y,\gamma) = \min\{F(t,x,y,k)|\, k \in \bar{K}[\gamma]\}, \tag{2.4.5}$$

$$F^+(t,x,y,\gamma) = \max\{F(t,x,y,k)|\, k \in \bar{K}[\gamma]\}. \tag{2.4.6}$$

Assume that $p(x) > 0$, $q(y) > 0$ and the $z^i(t,x,y,\gamma)$, $i \in \{-,+\}$, have continuous partials so that $z^i_t + p(x)z^i_{xx} + q(y)z^i_{yy}$ is continuous for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all γ. Define

$$\Gamma(t,x,y)[\gamma] = [z^-_t + p(x)z^-_{xx} + q(y)z^-_{yy}, z^+_t + p(x)z^+_{xx} + q(y)z^+_{yy}], \tag{2.4.7}$$

for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all γ. If, for each fixed $(t,x,y) \in \prod_{j=1}^3 I_j$, $\Gamma(t,x,y)[\gamma]$ defines the γ-cut of a fuzzy number, then it will be said that $z(t,x,y)$ is differentiable and is written

$$z_t[\gamma] + p(x)z_{xx}[\gamma] + q(y)z_{yy}[\gamma] = \Gamma(t,x,y)[\gamma], \tag{2.4.8}$$

for all $(t,x,y) \in \prod_{j=1}^3 I_j$ and all γ.

Sufficient conditions for $\Gamma(t,x,y)[\gamma]$ to define γ-cuts of a fuzzy number are [25]:

(i) $z^-_t(t,x,y,\gamma) + p(x)z^-_{xx}(t,x,y,\gamma) + q(y)z^-_{yy}(t,x,y,\gamma)$ is an increasing function of γ for each $(t,x,y) \in \prod_{j=1}^3 I_j$;

(ii) $z^+_t(t,x,y,\gamma) + p(x)z^+_{xx}(t,x,y,\gamma) + q(y)z^+_{yy}(t,x,y,\gamma)$ is a decreasing function of γ for each $(t,x,y) \in \prod_{j=1}^3 I_j$; and

(iii) $z^-_t(t,x,y,1) + p(x)z^-_{xx}(t,x,y,1) + q(y)z^-_{yy}(t,x,y,1) \leq$
$z^+_t(t,x,y,1) + p(x)z^+_{xx}(t,x,y,1) + q(y)z^+_{yy}(t,x,y,1)$ for $(t,x,y) \in \prod_{j=1}^3 I_j$.

Now we can suppose that the $z^i(t, x, y, \gamma)$ have continuous partials so

$$z_t^i + p(x) z_{xx}^i + q(y) z_{yy}^i,$$

is continuous on $\prod_{j=1}^3 I_j \times [0, 1]$, $i \in \{-, +\}$. Hence, if conditions (i)-(iii) above hold, $z(t, x, y)$ is differentiable.

For $z(t, x, y)$ to be a BFS of the fuzzy heat-like equation we need: (a) $z(t, x, y)$ differentiable; (b) Eq. (2.2.5) holds for $u(t, x, y) = z(t, x, y)$; and (c) $z(t, x, y)$ satisfies the initial and boundary conditions. Since no exist specified any particular initial and boundary conditions then only is checked if Eq. (2.2.5) holds.

$z(t, x, y)$ is a BFS (without the initial and boundary conditions) if $z(t, x, y)$ is differentiable and

$$z_t \oplus p(x) \odot z_{xx} \oplus q(y) \odot z_{yy} = F(t, x, y, \bar{K}), \tag{2.4.9}$$

or the following equations must hold

$$z_t^- + p(x) z_{xx}^- + q(y) z_{yy}^- = F^-(t, x, y, \gamma), \tag{2.4.10}$$

$$z_t^+ + p(x) z_{xx}^+ + q(y) z_{yy}^+ = F^+(t, x, y, \gamma), \tag{2.4.11}$$

for all $(t, x, y) \in \prod_{j=1}^3 I_j$ and all γ.

Now we will present a sufficient condition for the BFS to exist such as Buckley and Feuring. Since there are such a variety of possible initial and boundary conditions, hence we will omit them from the following theorem. One must separately check out the initial and boundary conditions. So, we will omit the constants $c_r, 1 \leq r \leq m$, from the problem. Therefore, Eq. (2.2.4) becomes $u(t, x, y) = g(t, x, y, k)$, so $z(t, x, y) = g(t, x, y, \bar{K})$.

Theorem 2.4.1. *Assume z(t,x) is differentiable.*

(a) If for all $i \in \{1, \ldots, n\}$ $g(t, x, k)$ and $F(t, x, k)$ are both increasing (or both decreasing) in k_i, for $(t, x) \in I_1 \times I_2$ and $k \in J$, then $z(t, x)$ is a BFS.

(b) If there is an $i \in \{1, \ldots, n\}$ so that for variable k_i, $g(t, x, k)$ is strictly increasing and $F(t, x, k)$ is strictly decreasing (or $g(t, x, k)$ is strictly decreasing and $F(t, x, k)$ is strictly increasing), for $(t, x) \in I_1 \times I_2$ and $k \in J$, then $z(t, x)$ is not a BFS.

Proof. See proof of theorem in [16]. □

Theorem 2.4.2. *Suppose $z(t, x, y)$ is differentiable.*

(a) If

$$p(x) > 0, \;\; q(y) > 0, \;\; (x, y) \in I_2 \times I_3, \tag{2.4.12}$$

and

$$\frac{\partial g}{\partial k_j} \frac{\partial F}{\partial k_j} > 0, \tag{2.4.13}$$

for $j = 1, \ldots, n$, then BFS=$z(t, x, y)$.

(b) If relations (2.4.12) do not hold or relation (2.4.13) does not hold for some j, then $z(t, x, y)$ is not a BFS.

Proof. Proof is similar to proof of Theorem 2.4.1. □

Therefore, if $z(t,x,y)$ is a BFS and it satisfies the initial and boundary conditions we will say that $z(t,x,y)$ is a BFS satisfying the initial and boundary conditions. If $z(t,x,y)$ is not a BFS, then we will consider the SS. Now let us define the SS [43]. Let

$$u(t,x,y)[\gamma] = [u^-(t,x,y,\gamma), u^+(t,x,y,\gamma)].$$

For example suppose $p(x) > 0$ and $q(y) < 0$, so consider the system of heat-like equations

$$u_t^- + p(x)u_{xx}^- + q(y)u_{yy}^+ = F^-(t,x,y,\gamma), \tag{2.4.14}$$

$$u_t^+ + p(x)u_{xx}^+ + q(y)u_{yy}^- = F^+(t,x,y,\gamma), \tag{2.4.15}$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and all $\gamma \in [0,1]$. We append to Eqs. (2.4.14) and (2.4.15) any initial and boundary conditions. For example, if it was $u(0,x,y) = \bar{C}$ then we add

$$u^-(0,x,y,\gamma) = c^-(\gamma), \tag{2.4.16}$$

$$u^+(0,x,y,\gamma) = c^+(\gamma), \tag{2.4.17}$$

where $\bar{C}[\gamma] = [c^-(\gamma), c^+(\gamma)]$. Let $u^i(t,x,y,\gamma), (i \in \{-,+\})$ solve Eqs. (2.4.14) and (2.4.15), plus initial and boundary conditions. If

$$[u^-(t,x,y,\gamma), u^+(t,x,y,\gamma)], \tag{2.4.18}$$

defines the γ-cut of a fuzzy number, for all $(t,x,y) \in \prod_{j=1}^{3} I_j$, then $u(t,x,y)$ is the SS.

We will say that derivative condition holds for fuzzy heat-like equation when Eqs. (2.4.12) and (2.4.13) are satisfied.

Theorem 2.4.3. *(1) If BFS=$z(t,x,y)$, then SS=$z(t,x,y)$.*

(2) If SS=$u(t,x,y)$ and the derivative condition holds, then BFS=$u(t,x,y)$.

Proof. (1) Follows from the definitions of BFS and SS.

(2) If SS=$u(t,x,y)$ then the Seikkala derivative [17] exists and since the derivative condition holds, therefore, the following Eqs. hold

$$u_t^- + p(x)u_{xx}^- + q(y)u_{yy}^- = F^-(t,x,y,\gamma), \tag{2.4.19}$$

$$u_t^+ + p(x)u_{xx}^+ + q(y)u_{yy}^+ = F^+(t,x,y,\gamma). \tag{2.4.20}$$

Also suppose one $k_j = k$ and $\frac{\partial g}{\partial k} < 0$, $\frac{\partial F}{\partial k} < 0$ (the other cases are similar and are omitted).

We see

$$z^-(t,x,y,\gamma) = g(t,x,y,k^+(\gamma)), \tag{2.4.21}$$

$$z^+(t,x,y,\gamma) = g(t,x,y,k^-(\gamma)), \tag{2.4.22}$$

$$F^-(t,x,y,\gamma) = F(t,x,y,k^+(\gamma)), \tag{2.4.23}$$

$$F^+(t,x,y,\gamma) = F(t,x,y,k^-(\gamma)). \tag{2.4.24}$$

Now look at Eqs. (2.4.10) and (2.4.11) also Eqs. (2.4.3) and (2.4.4), implies that

$$u^-(t,x,y,\gamma) = g(t,x,y,k^+(\gamma)) = z^-(t,x,y,\gamma),$$

$$u^+(t,x,y,\gamma) = g(t,x,y,k^-(\gamma)) = z^+(t,x,y,\gamma).$$

Therefore BFS=$u(t,x,y)$. □

Remark 2.4.1. The Theorem (2.4.2) holds for Eq. (2.2.3) and the proof is similar to Theorem 2.4.1.

Remark 2.4.2. Consider Eq. (2.2.1). Assume $z(t,x)$ is differentiable.

(a) If

$$p(x) > 0, \quad x \in I_2, \tag{2.4.25}$$

and

$$\frac{\partial g}{\partial k_j}\frac{\partial F}{\partial k_j} > 0, \tag{2.4.26}$$

for $j = 1, \ldots, n$, then BFS=$z(t,x)$.

(b) If relation (2.4.25) does not hold or relation (2.4.26) does not hold for some j, then $z(t,x)$ is not a BFS.

Now we consider the following illustrating examples.

Example 2.4.1. *We first consider the one-dimensional initial value problem*

$$u_t + \frac{1}{2}x^2 u_{xx} = k, \tag{2.4.27}$$

subject to the initial condition $u(0,x) = cx^2$ *and* $t \in [0, M_1), x \in (0, M_2)$. *Let* $k \in [0, J]$ *and* $c \in [0, L]$ *are constants.*

According to the VIM, a correction functional for Eq. (2.4.27) from Eq. (2.3.5) can be constructed as follows

$$u_{n+1}(t,x) = u_n(t,x) - \int_0^t \{(u_n(s,x))_s + \frac{1}{2}x^2(u_n(s,x))_{xx} - k\}\, ds.$$

Beginning with an initial approximation $u_0(t,x) = u(0,x) = cx^2$, we can obtain the following successive approximations

$$u_1(t,x) = kt + cx^2(1-t),$$
$$u_2(t,x) = kt + cx^2(1-t+\tfrac{t^2}{2!}),$$
$$u_3(t,x) = kt + cx^2(1-t+\tfrac{t^2}{2!}-\tfrac{t^3}{3!}),$$

and

$$u_n(t,x) = kt + cx^2(1-t+\frac{t^2}{2!}+\cdots+(-1)^n\frac{t^n}{n!}),\ n \geq 1.$$

The VIM admits the use of

$$u(t,x) = \lim_{n\longrightarrow\infty} u_n(t,x),$$

which gives the exact solution

$$u(t,x) = kt + cx^2e^{-t}.$$

Now we fuzzify $F(t,x,k) = k$ and $g(t,x,k,c) = kt + cx^2e^{-t}$. Clearly $F(t,x,\bar{K}) = \bar{K}$ so that $F^-(t,x,\gamma) = k^-(\gamma)$ and $F^+(t,x,\gamma) = k^+(\gamma)$. Also $g(t,x,\bar{K},\bar{C}) = \bar{K}\odot t \oplus \bar{C}\odot x^2e^{-t}$, therefore,

$$z^i(t,x,\gamma) = k^i(\gamma)t + c^i(\gamma)x^2e^{-t},$$

for $i \in \{-,+\}$, $\bar{K}[\gamma] = [k^-(\gamma), k^+(\gamma)]$ and $\bar{C}[\gamma] = [c^-(\gamma), c^+(\gamma)]$. $z(t,x)$ is differentiable because $z^i_t(t,x,\gamma) + \frac{1}{2}x^2 z^i_{xx}(t,x,\gamma) = k^i(\gamma)$, $i \in \{-,+\}$. Therefore, $z_t \oplus \frac{1}{2}x^2 \odot z_{xx} = \bar{K}$ and $z(t,x)$ is a fuzzy number for each $t \in (0,M_1)$ and $x \in (0,M_2)$. Since $p(x) > 0$, $\frac{\partial g}{\partial k} > 0$ and $\frac{\partial F}{\partial k} > 0$, Lemma (2.4.2) implies the result that $z(t,x)$ is a BFS. We easily see that

$$z^i(0,x,\gamma) = c^i(\gamma)x^2,$$

for $i \in \{-,+\}$, so $z(t,x)$ also satisfies the initial condition. The BFS that satisfies the initial condition may be written as

$$z(t,x) = \bar{K} \odot t \oplus \bar{C} \odot x^2 e^{-t},$$

for all $(t,x) \in [0, M_1) \times (0, M_2)$.

Example 2.4.2. *Consider the two-dimensional heat-like equation with variable coefficients as*

$$u_t(t,x,y) + \tfrac{1}{2}x^2 u_{xx}(t,x,y) + \tfrac{1}{2}y^2 u_{yy}(t,x,y) = kx^2 y,$$

$$u(0,x,y) = c_1 y^2 - c_2 x,$$

which $x, y \in (0,1)$, $t \in [0, M)$, $k \in [0, J]$ *and* $c_j \in [0, L_j]$, $j = 1, 2$.

Similarly we can establish an iteration formula in the form

$$u_{n+1} = u_n - \int_0^t \{(u_n)_s + \frac{1}{2}x^2 (u_n)_{xx} + \frac{1}{2}y^2 (u_n)_{yy} - kx^2 y\}\, ds. \tag{2.4.28}$$

We begin with an initial arbitrary approximation: $u_0(t,x,y) = u(0,x,y) = c_1 y^2 - c_2 x$, and using the iteration formula (2.4.28), we obtain the following successive approximations

$$u_1(t,x,y) = c_1 y^2 (1-t) - c_2 x + kx^2 yt,$$
$$u_2(t,x,y) = c_1 y^2 (1 - t + \tfrac{t^2}{2!}) - kx^2 y(-t + \tfrac{t^2}{2!}) - c_2 x,$$
$$u_3(t,x,y) = c_1 y^2 (1 - t + \tfrac{t^2}{2!} - \tfrac{t^3}{3!}) - kx^2 y(-t + \tfrac{t^2}{2!} - \tfrac{t^3}{3!}) - c_2 x,$$

and

$$u_n(t,x,y) = c_1 y^2 (1 - t + \cdots + \frac{(-1)^n t^n}{n!}) - kx^2 y(-t + \cdots + \frac{(-1)^n t^n}{n!}) - c_2 x,\ n \geq 1.$$

Then, the exact solution is given by

$$u(t,x,y) = g(t,x,y,k,c) = c_1 y^2 e^{-t} - kx^2 y(e^{-t} - 1) - c_2 x.$$

Fuzzify F and g producing their γ-cuts

$$\begin{aligned}
z^-(t,x,y,\gamma) &= c_1^-(\gamma) y^2 e^{-t} - k^-(\gamma) x^2 y(e^{-t} - 1) - c_2^+(\gamma) x, \\
z^+(t,x,y,\gamma) &= c_1^+(\gamma) y^2 e^{-t} - k^+(\gamma) x^2 y(e^{-t} - 1) - c_2^-(\gamma) x, \\
F^-(t,x,y,\gamma) &= k^-(\gamma) x^2 y, \\
F^+(t,x,y,\gamma) &= k^+(\gamma) x^2 y,
\end{aligned}$$

where $\bar{K} = [k^-(\gamma), k^+(\gamma)]$ and $\bar{C}_j[\gamma] = [c_j^-(\gamma), c_j^+(\gamma)], j = 1, 2$. We first check to see if $z(t,x,y)$ is differentiable. We compute

$$[z_t^- + \frac{1}{2} x^2 z_{xx}^- + \frac{1}{2} y^2 z_{yy}^-, z_t^+ + \frac{1}{2} x^2 z_{xx}^+ + \frac{1}{2} y^2 z_{yy}^+],$$

which are γ-cuts of $\bar{K} \odot x^2 y$ i.e. γ-cuts of a fuzzy number. Hence, $z(t,x,y)$ is differentiable.

Since the partial derivative F and g with respect to k, $p(x)$ and $q(y)$ are positive then Theorem (2.4.2) tells us that $z(t,x,y)$ is a BFS. The initial condition is

$$\begin{aligned}
z^-(0,x,y) &= c_1^-(\gamma) y^2 - c_2^+(\gamma) x, \\
z^+(0,x,y) &= c_1^+(\gamma) y^2 - c_2^-(\gamma) x,
\end{aligned}$$

which are true. Therefore, $z(t,x,y)$ is a BFS which also satisfies the initial condition. This BFS may be written

$$z(t,x,y) = \bar{C}_1 \odot y^2 e^{-t} \oplus \bar{K} \odot (-x^2 y(e^{-t} - 1)) \oplus \bar{C}_2 \odot (-x),$$

for all $x, y \in (0,1)$, $t \in [0, M)$.

Example 2.4.3. *We consider the one-dimensional heat-like model*

$$u_t(t,x) + (\tfrac{1}{2} - x)u_{xx}(t,x) = -kx^2t^2,$$
$$u(0,x) = cx^2, \tag{2.4.29}$$

which $t \in [0,1)$, $x \in (0, \frac{1}{2})$ *and the value of parameters* k *and* c *are in intervals* $[0,J]$ *and* $[0,L]$, *respectively.*

We can obtain the following iteration formula for the Eq. (2.4.29)

$$u_{n+1}(t,x) = u_n(t,x) - \int_0^t \{(u_n(s,x))_s + (\frac{1}{2} - x)(u_n(s,x))_{xx} + kx^2s^2\}ds. \tag{2.4.30}$$

We begin with an initial approximation: $u_0(t,x) = cx^2$. By (2.4.30), after two iterations the exact solution is given in the closed forms as

$$u(t,x) = g(t,x,k,c) = \frac{1}{12}kt^4 - \frac{1}{6}kxt^4 - \frac{1}{3}kx^2t^3 + cx^2 + 2cxt - ct.$$

Since $\frac{\partial F}{\partial k} = -x^2t^2 < 0$ and $\frac{\partial g}{\partial k} = \frac{1}{12}t^4 - \frac{1}{6}xt^4 - \frac{1}{3}x^2t^3 > 0$, for

$$0 < t < 1 \qquad and \qquad 0 < x < \frac{1}{4}(-t + \sqrt{4t + t^2})$$

then there is no BFS (Lemma (2.4.2)). We proceed to look for a SS. We must solve

$$u_t^-(t,x,\gamma) + (\tfrac{1}{2} - x)u_{xx}^-(t,x,\gamma) = -k^+(\gamma)x^2t^2,$$
$$u_t^+(t,x,\gamma) + (\tfrac{1}{2} - x)u_{xx}^+(t,x,\gamma) = -k^-(\gamma)x^2t^2,$$

subject to

$$u^i(0,x,\gamma) = c^i(\gamma)x^2,$$

for $i \in \{-,+\}$, $\bar{K}[\gamma] = [k^-(\gamma), k^+(\gamma)]$ and $\bar{C}[\gamma] = [c^-(\gamma), c^+(\gamma)]$. By VIM, the solution is

$$u^-(t,x,\gamma) = \tfrac{1}{12}k^+(\gamma)t^4 - \tfrac{1}{6}k^+(\gamma)xt^4 - \tfrac{1}{3}k^+(\gamma)x^2t^3 + c^-(\gamma)x^2 + 2c^-(\gamma)xt - c^-(\gamma)t,$$
$$u^+(t,x,\gamma) = \tfrac{1}{12}k^-(\gamma)t^4 - \tfrac{1}{6}k^-(\gamma)xt^4 - \tfrac{1}{3}k^-(\gamma)x^2t^3 + c^+(\gamma)x^2 + 2c^+(\gamma)xt - c^+(\gamma)t. \tag{2.4.31}$$

Now we denote

$$[u^-(t,x,\gamma), u^+(t,x,\gamma)],$$

defines γ-cuts of a fuzzy number on area $\Re$ as follows. Since $u^i(t,x,\gamma)$ are continuous and $u^-(t,x,1) = u^+(t,x,1)$ then we only require to check if $\frac{\partial u^-}{\partial \gamma} > 0$ and $\frac{\partial u^+}{\partial \gamma} < 0$. Since $\bar{K}$ and $\bar{C}$ are triangular fuzzy numbers, hence, we pick simple fuzzy parameter so that $k'^-(\gamma) = c'^-(\gamma) = b > 0$ and $k'^+(\gamma) = c'^+(\gamma) = -b$. The 'prime' denotes differentiation with respect to γ. Then, for a SS we need

$$\begin{aligned}
\tfrac{\partial u^-}{\partial \gamma} &= -\tfrac{1}{12}bt^4 + \tfrac{1}{6}bxt^4 + \tfrac{1}{3}bx^2t^3 + bx^2 + 2bxt - bt = \\
&\quad b(-\tfrac{1}{12}t^4 + \tfrac{1}{6}xt^4 + \tfrac{1}{3}x^2t^3 + x^2 + 2xt - t) > 0, \\
\tfrac{\partial u^+}{\partial \gamma} &= \tfrac{1}{12}bt^4 - \tfrac{1}{6}bxt^4 - \tfrac{1}{3}bx^2t^3 - bx^2 - 2bxt + bt = \\
&\quad -b(-\tfrac{1}{12}t^4 + \tfrac{1}{6}xt^4 + \tfrac{1}{3}x^2t^3 + x^2 + 2xt - t) < 0.
\end{aligned} \tag{2.4.32}$$

Therefore inequalities (2.4.32) hold if

$$-\frac{1}{12}t^4 + \frac{1}{6}xt^4 + \frac{1}{3}x^2t^3 + x^2 + 2xt - t > 0, \tag{2.4.33}$$

for $x \in (0, \frac{1}{2})$ and $t \in (0, 1)$. The inequality (2.4.33) holds if we have

$$\begin{aligned}
&0 < t < 1, \\
&\tfrac{-12t - t^4 + \sqrt{144t + 144t^2 + 60t^4 + 24t^5 + 4t^7 + t^8}}{12 + 4t^3} < x < \tfrac{1}{2}.
\end{aligned}$$

We find that

$$\max\{\frac{-12t - t^4 + \sqrt{144t + 144t^2 + 60t^4 + 24t^5 + 4t^7 + t^8}}{12 + 4t^3} \,|\, 0 < t < 1\} = 0.40103.$$

Hence we may choose $\Re$ by the above assumptions in form as

$$\Re = \{(t,x) |\, 0 < t < 1 \,\&\, 0.401031 \leq x < \frac{1}{2}\},$$

and the SS exists on $\Re$ in form Eqs. (2.4.31).

Example 2.4.4. *We consider the one-dimensional heat-like model,*

$$u_t(t,x) - u_{xx}(t,x) = -k\cos x,$$

$$u(0,x) = c\sin x,$$

which $x \in (0, \frac{\pi}{2})$, $t \in [0, M)$ and the value of parameters k and c are in intervals $[0, J]$ and $[0, L]$, respectively.

We can obtain the following iteration formula

$$u_{n+1}(t,x) = u_n(t,x) - \int_0^t \{(u_n(s,x))_s - (u_n(s,x))_{xx} + k\cos x\} ds. \tag{2.4.34}$$

We begin with an initial approximation: $u_0(t,x) = u(0,x) = c\sin x$. By (2.4.34), the following successive approximation are obtained

$$u_1(t,x) = c\sin x(1-t) - kt\cos x,$$
$$u_2(t,x) = c\sin x(1 - t + \tfrac{t^2}{2!}) + k\cos x(-t + \tfrac{t^2}{2!}),$$

and

$$u_n(t,x) = c\sin x(1 - t + \cdots + \frac{(-1)^n t^n}{n!}) + k\cos x(-t + \cdots + \frac{(-1)^n t^n}{n!}), n \geq 1.$$

We, therefore, obtain

$$u(t,x) = g(t,x,k,c) = ce^{-t}\sin x + k\cos x(e^{-t} - 1),$$

which is the exact solution. There is no BFS because $p(x) = -1 < 0$ (Lemma (2.4.2)). We proceed to look for a SS. We must solve

$$u_t^-(t,x,\gamma) - u_{xx}^+(t,x,\gamma) = -k^+(\gamma)\cos x,$$
$$u_t^+(t,x,\gamma) - u_{xx}^-(t,x,\gamma) = -k^-(\gamma)\cos x,$$

subject to

$$u^i(0,x,\gamma) = c^i(\gamma)\sin x,$$

$i \in \{-,+\}$, $\bar{K}[\gamma] = [k^-(\gamma), k^+(\gamma)]$ and $\bar{C}[\gamma] = [c^-(\gamma), c^+(\gamma)]$. The solution is

$$\begin{aligned} u^-(t,x,\gamma) &= c^-(\gamma)\cosh t \sin x - c^+(\gamma)\sinh t \sin x + \\ &\qquad k^-(\gamma)\cos x(\cosh t - 1) - k^+(\gamma)\cos x \sinh t, \\ u^+(t,x,\gamma) &= c^+(\gamma)\cosh t \sin x - c^-(\gamma)\sinh t \sin x + \\ &\qquad k^+(\gamma)\cos x(\cosh t - 1) - k^-(\gamma)\cos x \sinh t. \end{aligned}$$

We only need to check if $\frac{\partial u^-}{\partial \gamma} > 0$ and $\frac{\partial u^+}{\partial \gamma} < 0$, since the u^i are continuous and $u^-(t,x,1) = u^+(t,x,1)$. We pick simple fuzzy parameter so that $k'^-(\gamma) = c'^-(\gamma) = b > 0$ and $k'^+(\gamma) = c'^+(\gamma) = -b$. Then, for a SS we require

$$\begin{aligned} \tfrac{\partial u^-}{\partial \gamma} &= b\sin x(\cosh t + \sinh t) + b\cos x(\cosh t - 1 + \sinh t) > 0, \\ \tfrac{\partial u^+}{\partial \gamma} &= -b\sin x(\cosh t + \sinh t) - b\cos x(\cosh t - 1 + \sinh t) < 0. \end{aligned} \tag{2.4.35}$$

Since (2.4.35) holds for each $t \in (0, M)$ and $x \in (0, \frac{\pi}{2})$, therefore, $u(t,x)$ is SS and

$$u(t,x) = \bar{C} \odot \cosh t \sin x \oplus \bar{C} \odot (-\sinh t \sin x) \oplus \bar{K} \odot \cos x(\cosh t - 1) \oplus \bar{K} \odot (-\sinh t \cos x),$$

for all $t \in [0, M)$ and $x \in (0, \frac{\pi}{2})$.

2.5 Fuzzy wave-like equations

In here, we consider the wave-like equations in one and two dimensions which can be written in the forms

(a) One-dimensional:

$$u_{tt}(t,x) + p(x)u_{xx}(t,x) = F(t,x,k), \tag{2.5.1}$$

(b) Two-dimensional:

$$u_{tt}(t,x,y) + p(x)u_{xx}(t,x,y) + q(y)u_{yy}(t,x,y) = F(t,x,y,k), \qquad (2.5.2)$$

or

$$u_{tt}(t,x,y) + q(y)u_{xx}(t,x,y) + p(x)u_{yy}(t,x,y) = F(t,x,y,k), \qquad (2.5.3)$$

subject to certain initial and boundary conditions.

The initial and boundary conditions, in two-dimensional case, are introduced in a variety of forms such as $u(0,x,y) = c_1$, $u_t(0,x,y) = g_1(x,y,c_2)$, $u(M_1,x,y) = g_2(x,y,c_3,c_4)$,....

In this section, the method is illustrated for wave-like equation (2.5.2). For Eqs. (2.5.1) and (2.5.3), it is similar to (2.5.2), and we will omit them. In following lines, the components of Eq. (2.5.2) are enumerated:

- $I_1 = [0, M_1]$, $I_2 = [M_2, M_3]$ and $I_3 = [M_4, M_5]$ are three intervals, which M_j ($j = 2,3,4,5$) is negative or positive and $M_1 > 0$.
- $F(t,x,y,k)$, $u(t,x,y)$, $p(x)$ and $q(y)$ will be continuous functions for $(t,x,y) \in \prod_{j=1}^{3} I_j$.
- $p(x)$ and $q(y)$ have a finite number of roots for each $(x,y) \in I_2 \times I_3$.
- $k = (k_1,\ldots,k_n)$ and $c = (c_1,\ldots,c_m)$ are vectors of constants with k_j in interval J_j and c_r in interval L_r.

Suppose the constants k_j and c_r are imprecise in their values. We will model this uncertainty by substitute triangular fuzzy numbers for the k_j and c_r. If we fuzzify Eq. (2.5.2), then we obtain the fuzzy wave-like equation. In the same way, we can obtain and present strategy as sections 2.2 and 2.4 for the wave-like equations. Therefore, we only denote application of the VIM for them to several examples in this section.

According to the VIM, we construct a correction functional for Eq. (2.5.2) in the form

$$u_{n+1}(t,x,y) = u_n(t,x,y) + \int_0^t \lambda(s)\{(u_n)_{ss} + p(x)(\tilde{u}_n)_{xx} + q(y)(\tilde{u}_n)_{yy} - \tilde{F}\}\, ds, \quad (2.5.4)$$

where $n \geq 0$ and λ is a lagrange multiplier. Making stationary with respect to u_n for Eq. (2.5.4), we have

$$\lambda''(s) = 0,$$
$$1 - \lambda(s)|_{s=t} = 0,$$
$$\lambda(s)|_{s=t} = 0,$$

so, the lagrange multiplier is $\lambda = s - t$. Submitting the results into Eq. (2.5.4) leads to the following iteration formula

$$u_{n+1}(t,x,y) = u_n(t,x,y) + \int_0^t (s-t)\{(u_n)_{ss} + p(x)(u_n)_{xx} + q(y)(u_n)_{yy} - F\}\, ds. \quad (2.5.5)$$

Iteration formula starts with an initial approximation, for example $u_0(t,x,y) = u(0,x,y)$.

Remark 2.5.1. We can similarly represent Theorem (2.4.2), Theorem (2.4.3), Remark (4.2.1) and Lemma (2.4.2) to the fuzzy wave-like equations.

Now we represent the following illustrating examples.

Example 2.5.1. *Consider the one-dimensional wave-like equation with variable coefficients as*

$$u_{tt} + \frac{1}{2}x^2 u_{xx} = kxt, \quad (2.5.6)$$

with the initial conditions

$$u(0,x) = c_1 x,$$

$$u_t(0,x) = c_2 x^2,$$

where $x \in (0,1)$*,* $t \in [0,\pi)$*,* $k \in [0,J_1]$*,* $c_1 \in [0,L_1]$ *and* $c_2 \in [0,L_2]$ *are constants.*

The correction functional for Eq. (2.5.6) from Eq. (2.5.5) is given as

$$u_{n+1}(t,x) = u_n(t,x) + \int_0^t (s-t)\{(u_n(s,x))_{ss} + \frac{1}{2}x^2(u_n(s,x))_{xx} - kxs\}\, ds,$$

with an initial approximation $u_0(t,x) = u(0,x) = c_1 x + c_2 t x^2$, we can obtain the following successive approximations

$$u_1(t,x) = c_1 x + c_2 x^2(t - \tfrac{t^3}{3!}) + \tfrac{1}{6}kxt^3,$$
$$u_2(t,x) = c_1 x + c_2 x^2(t - \tfrac{t^3}{3!} + \tfrac{t^5}{5!}) + \tfrac{1}{6}kxt^3,$$

and

$$u_n(t,x) = c_1 x + c_2 x^2(t - \frac{t^3}{3!} + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!}) + \frac{1}{6}kxt^3,\ n \geq 1.$$

We, therefore, obtain

$$u(t,x) = g(t,x,k,c_1,c_2) = c_1 x + c_2 x^2 \sin t + \frac{1}{6}kxt^3,$$

which is the exact solution.

Fuzzify $F(t,x,k) = kxt$ and $g(t,x,k,c_1,c_2)$ producing their γ-cuts

$$z^-(t,x,\gamma) = c_1^-(\gamma)x + c_2^-(\gamma)x^2 \sin t + \tfrac{1}{6}k^-(\gamma)xt^3,$$
$$z^+(t,x,\gamma) = c_1^+(\gamma)x + c_2^+(\gamma)x^2 \sin t + \tfrac{1}{6}k^+(\gamma)xt^3,$$
$$F^-(t,x,\gamma) = k^-(\gamma)xt,$$
$$F^+(t,x,\gamma) = k^+(\gamma)xt,$$

where $\bar{K}[\gamma] = [k^-(\gamma), k^+(\gamma)]$ and $\bar{C}_j[\gamma] = [c_j^-(\gamma), c_j^+(\gamma)]$, $j = 1, 2$. $z(t, x)$ is differentiable because $z_{tt}^i + \frac{1}{2}x^2 z_{xx}^i$ for $i \in \{-, +\}$ are γ-cuts of $\bar{K} \odot tx$ i.e. γ-cuts of a fuzzy number. Due to $p(x) > 0$, $\frac{\partial g}{\partial k} > 0$ and $\frac{\partial F}{\partial k} > 0$, therefore $z(t, x)$ is a BFS. We easily see that

$$z^i(0, x, \gamma) = c_1^i(\gamma)x,$$

$$z_t^i(0, x, \gamma) = c_2^i(\gamma)x^2,$$

for $i \in \{-, +\}$, so $z(t, x)$ also holds the initial conditions. The BFS that holds the initial condition may be written as

$$z(t, x) = \bar{C}_1 \odot x \oplus \bar{C}_2 \odot x^2 \sin t \oplus \bar{K} \odot (\frac{1}{6}xt^3),$$

for all $x \in (0, 1)$ and $t \in [0, \pi)$.

Example 2.5.2. *Consider the two-dimensional initial value problem*

$$u_{tt}(t, x, y) + \frac{1}{2}x^2 u_{xx}(t, x, y) + \frac{1}{2}y^2 u_{yy}(t, x, y) = k_1 x^2 - k_2 y^2,$$

subject to the initial conditions

$$u(0, x, y) = 0,$$

$$u_t(0, x, y) = x^2 - y^2,$$

where $x, y \in (0, 1)$, $t \in [0, 2\pi)$, $k_1 \in [0, J_1]$ *and* $k_2 \in [0, J_2]$.

Similarly we obtain the following iteration formulation by the VIM

$$u_{n+1} = u_n + \int_0^t (s - t)\{(u_n)_{ss} + \frac{1}{2}x^2(u_n)_{xx} + \frac{1}{2}y^2(u_n)_{yy} - k_1 x^2 + k_2 y^2\}\, ds. \quad (2.5.7)$$

We select an initial approximation: $u_0(t,x,y) = u(0,x,y) = (x^2 - y^2)t$, by Eq. (2.5.7) can be obtained the following successive approximations

$$u_1(t,x,y) = k_1 x^2 \tfrac{t^2}{2!} - k_2 y^2 \tfrac{t^2}{2!} - x^2 \tfrac{t^3}{3!} + y^2 \tfrac{t^3}{3!} + t(x^2 - y^2),$$

$$u_2(t,x,y) = -k_1 x^2(-\tfrac{t^2}{2!} + \tfrac{t^4}{4!}) + k_2 y^2(-\tfrac{t^2}{2!} + \tfrac{t^4}{4!}) + x^2(-\tfrac{t^3}{3!} + \tfrac{t^5}{5!})$$
$$-y^2(-\tfrac{t^3}{3!} + \tfrac{t^5}{5!}) + t(x^2 - y^2),$$

and

$$u_n(t,x,y) = -k_1 x^2(-\tfrac{t^2}{2!} + \cdots + (-1)^n \tfrac{t^{2n}}{(2n)!}) + k_2 y^2(-\tfrac{t^2}{2!} + \cdots + (-1)^n \tfrac{t^{2n}}{(2n)!})$$
$$+(x^2 - y^2)(-\tfrac{t^3}{3!} + \cdots + (-1)^n \tfrac{t^{2n+1}}{(2n+1)!}) + t(x^2 - y^2), \;\; n \geq 1$$

we, so, obtain the following solution

$$u(t,x,y) = g(t,x,y,k_1,k_2) = -k_1 x^2(\cos t - 1) + k_2 y^2(\cos t - 1) + (x^2 - y^2)\sin t,$$

which is the exact solution.

Now we fuzzify F and g. Clearly $F(t,x,y,\bar{K}_1,\bar{K}_2) = \bar{K}_1 \odot x^2 \oplus \bar{K}_2 \odot (-y^2)$ so that

$$F^-(t,x,y,\gamma) = k_1^-(\gamma)x^2 - k_2^+(\gamma)y^2,$$

$$F^+(t,x,y,\gamma) = k_1^+(\gamma)x^2 - k_2^-(\gamma)y^2.$$

Also, obtain for g

$$z^-(t,x,y,\gamma) = -k_1^-(\gamma)x^2(\cos t - 1) + k_2^+(\gamma)y^2(\cos t - 1) + (x^2 - y^2)\sin t,$$

$$z^+(t,x,y,\gamma) = -k_1^+(\gamma)x^2(\cos t - 1) + k_2^-(\gamma)y^2(\cos t - 1) + (x^2 - y^2)\sin t,$$

where $\bar{K}_1[\gamma] = [k_1^-(\gamma), k_1^+(\gamma)]$ and $\bar{K}_2[\gamma] = [k_2^-(\gamma), k_2^+(\gamma)]$. We compute

$$[z_{tt}^- + \frac{1}{2}x^2 z_{xx}^- + \frac{1}{2}y^2 z_{yy}^-, z_{tt}^+ + \frac{1}{2}x^2 z_{xx}^+ + \frac{1}{2}y^2 z_{yy}^+],$$

which are γ-cuts of $\bar{K}_1 \odot x^2 \oplus \bar{K}_2 \odot (-y^2)$ i.e. γ-cuts of a fuzzy number, therefore, $z(t,x,y)$ is differentiable. Due to

$$\tfrac{\partial F}{\partial k_1} > 0, \;\; \tfrac{\partial F}{\partial k_2} < 0, \;\; p(x) > 0,$$

$$\tfrac{\partial g}{\partial k_1} > 0, \;\; \tfrac{\partial g}{\partial k_2} < 0, \;\; q(y) > 0,$$

then by Theorem (2.4.2), $z(t,x,y)$ is a BFS. The initial conditions are true and as

$$z^-(0,x,y) = 0,$$
$$z^+(0,x,y) = 0,$$
$$z_t^-(0,x,y) = x^2 - y^2,$$
$$z_t^+(0,x,y) = x^2 - y^2.$$

Hence, $z(t,x,y)$ is a BFS which also satisfies the initial condition. This BFS may be written

$$z(t,x,y) = \bar{K}_1 \odot x^2(1-\cos t) \oplus \bar{K}_2 \odot y^2(\cos t - 1) \oplus (x^2 - y^2)\sin t,$$

for $x, y \in (0,1)$ and $t \in [0, 2\pi)$.

Example 2.5.3. *We consider the one-dimensional wave-like equation*

$$\begin{aligned} &u_{tt}(t,x) + xu_{xx}(t,x) = -kx^2, \\ &u(0,x) = cx^2, \\ &u_t(0,x) = 1, \end{aligned} \tag{2.5.8}$$

which $t \in [0,1)$, $x \in (-1,1)$ *and the value of parameters* k *and* c *are in intervals* $[0,J]$ *and* $[0,L]$, *respectively.*

The correction functional for the Eq. (2.5.8)

$$u_{n+1}(t,x) = u_n(t,x) + \int_0^t (s-t)\{(u_n(s,x))_{ss} + x(u_n(s,x))_{xx} + kx^2\}ds. \tag{2.5.9}$$

Selecting the initial approximation: $u_0(t,x) = cx^2 + t$. By (2.5.9), after two iterations the exact solution is given in the closed forms as

$$u(t,x) = g(t,x,k,c) = \frac{1}{12}kxt^4 - \frac{1}{2}kx^2t^2 + cx^2 - cxt^2 + t.$$

There is no BFS because $\frac{\partial F}{\partial k} = -x^2 < 0$ and $\frac{\partial g}{\partial k} = \frac{1}{12}xt^4 - \frac{1}{2}x^2t^2 > 0$, when $0 < x < \frac{1}{6}$ and $2.44949\sqrt{x} < t < 1$. We proceed to look for a SS. We consider two states.

Case (1): suppose $x \in [0, 1)$ and $t \in [0, 1)$, we must solve

$$u^-_{tt}(t, x, \gamma) + xu^-_{xx}(t, x, \gamma) = -k^+(\gamma)x^2,$$
$$u^+_{tt}(t, x, \gamma) + xu^+_{xx}(t, x, \gamma) = -k^-(\gamma)x^2,$$

subject to

$$u^i(0, x, \gamma) = c^i(\gamma)x^2 + t,$$

for $i \in \{-, +\}$, $\bar{K}[\gamma] = [k^-(\gamma), k^+(\gamma)]$ and $\bar{C}[\gamma] = [c^-(\gamma), c^+(\gamma)]$. By VIM, the solution is

$$\begin{aligned} u^-(t, x, \gamma) &= \tfrac{1}{12}k^+(\gamma)xt^4 - \tfrac{1}{2}k^+(\gamma)x^2t^2 + c^-(\gamma)x^2 - c^-(\gamma)xt^2 + t, \\ u^+(t, x, \gamma) &= \tfrac{1}{12}k^-(\gamma)xt^4 - \tfrac{1}{2}k^-(\gamma)x^2t^2 + c^+(\gamma)x^2 - c^+(\gamma)xt^2 + t. \end{aligned} \tag{2.5.10}$$

Now we show

$$[u^-(t, x, \gamma), u^+(t, x, \gamma)],$$

defines γ-cuts of a fuzzy number. Thus we only need to check if $\frac{\partial u^-}{\partial \gamma} > 0$ and $\frac{\partial u^+}{\partial \gamma} < 0$, since $u^i(t, x, \gamma)$ are continuous and $u^-(t, x, 1) = u^+(t, x, 1)$. There is a region $\Re$ contained in $[0, 1)\times[0, 1)$ for which the SS exists and in $[0, 1)\times[0, 1)-\Re$ there may be no SS. Since $\bar{K}$ and $\bar{C}$ are triangular fuzzy numbers, therefore, we pick simple fuzzy parameter so that $k'^-(\gamma) = c'^-(\gamma) = b > 0$ and $k'^+(\gamma) = c'^+(\gamma) = -b$. The 'prime' denotes differentiation with respect to γ. Then, for a SS we require

$$\frac{\partial u^-}{\partial \gamma} = -\tfrac{1}{12}bxt^4 + \tfrac{1}{2}bx^2t^2 + bx^2 - bxt^2 = b(-\tfrac{1}{12}xt^4 + \tfrac{1}{2}x^2t^2 + x^2 - xt^2) > 0,$$
$$\frac{\partial u^+}{\partial \gamma} = \tfrac{1}{12}bxt^4 - \tfrac{1}{2}bx^2t^2 - bx^2 + bxt^2 = -b(-\tfrac{1}{12}xt^4 + \tfrac{1}{2}x^2t^2 + x^2 - xt^2) < 0.$$

Thus we must have

$$-\frac{1}{12}xt^4 + \frac{1}{2}x^2t^2 + x^2 - xt^2 > 0, \tag{2.5.11}$$

for $t \in [0,1)$ and $x \in (0,1)$. The inequality (2.5.11) holds if

$$0 \leq t < 1, \quad \frac{12t^2 + t^4}{12 + 6t^2} < x < 1.$$

So under the above assumptions we may choose

$$\Re = \{(t,x) |\, 0 \leq t < 1 \,\&\, \frac{12t^2 + t^4}{12 + 6t^2} < x < 1\},$$

and the SS exists on $\Re$ in form Eqs. (2.5.10).

Case 2: assume $t \in [0,1)$ and $x \in (-1,0]$, we must solve

$$u^-_{tt}(t,x,\gamma) + xu^+_{xx}(t,x,\gamma) = -k^+(\gamma)x^2,$$
$$u^+_{tt}(t,x,\gamma) + xu^-_{xx}(t,x,\gamma) = -k^-(\gamma)x^2,$$

subject to

$$u^i(0,x,\gamma) = c^i(\gamma)x^2 + t,$$

for $i \in \{-,+\}$, $\bar{K}[\gamma] = [k^-(\gamma), k^+(\gamma)]$ and $\bar{C}[\gamma] = [c^-(\gamma), c^+(\gamma)]$. By VIM, the solution is

$$\begin{aligned} u^-(t,x,\gamma) &= \tfrac{1}{12}k^-(\gamma)xt^4 - \tfrac{1}{2}k^+(\gamma)x^2t^2 + c^-(\gamma)x^2 - c^+(\gamma)xt^2 + t, \\ u^+(t,x,\gamma) &= \tfrac{1}{12}k^+(\gamma)xt^4 - \tfrac{1}{2}k^-(\gamma)x^2t^2 + c^+(\gamma)x^2 - c^-(\gamma)xt^2 + t. \end{aligned} \tag{2.5.12}$$

Now we represent

$$[u^-(t,x,\gamma), u^+(t,x,\gamma)],$$

defines γ-cuts of a fuzzy number. Thus what we need to check is $\frac{\partial u^-}{\partial \gamma} > 0$ and $\frac{\partial u^+}{\partial \gamma} < 0$, since $u^i(t,x,\gamma)$ are continuous and $u^-(t,x,1) = u^+(t,x,1)$. Similar to case (1) we pick simple fuzzy parameter so that $k'^{-}(\gamma) = c'^{-}(\gamma) = b > 0$ and $k'^{+}(\gamma) = c'^{+}(\gamma) = -b$, Then for a SS of Eqs. (2.5.12) we require

$$\frac{\partial u^-}{\partial \gamma} = \tfrac{1}{12}bxt^4 + \tfrac{1}{2}bx^2t^2 + bx^2 + bxt^2 = b(\tfrac{1}{12}xt^4 + \tfrac{1}{2}x^2t^2 + x^2 + xt^2) > 0,$$
$$\frac{\partial u^+}{\partial \gamma} = -\tfrac{1}{12}bxt^4 - \tfrac{1}{2}bx^2t^2 - bx^2 - bxt^2 = -b(\tfrac{1}{12}xt^4 + \tfrac{1}{2}x^2t^2 + x^2 + xt^2) < 0.$$

Therefore, we must have

$$\frac{1}{12}xt^4 + \frac{1}{2}x^2t^2 + x^2 + xt^2 > 0, \tag{2.5.13}$$

for $t \in [0,1)$ and $x \in (-1,0)$. The inequality (2.5.13) holds if

$$0 \le t < 1, \quad -1 < x < -\frac{12t^2 + t^4}{12 + 6t^2}.$$

thus, by the above assumptions we may choose

$$\Re = \{(t,x)|\, 0 \le t < 1\,\&\, -1 < x < -\frac{12t^2 + t^4}{12 + 6t^2}\},$$

and the SS exists on $\Re$ in form Eqs. (2.5.12).

Example 2.5.4. *We consider the one-dimensional wave-like equation*

$$\begin{aligned} &u_{tt}(t,x) - u_{xx}(t,x) = kte^x, \\ &u(0,x) = c\cos x, \end{aligned} \tag{2.5.14}$$

which $t \in [0, \frac{\pi}{2})$, $x \in (0, \frac{\pi}{2})$ *and the value of parameters* k *and* c *are in intervals* $[0, J]$ *and* $[0, L]$, *respectively.*

The correction functional for the Eq. (2.5.14)

$$u_{n+1}(t,x) = u_n(t,x) + \int_0^t (s-t)\{(u_n(s,x))_{ss} - (u_n(s,x))_{xx} - kse^x\}ds. \tag{2.5.15}$$

Selecting the initial approximation: $u_0(t,x) = u(0,x) = c\cos x$. By (2.5.15), the following successive approximation are obtained

$$u_1(t,x) = ke^x\tfrac{t^3}{3!} + c\cos x(1-\tfrac{t^2}{2!}),$$
$$u_2(t,x) = ke^x(\tfrac{t^3}{3!}+\tfrac{t^5}{5!}) + c\cos x(1-\tfrac{t^2}{2!}+\tfrac{t^4}{4!}),$$

and

$$u_n(t,x) = ke^x(\frac{t^3}{3!}+\cdots+\frac{t^{2n+1}}{(2n+1)!}) + c\cos x(1-\frac{t^2}{2!}+\cdots+(-1)^n\frac{t^{2n}}{(2n)!}),\ n\geq 1.$$

Thus, the exact solution is given in the closed form as

$$u(t,x) = g(t,x,k,c) = ke^x(\sinh t - t) + c\cos x\cos t.$$

There is no BFS because $p(x) = -1 < 0$. We proceed to look for a SS. We must solve

$$u^-_{tt}(t,x,\gamma) - u^+_{xx}(t,x,\gamma) = k^-(\gamma)te^x,$$
$$u^+_{tt}(t,x,\gamma) - u^-_{xx}(t,x,\gamma) = k^+(\gamma)te^x,$$

subject to

$$u^i(0,x,\gamma) = c^i(\gamma)\cos x,$$

for $i \in \{-,+\}$, $\bar{K}[\gamma] = [k^-(\gamma), k^+(\gamma)]$ and $\bar{C}[\gamma] = [c^-(\gamma), c^+(\gamma)]$. By VIM, the solution is

$$u^-(t,x,\gamma) = \tfrac{1}{2}k^-(\gamma)e^x(\sinh t - \sin t) + \tfrac{1}{2}k^+(\gamma)e^x(\sinh t + \sin t - 2t) +$$
$$\tfrac{1}{2}c^-(\gamma)\cos x(\cos t + \cosh t) - \tfrac{1}{2}c^+(\gamma)\cos x(\cosh t - \cos t),$$
$$u^+(t,x,\gamma) = \tfrac{1}{2}k^+(\gamma)e^x(\sinh t - \sin t) + \tfrac{1}{2}k^-(\gamma)e^x(\sinh t + \sin t - 2t) +$$
$$\tfrac{1}{2}c^+(\gamma)\cos x(\cos t + \cosh t) - \tfrac{1}{2}c^-(\gamma)\cos x(\cosh t - \cos t).$$

We only need to check if $\frac{\partial u^-}{\partial\gamma} > 0$ and $\frac{\partial u^+}{\partial\gamma} < 0$, since the u^i are continuous and $u^-(t,x,1) = u^+(t,x,1)$. We pick simple fuzzy parameter so that $k'^-(\gamma) =$

$c'^{-}(\gamma) = b > 0$ and $k'^{+}(\gamma) = c'^{+}(\gamma) = -b$. Then, for a SS we require

$$\begin{aligned} \tfrac{\partial u^-}{\partial \gamma} &= be^x(t - \sin t) + b\cos x \cosh t > 0, \\ \tfrac{\partial u^+}{\partial \gamma} &= -be^x(t - \sin t) - b\cos x \cosh t < 0. \end{aligned} \tag{2.5.16}$$

Since (2.5.16) holds for each $t \in [0, \frac{\pi}{2})$ and $x \in (0, \frac{\pi}{2})$, therefore, $u(t, x)$ is SS and

$$\begin{aligned} u(t,x) = \tfrac{1}{2}e^x(\sinh t - \sin t) \odot \bar{K} \oplus \tfrac{1}{2}e^x(\sinh t + \sin t - 2t) \odot \bar{K} \\ \oplus \tfrac{1}{2}\cos x(\cos t + \cosh t) \odot \bar{C} \oplus (-\tfrac{1}{2}\cos x(\cosh t - \cos t)) \odot \bar{C}, \end{aligned}$$

for $k \in [0, J]$, $c \in [0, L]$, $t \in [0, \frac{\pi}{2})$ and $x \in (0, \frac{\pi}{2})$.

Chapter 3

Fuzzy Partial Differential Equations under Hukuhara Derivative

Mathematical uncertain or fuzzy models have attracted much attention in various fields of applied science [23, 33]. The general ideas and essential features of these models are their most applications, for instance, in the fuzzy integral equations [9, 10, 38, 40, 41] and the fuzzy differential equations [7, 17, 43]. There is an interesting growth in fuzzy partial differential equations particularly in the past decade. In general, several systems are mostly related to uncertainty and inexactness. The problem of inexactness is discussed in general exact science, and that of uncertainty is discussed as vagueness or fuzzy. For fuzzy concepts, recently, the fuzzy partial differential equations have been thoroughly studied [16, 47, 53].

3.1 Introduction

In this chapter, the existence and uniqueness theorems for a solution to an uncertain characteristic Cauchy reaction-diffusion problem is studied by Adomian decomposition method. Sufficient conditions are presented for uniform convergence of the proposed method. Also, some illustrative examples are given for solving the problem.

3.2 Uncertain Characteristic Cauchy Problem (UCCP)

Reaction-diffusion equations describe a wide variety of nonlinear systems in physics, chemistry, ecology, biology and engineering [15, 18, 45]. Consider the characteristic Cauchy reaction-diffusion equation

$$u_t(t,x) = \eta\, u_{xx}(t,x) + \phi\, u(t,x), \tag{3.2.1}$$

subject to

$$u(0,x) = f(x), \tag{3.2.2}$$

where

- $u(t,x)$ is the concentration parameter, and $(t,x) \in \Omega = [0,+\infty) \times \mathbb{R}$,
- ϕ is the reaction parameter. In this chapter, we will consider three cases as follows:

- The case $\phi = constant$,
- The case $\phi = p(x)$,
- The case $\phi = p(t)$,

- $\eta > 0$ is the finite diffusion coefficient.

In this section, we assume $f(x)$ is the fuzzy extension of real-valued differentiable function, i.e. fuzzy function. Therefore, we will have a fuzzy problem by the Eqs. (3.2.1) and (3.2.2) called an UCCP. We consider ϕ as a known crisp function and, without losing generality, assume $\phi < 0$. Now an UCCP, using the parametric representation of fuzzy numbers is translated as system of equations in the crisp case.

If $f : \mathbb{R} \to E$ is a fuzzy function with the parametric representation of $f(x)$ as

$$f(x) = (f^-(x, \gamma), f^+(x, \gamma)),$$

then in the Eq. (3.2.1) $u : \Omega \to (E, D)$ is a fuzzy function that

$$u(t, x) = (u^-(t, x, \gamma), u^+(t, x, \gamma)).$$

Thus by assumptions $\eta > 0$ and $\phi < 0$, we can obtain $u(t, x)$ by solving the following system in the crisp case

$$\begin{aligned}
u_t^-(t, x, \gamma) &= \eta\, u_{xx}^-(t, x, \gamma) + \phi\, u^+(t, x, \gamma),\\
u_t^+(t, x, \gamma) &= \eta\, u_{xx}^+(t, x, \gamma) + \phi\, u^-(t, x, \gamma),\\
u^\pm(0, x, \gamma) &= f^\pm(x, \gamma).
\end{aligned} \tag{3.2.3}$$

The approximate solution of this system is found by the Adomian Decomposition Method (ADM) [1, 4, 5, 6, 34, 51] and an approximation is obtained for fuzzy solution of UCCP.

3.3 The Adomian decomposition method

Defining the partial differential operators $L_t = \frac{\partial}{\partial t}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$. Consider the partial differential equation is written in an operator form

$$L_t u = \eta\, L_{xx} u + \phi\, I\, u, \tag{3.3.1}$$

subject to

$$u(0, x) = g(x), \tag{3.3.2}$$

where I is the identity operator.

Let us formally define the inverse integral operator [4, 5],

$$L_t^{-1} = \int_0^t ds.$$

Applying the inverse operator L_t^{-1} to the Eq. of (3.3.1), and using the initial condition (3.3.2) yields

$$u = g(x) + \eta\, L_t^{-1}(L_{xx} u) + L_t^{-1}(\phi\, I\, u). \tag{3.3.3}$$

The linear terms $u(t, x)$ can be decomposed by an infinite series of components

$$u(t, x) = \sum_{k=0}^{\infty} u_k(t, x). \tag{3.3.4}$$

Because there are not nonlinear terms in Eq. (3.3.1), therefore it is not expressed the infinite series of the so-called Adomian polynomials. Substituting (3.3.4) into (3.3.3) gives

$$\sum_{k=0}^{\infty} u_k = g(x) + \eta\, L_t^{-1}(L_{xx} \sum_{k=0}^{\infty} u_k) + L_t^{-1}(\phi \sum_{k=0}^{\infty} u_k). \tag{3.3.5}$$

In result, the Eq. (3.3.1) is transformed into a set of recursive relations given by

$$\begin{aligned} &u_0(t, x) = g(x), \\ &u_{k+1}(t, x) = \eta\, L_t^{-1}(L_{xx} u_k) + L_t^{-1}(\phi\, u_k), \quad k \geq 0. \end{aligned} \tag{3.3.6}$$

We assume $\varphi_k(t,x) = \sum_{i=0}^{k} u_i(t,x)$, obviously we have

$$u(t,x) = \lim_{k\to\infty} \varphi_k,$$

therefore we rewrite successive iterations (3.3.6) as follows

$$\begin{aligned} &\varphi_0(t,x) = g(x), \\ &\varphi_{k+1}(t,x) = g(x) + \textstyle\sum_{i=1}^{k+1} \int_0^t (\eta\, L_{xx} u_{i-1} + \phi\, u_{i-1})\, ds, \quad k \geq 0. \end{aligned} \tag{3.3.7}$$

3.4 Existence and uniqueness of fuzzy solution

In this section, the existence and uniqueness of fuzzy solution is obtained using successive iterations of ADM. Also, we will show uniform convergence of ADM.

Consider the UCCP

$$L_t u(t,x) = \eta \odot L_{xx} u(t,x) \oplus \phi \odot u(t,x), \tag{3.4.1}$$

subject to

$$\begin{aligned} &u(0,x) = f(x), \\ &(t,x) \in \Omega = [0,+\infty) \times \mathbb{R}, \end{aligned} \tag{3.4.2}$$

where $L_t = \frac{\partial}{\partial t}$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$ define the partial differential operators and $\eta > 0$ is a constant and $f : \mathbb{R} \to E$ is a fuzzy function and ϕ is defined as Section 3.2.

Definition 3.4.1. The fuzzy function $u : \Omega \to (E, D)$ is a solution of the Eqs. (3.4.1) and (3.4.2) if and only if

$$u_t^-(t,x,\gamma) = \eta\, u_{xx}^-(t,x,\gamma) + \phi\, u^+(t,x,\gamma),$$

$$u_t^+(t,x,\gamma) = \eta\, u_{xx}^+(t,x,\gamma) + \phi\, u^-(t,x,\gamma),$$

and

$$u^-(0,x,\gamma) = f^-(x,\gamma),$$

$$u^+(0,x,\gamma) = f^+(x,\gamma).$$

Theorem 3.4.1. *Let $\beta \geq 1$. Assume that the Eqs. (3.4.1) and (3.4.2) satisfy the following conditions*

(i) $f : \mathbb{R} \to E$ is continuous and bounded,

(ii) ϕ is a continuous function,

(iii) $u, v, u_{xx}, v_{xx} : \Omega \to (E, D)$ are continuous fuzzy functions on Ω and satisfy the following inequalities

$$D(L_{xx}u, L_{xx}v) \leq \theta e^{-\beta t} D(u,v), \tag{3.4.3}$$

$$D(\phi \odot u, \phi \odot v) \leq \epsilon e^{-\beta t} D(u,v), \tag{3.4.4}$$

where $0 < \theta < \frac{1}{4\eta}$, $0 < \epsilon < \frac{1}{4}$.

Then there exists a unique fuzzy solution $u(t,x) : \Omega \to (E,D)$ of (3.4.1) and the successive iterations

$$\varphi_0(t,x) = f(x),$$

$$\varphi_{k+1}(t,x) = f(x) \oplus \sum_{i=1}^{k+1} \int_0^t (\eta \odot L_{xx}u_{i-1}(s,x) \oplus \phi \odot u_{i-1}(s,x))\, ds, \quad (k \geq 0), \tag{3.4.5}$$

are uniformly convergent to $u(t,x)$ on Ω.

In this section $\sum$ means the sum of fuzzy numbers for each (t, x) in domain. Before proving Theorem 3.4.1, at first we represent the following lemma.

Lemma 3.4.2. *If the conditions of Theorem 3.4.1 are satisfied and*

$$\begin{aligned} &u_0(t,x) = f(x), \\ &u_k(t,x) = \int_0^t (\eta \odot L_{xx}u_{k-1}(s,x) \oplus \phi \odot u_{k-1}(s,x))\, ds, \quad (k \geq 1), \end{aligned} \tag{3.4.6}$$

then

(I) $u_k(t,x)$ *is bounded,*

(II) $u_k(t,x)$ *is continuous.*

Now we will prove Theorem 3.4.1.

Proof. It is easy to see that all $\varphi_k(t,x)$ are bounded on Ω. Indeed, $\varphi_0(t,x) = f(x)$ is bounded by the hypothesis. Assume $\varphi_{k-1}(t,x)$ is bounded. From (3.4.5) we have

$$\begin{aligned} D(\varphi_k(t,x), \tilde{0}) &= D(f(x) \oplus \textstyle\sum_{i=1}^{k} \int_0^t (\eta \odot L_{xx}u_{i-1}(s,x) \oplus \phi \odot u_{i-1}(s,x))\, ds, \tilde{0}) \\ &= D(f(x) \oplus \textstyle\sum_{i=1}^{k-1} \int_0^t (\eta \odot L_{xx}u_{i-1}(s,x) \oplus \phi \odot u_{i-1}(s,x))\, ds \oplus \\ &\quad \textstyle\int_0^t (\eta \odot L_{xx}u_{k-1}(s,x) \oplus \phi \odot u_{k-1}(s,x))\, ds, \tilde{0}) \\ &= D(\varphi_{k-1}(t,x) \oplus \textstyle\int_0^t (\eta \odot L_{xx}u_{k-1}(s,x) \oplus \phi \odot u_{k-1}(s,x))\, ds, \tilde{0}) \\ &\leq D(\varphi_{k-1}(t,x), \tilde{0}) + D(\textstyle\int_0^t (\eta \odot L_{xx}u_{k-1}(s,x) \oplus \phi \odot u_{k-1}(s,x))\, ds, \tilde{0}), \end{aligned}$$

using induction and Lemma 3.4.2 part (I) we obtain that $\varphi_k(t,x)$ is bounded. Thus, $\{\varphi_k(t,x)\}$ is a sequence of bounded functions on Ω.

Next we prove that $\varphi_k(t,x)$ are continuous on Ω. For this purpose, without losing generality, we assume $\phi = p(x)$ then by Lemma 3.4.2 part (II) for $-\infty < x \leq \hat{x} < +\infty$ and $0 \leq t \leq \hat{t} < +\infty$, we have

$$
\begin{aligned}
&D(\varphi_k(t,x), \varphi_k(\hat{t},\hat{x})) \\
&\leq D(f(x), f(\hat{x})) + \\
&\quad D(\textstyle\sum_{i=1}^k \int_0^t (\eta \odot L_{xx} u_{i-1}(s,x) \oplus p(x) \odot u_{i-1}(s,x))\, ds, \\
&\quad\quad \textstyle\sum_{i=1}^k \int_0^{\hat{t}} (\eta \odot L_{\hat{x}\hat{x}} u_{i-1}(s,\hat{x}) \oplus p(\hat{x}) \odot u_{i-1}(s,\hat{x}))\, ds) \\
&\leq D(f(x), f(\hat{x})) + D(\textstyle\sum_{i=1}^k \int_0^t (\eta \odot L_{xx} u_{i-1}(s,x) \oplus p(x) \odot u_{i-1}(s,x))\, ds, \\
&\quad\quad \textstyle\sum_{i=1}^k \int_0^t (\eta \odot L_{\hat{x}\hat{x}} u_{i-1}(s,\hat{x}) \oplus p(\hat{x}) \odot u_{i-1}(s,\hat{x}))\, ds) + \\
&\quad D(\textstyle\sum_{i=1}^k \int_t^{\hat{t}} (\eta \odot L_{\hat{x}\hat{x}} u_{i-1}(s,\hat{x}) \oplus p(\hat{x}) \odot u_{i-1}(s,\hat{x}))\, ds, \tilde{0}) \\
&\leq D(f(x), f(\hat{x})) + D(\textstyle\sum_{i=1}^k \int_0^t \eta \odot L_{xx} u_{i-1}(s,x)\, ds, \sum_{i=1}^k \int_0^t \eta \odot L_{\hat{x}\hat{x}} u_{i-1}(s,\hat{x})\, ds) + \\
&\quad D(\textstyle\sum_{i=1}^k \int_0^t p(x) \odot u_{i-1}(s,x)\, ds, \sum_{i=1}^k \int_0^t p(\hat{x}) \odot u_{i-1}(s,\hat{x})\, ds) + \\
&\quad D(\textstyle\sum_{i=1}^k \int_t^{\hat{t}} \eta \odot L_{\hat{x}\hat{x}} u_{i-1}(s,\hat{x})\, ds, \tilde{0}) + D(\sum_{i=1}^k \int_t^{\hat{t}} p(\hat{x}) \odot u_{i-1}(s,\hat{x})\, ds, \tilde{0}) \\
&\leq D(f(x), f(\hat{x})) + \eta t \textstyle\sum_{i=1}^k \sup_{t\in[0,+\infty), x, \hat{x}\in\mathbb{R}} D(L_{xx} u_{i-1}(t,x), L_{\hat{x}\hat{x}} u_{i-1}(t,\hat{x})) + \\
&\quad t \textstyle\sum_{i=1}^k \sup_{t\in[0,+\infty), x, \hat{x}\in\mathbb{R}} D(p(x) \odot u_{i-1}(t,x), p(\hat{x}) \odot u_{i-1}(t,\hat{x})) + \\
&\quad \eta\theta \sup_{t,\hat{t}\in[0,+\infty), \hat{x}\in\mathbb{R}} D(\textstyle\sum_{i=1}^k u_{i-1}(t,\hat{x}), \tilde{0}) \int_t^{\hat{t}} e^{-\beta s} ds \\
&\quad + \epsilon \sup_{t,\hat{t}\in[0,+\infty), \hat{x}\in\mathbb{R}} D(\textstyle\sum_{i=1}^k u_{i-1}(t,\hat{x}), \tilde{0}) \int_t^{\hat{t}} e^{-\beta s} ds.
\end{aligned}
$$

By hypotheses we have

$$D(\varphi_k(t,x),\varphi_k(\hat{t},\hat{x}))\to 0 \quad as \quad (t,x)\to(\hat{t},\hat{x}).$$

Thus the sequence $\{\varphi_k(t,x)\}$ is continuous on Ω.

The uniform convergence of the sequence $\{\varphi_k(t,x)\}$ is proved as follows. Relations (3.4.3), (3.4.4) and their analogue components corresponding to $k+1$, for $k\geq 1$ will result in the following

$$\begin{aligned}
&D(\varphi_{k+1}(t,x),\varphi_k(t,x))\\
&\quad = D(f(x)\oplus \textstyle\sum_{i=1}^{k+1}\int_0^t(\eta\odot L_{xx}u_{i-1}(s,x)\oplus\phi\odot u_{i-1}(s,x))\,ds,\\
&\qquad f(x)\oplus \textstyle\sum_{i=1}^{k}\int_0^t(\eta\odot L_{xx}u_{i-1}(s,x)\oplus\phi\odot u_{i-1}(s,x))\,ds)\\
&\quad = D(\varphi_k(t,x)\oplus \textstyle\int_0^t(\eta\odot L_{xx}u_k(s,x)\oplus\phi\odot u_k(s,x))\,ds,\varphi_k(t,x))\\
&\quad = D(\textstyle\int_0^t(\eta\odot L_{xx}u_k(s,x)\oplus\phi\odot u_k(s,x))\,ds,\tilde{0})\\
&\quad \leq \eta\theta\sup_{(t,x)\in\Omega}D(u_k(t,x),\tilde{0})\textstyle\int_0^t e^{-\beta s}ds\\
&\qquad +\epsilon\sup_{(t,x)\in\Omega}D(u_k(t,x),\tilde{0})\textstyle\int_0^t e^{-\beta s}ds\\
&\quad \leq (\tfrac{\eta\theta}{\beta}+\tfrac{\epsilon}{\beta})\sup_{(t,x)\in\Omega}D(u_k(t,x),\tilde{0})\\
&\quad \leq \tfrac{1}{2\beta}\sup_{(t,x)\in\Omega}D(u_k(t,x),\tilde{0}).
\end{aligned}$$

Therefore we have

$$\sup_{(t,x)\in\Omega}D(\varphi_{k+1}(t,x),\varphi_k(t,x))\leq\frac{1}{2\beta}\sup_{(t,x)\in\Omega}D(u_k(t,x),\tilde{0}). \tag{3.4.7}$$

On the other hand, from (3.4.6) we can obtain for $k \geq 1$,

$$D(u_k(t,x),\tilde{0}) = D(\int_0^t(\eta \odot L_{xx}u_{k-1}(s,x) \oplus \phi \odot u_{k-1}(s,x))\,ds,\tilde{0})$$

$$\leq D(\int_0^t \eta \odot L_{xx}u_{k-1}(s,x)\,ds,\tilde{0}) + D(\int_0^t \phi \odot u_{k-1}(s,x)\,ds,\tilde{0})$$

$$\leq \eta\theta \sup_{(t,x)\in\Omega} D(u_{k-1}(t,x),\tilde{0}) \int_0^t e^{-\beta s}ds$$

$$+\epsilon \sup_{(t,x)\in\Omega} D(u_{k-1}(t,x),\tilde{0}) \int_0^t e^{-\beta s}ds$$

$$\leq \tfrac{1}{2\beta} \sup_{(t,x)\in\Omega} D(u_{k-1}(t,x),\tilde{0})$$

$$\vdots$$

$$\leq \tfrac{1}{(2\beta)^k} \sup_{(t,x)\in\Omega} D(u_0(t,x),\tilde{0}) = \tfrac{1}{(2\beta)^k} \sup_{x\in\mathbb{R}} D(f(x),\tilde{0}).$$

Thus we have

$$\sup_{(t,x)\in\Omega} D(u_k(t,x),\tilde{0}) \leq \frac{Q}{(2\beta)^k}, \tag{3.4.8}$$

where $Q = \sup_{x\in\mathbb{R}} D(f(x),\tilde{0})$. Furthermore, from (3.4.7), we obtain

$$\sup_{(t,x)\in\Omega} D(\varphi_{k+1}(t,x),\varphi_k(t,x)) \leq \frac{Q}{(2\beta)^{k+1}},$$

which denotes that the series $\sum_{k=0}^{+\infty} D(\varphi_{k+1}(t,x),\varphi_k(t,x))$ is dominated, uniformly on Ω, by the series $\frac{Q}{2\beta}\sum_{k=0}^{+\infty}\frac{1}{(2\beta)^k}$. But $\beta \geq 1$ guarantees the convergence of the last series, implying the uniform convergence of the sequence $\{\varphi_k(t,x)\}$. If we show $u(t,x) = \lim_{k\to+\infty}\varphi_k(t,x)$, then $u(t,x)$ satisfies (3.4.1). It is obviously continuous and bounded on Ω. The two preceding summations are the usual summations.

To prove the uniqueness, suppose $u(t,x)$ and $v(t,x)$ be two continuous solutions of (3.4.1) on Ω, then

$$0 \le D(u(t,x), v(t,x)) = D(u(t,x) \oplus \varphi_k(t,x), v(t,x) \oplus \varphi_k(t,x))$$

$$\le D(u(t,x), \varphi_k(t,x)) + D(v(t,x), \varphi_k(t,x)),$$

and since $\varphi_k(t,x)$ is convergent to solution of (3.4.1), therefore

$$D(u(t,x), \varphi_k(t,x)) \to 0,$$

$$D(v(t,x), \varphi_k(t,x)) \to 0,$$

when $k \to +\infty$, $D(u(t,x), v(t,x)) = 0$ that is $u(t,x) = v(t,x)$. This finishes the proof of Theorem 3.4.1. □

Theorem 3.4.3. *Assume $0 < \beta < 1$ and the Eqs. (3.4.1) and (3.4.2) satisfy the following conditions*

(i) $f : \mathbb{R} \to E$ is continuous and bounded,

(ii) ϕ is a continuous function such that

$$\int_0^t |\phi| \, ds \le \frac{\beta}{2}, \tag{3.4.9}$$

(iii) $u, v, u_{xx}, v_{xx} : \Omega \to (E, D)$ are continuous fuzzy functions and satisfy the following inequality

$$D(L_{xx}u, L_{xx}v) \le \theta e^{-\beta t} \, D(u, v), \tag{3.4.10}$$

where $0 < \theta < \frac{\beta^2}{2\eta}$.

Then there exists a unique fuzzy solution $u(t,x) : \Omega \to (E,D)$ of (3.4.1) and the successive iterations

$$\varphi_0(t,x) = f(x),$$

$$\varphi_{k+1}(t,x) = f(x) \oplus \sum_{i=1}^{k+1} \int_0^t (\eta \odot L_{xx} u_{i-1}(s,x) \oplus \phi \odot u_{i-1}(s,x))\, ds, \qquad (k \geq 0), \tag{3.4.11}$$

are uniformly convergent to $u(t,x)$ on Ω.

Proof. Proving the uniqueness of solution and that $\varphi_k(t,x)$ is bounded and continuous, is similar to proving Theorem 3.4.1.

Now, by the aforementioned hypotheses the uniform convergence of the sequence $\{\varphi_k(t,x)\}$ is proved as follows. By relation (3.4.10) and their analogue components corresponding to $k+1$, for $k \geq 1$ will result in the following

$$D(\varphi_{k+1}(t,x), \varphi_k(t,x))$$

$$= D(f(x) \oplus \sum_{i=1}^{k+1} \int_0^t (\eta \odot L_{xx} u_{i-1}(s,x) \oplus \phi \odot u_{i-1}(s,x))\, ds,$$

$$f(x) \oplus \sum_{i=1}^{k} \int_0^t (\eta \odot L_{xx} u_{i-1}(s,x) \oplus \phi \odot u_{i-1}(s,x))\, ds)$$

$$= D(\varphi_k(t,x) \oplus \int_0^t (\eta \odot L_{xx} u_k(s,x) \oplus \phi \odot u_k(s,x))\, ds, \varphi_k(t,x))$$

$$= D(\int_0^t (\eta \odot L_{xx} u_k(s,x) \oplus \phi \odot u_k(s,x))\, ds, \tilde{0})$$

$$\leq \eta\theta \sup_{(t,x)\in\Omega} D(u_k(t,x), \tilde{0}) \int_0^t e^{-\beta s} ds + \sup_{(t,x)\in\Omega} D(u_k(t,x), \tilde{0}) \int_0^t |\phi|\, ds$$

$$\leq \frac{\eta\theta}{\beta} \sup_{(t,x)\in\Omega} D(u_k(t,x), \tilde{0}) + \frac{\beta}{2} \sup_{(t,x)\in\Omega} D(u_k(t,x), \tilde{0})$$

$\leq (\frac{\eta\theta}{\beta} + \frac{\beta}{2}) \sup_{(t,x)\in\Omega} D(u_k(t,x), \tilde{0})$

$\leq \beta \sup_{(t,x)\in\Omega} D(u_k(t,x), \tilde{0})$.

Thus we obtain

$$\sup_{(t,x)\in\Omega} D(\varphi_{k+1}(t,x), \varphi_k(t,x)) \leq \beta \sup_{(t,x)\in\Omega} D(u_k(t,x), \tilde{0}). \qquad (3.4.12)$$

As (3.4.8), we can obtain

$$\sup_{(t,x)\in\Omega} D(u_k(t,x), \tilde{0}) \leq Q\beta^k,$$

which $Q = \sup_{x\in\mathbb{R}} D(f(x), \tilde{0})$. Moreover, from (3.4.12) we have

$$\sup_{(t,x)\in\Omega} D(\varphi_{k+1}(t,x), \varphi_k(t,x)) \leq Q\beta^{k+1},$$

where shows that the series $\sum_{k=0}^{+\infty} D(\varphi_{k+1}(t,x), \varphi_k(t,x))$ is dominated, uniformly on Ω, by the series $Q\beta \sum_{k=0}^{+\infty} \beta^k$. But $0 < \beta < 1$ guarantees the convergence of the last series, implying the uniform convergence of the sequence $\{\varphi_k(t,x)\}$. If we show $u(t,x) = \lim_{k\to+\infty} \varphi_k(t,x)$, then $u(t,x)$ satisfies (3.4.1). It is obviously continuous and bounded on Ω. The two preceding summations are the usual summations. This finishes the proof of Theorem 3.4.3. □

Theorem 3.4.4. *If the conditions of Theorem 3.4.1 hold, then $u(t,x) \in E$ for each $(t,x) \in \Omega$.*

Proof. We prove that $u(t,x)$ holds in Lemma 1.3.1 for each $(t,x) \in \Omega$. The successive approximations $\varphi_0(t,x) = f(x) \in E$ for all $x \in \mathbb{R}$,

$$\varphi_{k+1}(t,x) = f(x) \oplus \sum_{i=1}^{k+1} \int_0^t (\eta \odot L_{xx} u_{i-1}(s,x) \oplus \phi \odot u_{i-1}(s,x))\, ds, \qquad (k \geq 0),$$

where the integral is the fuzzy integral, define a sequence of fuzzy numbers $\varphi_k(t,x) \in E$ for each $(t,x) \in \Omega$. Hence

$$\varphi_k(t,x)[\gamma^1] \supset \varphi_k(t,x)[\gamma^2], \;\; if \;\; 0 < \gamma_1 \leq \gamma_2 \leq 1,$$

which implies that

$$[u^-(t,x,\gamma^1), u^+(t,x,\gamma^1)] \supset [u^-(t,x,\gamma^2), u^+(t,x,\gamma^2)], \;\; if \;\; 0 < \gamma_1 \leq \gamma_2 \leq 1,$$

since, by the convergence of sequence $\varphi_k(t,x)$, the end points of $\varphi_k(t,x)[\gamma]$ converge to $u^-(t,x,\gamma)$ and $u^+(t,x,\gamma)$, respectively. Thus the inclusion property (i) of Lemma 1.3.1 holds for the intervals $[u^-(t,x,\gamma), u^+(t,x,\gamma)]$, $\gamma \in (0,1]$.

For the proof of the continuity property (ii) of Lemma 1.3.1, let (γ_k) be a non-decreasing sequence in $(0,1]$ converging to γ. Then $\varphi_0^-(t,x,\gamma_k) \to \varphi_0^-(t,x,\gamma)$ and $\varphi_0^+(t,x,\gamma_k) \to \varphi_0^+(t,x,\gamma)$ because $\varphi_0(t,x) \in E$ for all $(t,x) \in \Omega$. But then, by the continuous dependence on the initial approximation of the solution, $u^-(t,x,\gamma_k) \to u^-(t,x,\gamma)$ and $u^+(t,x,\gamma_k) \to u^+(t,x,\gamma)$, i.e. (ii) holds for the intervals

$$[u^-(t,x,\gamma), u^+(t,x,\gamma)], \;\; \gamma \in (0,1].$$

Hence, by Lemma 1.3.1, $u(t,x) \in E$. ◻

Theorem 3.4.5. *If the conditions of Theorem 3.4.1 hold, then the largest interval of existence of any fuzzy solution $u(t,x)$ of (3.4.1) is Ω and the limit*

$$\lim_{x\to\pm\infty}\lim_{t\to+\infty} u(t,x) = \xi \in E,$$

exists.

Proof. Obviously, using Theorem 3.4.1, $u(t,x)$ is continuous and bounded on Ω then the limits

$$[\xi^-(\gamma), \xi^+(\gamma)] = [\lim_{x\to\pm\infty}\lim_{t\to+\infty} u^-(t,x,\gamma), \lim_{x\to\pm\infty}\lim_{t\to+\infty} u^+(t,x,\gamma)],$$

exist. On the other hand via Theorem 3.4.4, we can obtain that the intervals $[\xi^-(\gamma), \xi^+(\gamma)]$, $0 < \gamma \leq 1$, define a fuzzy number $\xi \in E$. □

Remark 3.4.1. We can analogously obtain the result of Theorem 3.4.4 and Theorem 3.4.5 using Theorem 3.4.3.

We illustrate the method by the following examples. Note that in all examples $\gamma \in [0,1]$.

Example 3.4.1. *Consider the Eq. (3.4.1) with $\eta = 1$ and $\phi = p(x) = -\frac{x^2+5}{x^2+7}$ i.e.*

$$u_t(t,x) = u_{xx}(t,x) \oplus (-\frac{x^2+5}{x^2+7}) \odot u(t,x), \quad (t,x) \in \Omega. \tag{3.4.13}$$

The fuzzy initial condition is given as

$$\begin{aligned} u^-(0,x,\gamma) &= 3e^{-4}(x^2+7)(-\gamma^2+3\gamma-2), \\ u^+(0,x,\gamma) &= 3e^{-4}(x^2+7)(\gamma^2-3\gamma+2), \end{aligned} \tag{3.4.14}$$

i.e. $u(0,x) = (u^-(0,x,\gamma), u^+(0,x,\gamma))$ and is plotted in Figure 1.

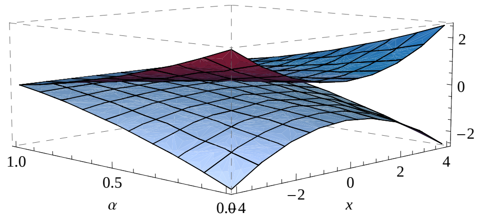

Fig. 1. The fuzzy initial condition in Example 3.4.1.

The exact fuzzy solution $u(t,x) = (u^-(t,x,\gamma), u^+(t,x,\gamma))$ in this case is given by

$$\begin{aligned} u^-(t,x,\gamma) &= 3e^{t-4}(x^2+7)(-\gamma^2+3\gamma-2), \\ u^+(t,x,\gamma) &= 3e^{t-4}(x^2+7)(\gamma^2-3\gamma+2). \end{aligned} \tag{3.4.15}$$

We see $u^-(t,x,\gamma)$ and $u^+(t,x,\gamma)$ represent a valid fuzzy number on Ω, but note that the H-derivative of (3.4.15) in that case with respect to x is given by

$$\begin{aligned} u_x^-(t,x,\gamma) &= 6xe^{t-4}(-\gamma^2+3\gamma-2), \\ u_x^+(t,x,\gamma) &= 6xe^{t-4}(\gamma^2-3\gamma+2), \end{aligned}$$

and it gives a fuzzy number for $(t,x) \in [0,+\infty) \times [0,+\infty)$. Then it is again H-differentiable on $[0,+\infty) \times [0,+\infty)$ and

$$\begin{aligned} u_{xx}^-(t,x,\gamma) &= 6e^{t-4}(-\gamma^2+3\gamma-2), \\ u_{xx}^+(t,x,\gamma) &= 6e^{t-4}(\gamma^2-3\gamma+2). \end{aligned}$$

Also the H-derivative of (3.4.15) in that case with respect to t is given by

$$\begin{aligned} u_t^-(t,x,\gamma) &= 3e^{t-4}(x^2+7)(-\gamma^2+3\gamma-2), \\ u_t^+(t,x,\gamma) &= 3e^{t-4}(x^2+7)(\gamma^2-3\gamma+2), \end{aligned}$$

and it gives a fuzzy number for $(t,x) \in \Omega$.

We thus see that $u(t,x)$ defined by (3.4.15) is H-differentiable with respect to x and t for $(t,x) \in [0,+\infty) \times [0,+\infty)$ and satisfy (3.4.13) and (3.4.14).

Suppose $u(t,x) = (u^-(t,x,\gamma), u^+(t,x,\gamma))$, therefore the system of equations (3.2.3) is as follows

$$\begin{aligned} u_t^-(t,x,\gamma) &= u_{xx}^-(t,x,\gamma) - \tfrac{x^2+5}{x^2+7}\, u^+(t,x,\gamma), \\ u_t^+(t,x,\gamma) &= u_{xx}^+(t,x,\gamma) - \tfrac{x^2+5}{x^2+7}\, u^-(t,x,\gamma), \end{aligned} \tag{3.4.16}$$

with the initial conditions (3.4.14). We apply the ADM with the Eq. (3.4.16). Therefore, we calculate u_k^-s and u_k^+s for $k = 1, \ldots, 20$ and consider u_{20}^- and u_{20}^+ as a approximation of exact solution, (3.4.15). The numerical results by this approximation are summarized in Table 1 and the error function is plotted in Fig. 2.

Table 1
Comparison of the exact and approximate values in Example 3.4.1

x	t	Exact values ($\gamma = 0.5$)	Approximate values by the ADM ($\gamma = 0.5$)	Error (D)
-4	0	(-0.94783,0.94783)	(-0.94783,0.94783)	0
-3	0.3	(-0.89005,0.89005)	(-0.89005,0.89005)	1.11022×10^{-16}
0	0.75	(-0.61069,0.61069)	(-0.61069,0.61069)	4.44089×10^{-16}
2	1	(-1.23223,1.23223)	(-1.23223,1.23223)	4.44089×10^{-16}
4	1.5	(-4.2479,4.2479)	(-4.2479,4.2479)	1.77636×10^{-15}

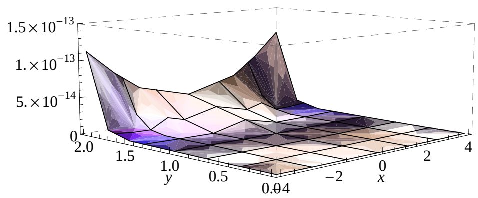

Fig. 2. Plot of D in Example 3.4.1.

In the following examples it can be seen that a monotonic convergence of the ADM exist without considering the exact solution. The sequence of partial sums u_k^- and u_k^+ are showed for various values of k. Form figure and table in examples it can be seen that convergence is achieved rapidly within $15-20$ terms taken the decomposition series solution.

Example 3.4.2. *Consider* $\eta = \frac{1}{6}$ *and* $\phi = -1$, *the Eq. (3.4.1) is written the following as*

$$u_t(t,x) = \frac{1}{6} \odot u_{xx}(t,x) \oplus (-1) \odot u(t,x), \quad (t,x) \in \Omega. \tag{3.4.17}$$

The parametric representation of the fuzzy initial condition is as follows

$$\begin{aligned} u^-(0,x,\gamma) &= \tfrac{e^x}{e^x+1}(-\gamma^2+2\gamma+3), \\ u^+(0,x,\gamma) &= \tfrac{e^x}{e^x+1}(\gamma^2-3\gamma+6). \end{aligned} \tag{3.4.18}$$

If $u(t,x) = (u^-(t,x,\gamma), u^+(t,x,\gamma))$, the system of equations (3.2.3) is sufficient to be solved, i.e.

$$\begin{aligned} u_t^-(t,x,\gamma) &= \tfrac{1}{6}u_{xx}^-(t,x,\gamma) - u^+(t,x,\gamma), \\ u_t^+(t,x,\gamma) &= \tfrac{1}{6}u_{xx}^+(t,x,\gamma) - u^-(t,x,\gamma), \end{aligned} \tag{3.4.19}$$

with the initial conditions (3.4.18). We obtain approximate solution of (3.4.17) by the ADM. Thus we calculate u_k^-s and u_k^+s for $k = 1, \dots, 20$ and consider u_j^- and u_j^+ that $j \in \{15, 16, \dots, 20\}$ as a approximation of the exact solution. Table 2 denotes the numerical results and Fig. 3 shows the error function.

Table 2

Comparison of the approximate values by the ADM in Example 3.4.2

x	t	Approximate values ($k=15, \gamma=0.5$)	Approximate values ($k=20, \gamma=0.5$)	Error (D)
-3	0	(0.17785,0.22527)	(0.17785,0.22527)	0
-2.5	0.25	(0.20901,0.30960)	(0.20901,0.30960)	5.55112×10^{-17}
0	0.5	(0.8767,1.70106)	(0.8767,1.70106)	4.44089×10^{-16}
2.5	1	(0.18681,2.6718)	(0.18681,2.6718)	6.5578×10^{-10}
3	1.5	(-1.2174,3.0038)	(-1.2174,3.0038)	2.86757×10^{-8}

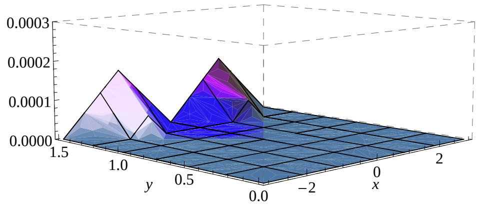

Fig. 3. Plot of D in Example 3.4.2.

Example 3.4.3. *We solve the Eq. (3.4.1) with* $\eta = 1$ *and* $\phi = p(t) = -e^{-t}$ *i.e.*

$$u_t(t,x) = u_{xx}(t,x) \oplus (-e^{-t}) \odot u(t,x), \quad (t,x) \in \Omega. \tag{3.4.20}$$

The fuzzy initial condition is written as

$$\begin{aligned} u^-(0,x,\gamma) &= e^{x-4}\gamma, \\ u^+(0,x,\gamma) &= e^{x-4}(2-\gamma). \end{aligned} \tag{3.4.21}$$

Assume $u(t,x) = (u^-(t,x,\gamma), u^+(t,x,\gamma))$, hence the system of equations (3.4.20) is as follows

$$\begin{aligned} u_t^-(t,x,\gamma) &= u_{xx}^-(t,x,\gamma) - e^{-t}\, u^+(t,x,\gamma), \\ u_t^+(t,x,\gamma) &= u_{xx}^+(t,x,\gamma) - e^{-t}\, u^-(t,x,\gamma), \end{aligned} \tag{3.4.22}$$

with the initial conditions (3.4.21). We obtain approximate solution of (3.4.20) by the ADM. Thus we calculate u_k^-s and u_k^+s for $k = 1, \ldots, 20$ and consider u_j^- and u_j^+ that $j \in \{15, 16, \ldots, 20\}$ as a approximation of the exact solution. Table 3 and Fig. 4 show the numerical results and the error function, respectively.

Table 3

Comparison of the approximate values by the ADM in Example 3.4.3

x	t	Approximate values ($k = 15, \gamma = 0.5$)	Approximate values ($k = 20, \gamma = 0.5$)	Error (D)
-1	0	(0.00337,0.010107)	(0.00337,0.010107)	0
-0.5	0.25	(0.00254,0.020331)	(0.00254,0.020331)	3.46945×10^{-18}
0	0.5	(-0.0020,0.042752)	(-0.0020,0.042752)	1.52656×10^{-16}
0.5	0.75	(-0.0165,0.091894)	(-0.0165,0.091894)	7.86315×10^{-14}
1	1	(-0.0554,0.199249)	(-0.0554,0.199249)	6.66961×10^{-12}

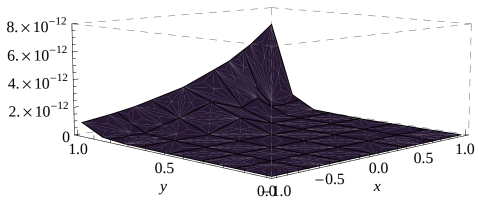

Fig. 4. Plot of D in Example 3.4.3.

Chapter 4

Fuzzy Partial Differential Equations under Generalized Parital Derivative

The derivative of fuzzy functions is defined with variety modes. For instance, Blasi differential, Fréchet differential (see [20]) and H-derivative [42]. The H-derivative is a popular and practical concept, and an its important application is exhibited in differential equations which in fuzzy setting are a natural way to model uncertainty systems [2, 3, 7, 24, 31, 53], also see chapter 3.

Under H-derivative, mainly existence and uniqueness theorems of the solution for a fuzzy differential equation are obtained (see e.g. [30, 32, 43, 46, 52]). This concept of derivative in fuzzy differential equations leads to solutions where have an increasing support. This shortcoming is solved by interpreting a fuzzy differential equation as a system of differential inclusions (see e.g. [21, 29]). The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy function. In another approach in [16, 17] instead of derivative concept, the extension principle is used to extend crisp differential equations to the fuzzy case.

For solving the above mentioned shortcomings, the strongly generalized differentiability concept has been introduced in [13] and studied in [12, 14, 11] by Bede and others. Indeed, the strongly generalized derivative is defined for a larger class of fuzzy functions than the Hukuhara derivative. In [12], Bede and Gal studied several characterizations under this interpretation for a fuzzy function from $\mathbb{R}$ into E, and as an application obtained the existence and uniqueness theorems of the solutions for a fuzzy differential equation.

4.1 Introduction

In this chapter, the strongly generalized differentiability concept Bede and Gal in [12] is studied and extended for fuzzy functions from $\mathbb{R}^n$ into E, and is discussed characterizations of generalized directional derivative. Also an application of generalized directional derivative are given to fuzzy partial differential equations. At last, using successive iterations of the Adomian decomposition method, existence theorems of solutions for a fuzzy partial differential equation involving generalized differentiability are obtained and several illustrative examples are given.

4.2 Generalized directional derivative

Note that Definition 1.4.5 of the directional derivative is very restrictive, therefore we introduce a more general definition for fuzzy functions in the direction $y \in \mathbb{R}^n$ at $x_0 \in M$. This restriction obtains from: let $\bar{C} \in E$ and $g : M(\subset \mathbb{R}^n) \to \mathbb{R}^{>0}$ be differentiable in the directional y at x_0. Define $F : M(\subset \mathbb{R}^n) \to E$ by $F(x) = \bar{C} \odot g(x)$, for all $x \in M$. At first, we assume $D_y g(x_0) > 0$, i.e. the usual directional derivative g in the direction y at x_0. Then $D_y g(x_0) = \lim_{h\to 0^+} \frac{g(x_0+hy)-g(x_0)}{h}$, it follows that for $h > 0$ sufficiently small we have $g(x_0+hy)-g(x_0) = w(x_0, y, h) > 0$. By multiplying $\bar{C} \in E$, it results $\bar{C} \odot g(x_0+hy) = \bar{C} \odot g(x_0) \oplus \bar{C} \odot w(x_0, y, h)$, i.e. there exists the H-difference $F(x_0+hy) -_H F(x_0)$. In the same way as above, by $D_y g(x_0) = \lim_{h\to 0^+} \frac{g(x_0)-g(x_0-hy)}{h}$, we also obtain that there exists the H-difference $F(x_0) -_H F(x_0 - hy)$. Therefore, this simple reasoning denotes that $D_y F(x_0) = \bar{C} \odot D_y g(x_0)$.

Now, if we assume $D_y g(x_0) < 0$, we easily see that we cannot use the above type of reasoning to prove that the H-differences $F(x_0+hy) -_H F(x_0)$, $F(x_0) -_H F(x_0-hy)$ and the directional derivative $D_y F(x_0)$ exist. Then via Definition 1.4.5 we can not say that exists $D_y F(x_0)$. To avoid this disadvantage, we introduce some generalized conceptions of differentiability in this section as follows.

Definition 4.2.1. Let $F : M(\subset \mathbb{R}^n) \to E$, $x_0 \in M$, $y \in \mathbb{R}^n$ and there exists $\delta > 0$. We say that F is strongly generalized directional differentiable in the direction y at x_0, if there exists an element $D_y F(x_0) \in E$ such that

(i) for all $h \in (0, \delta)$ sufficiently small, there exist $x_0 + hy, x_0 - hy \in M$, $F(x_0 +$

$hy) -_H F(x_0)$, $F(x_0) -_H F(x_0 - hy)$ and the limits (in the metric D)

$$\lim_{h\to 0^+} \frac{F(x_0 + hy) -_H F(x_0)}{h} = \lim_{h\to 0^+} \frac{F(x_0) -_H F(x_0 - hy)}{h} = D_y F(x_0),$$

or

(ii) for all $h \in (0, \delta)$ sufficiently small, there exist $x_0 + hy, x_0 - hy \in M$, $F(x_0) -_H F(x_0 + hy)$, $F(x_0 - hy) -_H F(x_0)$ and the limits

$$\lim_{h\to 0^+} \frac{F(x_0) -_H F(x_0 + hy)}{(-h)} = \lim_{h\to 0^+} \frac{F(x_0 - hy) -_H F(x_0)}{(-h)} = D_y F(x_0),$$

or

(iii) for all $h \in (0, \delta)$ sufficiently small, there exist $x_0 + hy, x_0 - hy \in M$, $F(x_0 + hy) -_H F(x_0)$, $F(x_0 - hy) -_H F(x_0)$ and the limits

$$\lim_{h\to 0^+} \frac{F(x_0 + hy) -_H F(x_0)}{h} = \lim_{h\to 0^+} \frac{F(x_0 - hy) -_H F(x_0)}{(-h)} = D_y F(x_0),$$

or

(iv) for all $h \in (0, \delta)$ sufficiently small, there exist $x_0 + hy, x_0 - hy \in M$, $F(x_0) -_H F(x_0 + hy)$, $F(x_0) -_H F(x_0 - hy)$ and the limits

$$\lim_{h\to 0^+} \frac{F(x_0) -_H F(x_0 + hy)}{(-h)} = \lim_{h\to 0^+} \frac{F(x_0) -_H F(x_0 - hy)}{h} = D_y F(x_0).$$

(h and $(-h)$ at denominators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$, respectively.)

Remark 4.2.1. The Definition 4.2.1 is not contradictory, i.e. if for F and x_0, at least two from the possibilities (i)-(iv) simultaneously hold, then we do not obtain a contradiction. For example, we assume the senses (ii) and (iv) simultaneously hold. Then by $F(x_0) = F(x_0+hy)\oplus u$, $F(x_0-hy) = F(x_0)\oplus v$, $F(x_0) = F(x_0-hy)\oplus w$, with $u, v, w \in E$, we have $F(x_0) = F(x_0) \oplus (v \oplus w)$, i.e. $v \oplus w = \tilde{0}$ where implies:(1) $v = w = \tilde{0}$, case when $D_yF(x_0) = \tilde{0}$ or (2) $v, w \in \mathbb{R}$ and $v = -w$, case when $D_yF(x_0) \in \mathbb{R}$. But in all these cases we can easily see where all the limits in the preceding definition are equal. The same as consequence obtains for any other combination from (i) to (iv).

Theorem 4.2.1. *Let $F : M(\subset \mathbb{R}^n) \to E$ be the strongly generalized directional differentiable in the direction $y \in \mathbb{R}^n$ on each point $x \in M$ in the sense of Definition 4.2.1(iii) or 4.2.1(iv). Then $D_yF(x) \in \mathbb{R}$ for all $x \in M$.*

Proof. Assume F is differentiable in the direction y at x according to Definition 4.2.1(iv). Then for $h > 0$ sufficiently small, there exist H-differences $F(x) -_H F(x+hy)$ and $F(x) -_H F(x - hy)$. Hence, we have

$$F(x) = F(x + hy) \oplus u(x, y, h), \tag{4.2.1}$$

and

$$F(x) = F(x - hy) \oplus v(x, y, h), \tag{4.2.2}$$

for $h > 0$ sufficiently small. If we replace in (4.2.2) $x = x + hy$, we obtain

$$F(x + hy) = F(x) \oplus v(x + hy, y, h), \tag{4.2.3}$$

for $h > 0$ sufficiently small. From (4.2.1) and (4.2.3) we have $u(x, y, h) \oplus v(x + hy, y, h) = \tilde{0}$. By Theorem 1.3.2(2) we obtain $u(x, y, h), v(x + hy, y, h) \in \mathbb{R}$ for $h > 0$ sufficiently small. Then it is easy to see that $D_y F(x) = \lim_{h \to 0^+} \frac{u(x,y,h)}{h} \in \mathbb{R}$. If F is differentiable in the direction y at x according to Definition 4.2.1(iii), the reasonings are similar. $\square$

Theorem 4.2.2. *Let $F : M(\subset \mathbb{R}^n) \to E$ be a fuzzy function, $x_0 \in M$, $y \in \mathbb{R}^n$ and there exists $\delta > 0$. Then*

(1) F is (i)-differentiable in the sense of Definition 4.2.1 in the direction y at x_0 iff there exists $\delta > 0$ such that $x_0 + hy, x_0 - hy \in M$ and H-differences $F(x_0 + hy) -_H F(x_0)$ and $F(x_0) -_H F(x_0 - hy)$ exist for any $h \in (0, \delta)$. And there exists $A[\gamma] \subset \mathbb{R}$ for any $\gamma \in [0, 1]$ such that in the metric D, the interval valued functions

$$G_+^{x_0,y}(h, \gamma) = [\frac{F(x_0 + hy) -_H F(x_0)}{h}][\gamma],$$

and

$$G_-^{x_0,y}(h, \gamma) = [\frac{F(x_0) -_H F(x_0 - hy)}{h}][\gamma],$$

uniformly converge to $A[\gamma]$ with respect to γ on $[0, 1]$ as $h \to 0^+$.

(2) F is (ii)-differentiable in the sense of Definition 4.2.1 in the direction y at x_0 iff there exists $\delta > 0$ such that $x_0 + hy, x_0 - hy \in M$ and H-differences $F(x_0 + hy) -_H F(x_0)$ and $F(x_0) -_H F(x_0 - hy)$ exist for any $h \in (0, \delta)$. And there exists $A[\gamma] \subset \mathbb{R}$ for any $\gamma \in [0, 1]$ such that in the metric D, the interval valued functions

$$G_+^{x_0,y}(h, \gamma) = [\frac{F(x_0) -_H F(x_0 + hy)}{(-h)}][\gamma],$$

and

$$G_-^{x_0,y}(h, \gamma) = [\frac{F(x_0 - hy) -_H F(x_0)}{(-h)}][\gamma],$$

uniformly converge to $A[\gamma]$ with respect to γ on $[0, 1]$ as $h \to 0^+$.

Proof. The case (1) has been proved in Theorem 3.1 in [48]. The result for case (2) is obtained as the case (1). □

Theorem 4.2.3. *If $g : M(\subset \mathbb{R}^n) \to \mathbb{R}$ is differentiable in the direction $y \in \mathbb{R}^n$ on M such that $D_y g(x)$ has at most a finite number of roots in M and $\bar{C} \in E$, then $F(x) = \bar{C} \odot g(x)$ is strongly generalized directional differentiable in the direction y on M and $D_y F(x) = \bar{C} \odot D_y g(x)$, $\forall x \in M$.*

Proof. For $x_0 \in M$ we have the following cases:

(i) $g(x_0) < 0$, $D_y g(x_0) > 0$; (ii) $g(x_0) < 0$, $D_y g(x_0) < 0$; (iii) $g(x_0) < 0$, $D_y g(x_0) = 0$; (iv) $g(x_0) > 0$, $D_y g(x_0) > 0$; (v) $g(x_0) > 0$, $D_y g(x_0) < 0$; (vi) $g(x_0) > 0$, $D_y g(x_0) = 0$;

(vii) $g(x_0) = 0$, $D_y g(x_0) > 0$; (viii) $g(x_0) = 0$, $D_y g(x_0) < 0$; (ix) $g(x_0) = 0$, $D_y g(x_0) = 0$;

Case (i): Let $D_y g(x_0) = \lim_{h\to 0^+} \frac{g(x_0)-g(x_0+hy)}{(-h)} = \lim_{h\to 0^+} \frac{g(x_0-hy)-g(x_0)}{(-h)}$. For $h > 0$ sufficiently small, $g(x_0+hy) < 0$, $g(x_0-hy) < 0$, $g(x_0)-g(x_0+hy) = \alpha_1(x_0, y, h) < 0$, $g(x_0-hy)-g(x_0) = \alpha_2(x_0, y, h) < 0$, i.e. $g(x_0) = g(x_0+hy)+\alpha_1(x_0, y, h)$, $g(x_0-hy) = g(x_0) + \alpha_2(x_0, y, h)$. With multiply $\bar{C} \in E$, we obtain that there exist $F(x_0) -_H F(x_0 + hy)$, $F(x_0 - hy) -_H F(x_0)$ and that $D_y F(x_0) = \lim_{h\to 0^+} \frac{F(x_0)-_H F(x_0+hy)}{(-h)} = \lim_{h\to 0^+} \frac{F(x_0-hy)-_H F(x_0)}{(-h)} = \bar{C} \odot D_y g(x_0)$, i.e. F is strongly generalized directional differentiable in the direction y at x_0 by Definition 4.2.1(ii).

Case (ii): Let $D_y g(x_0) = \lim_{h\to 0^+} \frac{g(x_0+hy)-g(x_0)}{h} = \lim_{h\to 0^+} \frac{g(x_0)-g(x_0-hy)}{h}$. For $h > 0$ sufficiently small, as above $g(x_0 + hy) < 0$, $g(x_0 - hy) < 0$, $g(x_0 + hy) - g(x_0) = \alpha_1(x_0, y, h) < 0$, $g(x_0) - g(x_0 - hy) = \alpha_2(x_0, y, h) < 0$. With multiply $\bar{C} \in E$, we obtain that there exist $F(x_0 + hy) -_H F(x_0)$, $F(x_0) -_H F(x_0 - hy)$ and that $D_y F(x_0) = \bar{C} \odot D_y g(x_0)$ according to Definition 4.2.1(i).

Case (iii): At first, assume x_0 is an extremum point. If x_0 is a maximum point then for $h > 0$ sufficiently small we have $g(x_0 + hy) \le g(x_0) < 0$, $g(x_0 - hy) \le g(x_0) < 0$, i.e. $g(x_0 + hy) - g(x_0) = \alpha_1(x_0, y, h) \le 0$, $g(x_0 - hy) - g(x_0) = \alpha_2(x_0, y, h) \le 0$. With multiply $\bar{C} \in E$, we easily obtain that F is differentiable in the direction y at x_0 according to Definition 4.2.1(iii), and that $D_y F(x_0) = \bar{C} \odot D_y g(x_0) = \bar{C} \odot \tilde{0} = \tilde{0}$.

If x_0 is a minimum point, then the reasonings are similar.

Now, if $D_y g(x_0) = 0$ but x_0 is not an extremum point, then there exists a set $\prod_{i=1}^{n}(x_0 - \delta, x_0 + \delta) \subset M$, such that $D_y g(x) \neq 0,\ \forall x \in \prod_{i=1}^{n}(x_0 - \delta, x_0 + \delta)\backslash\{x_0\}$ and it follows that $D_y g(x) > 0,\ \forall x \in \prod_{i=1}^{n}(x_0 - \delta, x_0 + \delta)\backslash\{x_0\}$ or $D_y g(x) < 0,\ \forall x \in \prod_{i=1}^{n}(x_0 - \delta, x_0 + \delta)\backslash\{x_0\}$.

If $D_y g(x) > 0,\ \forall x \in \prod_{i=1}^{n}(x_0 - \delta, x_0 + \delta)\backslash\{x_0\}$, it follows $g(x_0) - g(x_0 + hy) = \alpha_1(x_0, y, h) \leq 0$, $g(x_0 - hy) - g(x_0) = \alpha_2(x_0, y, h) \leq 0$ and F is differentiable in the direction y at x_0 according to Definition 4.2.1(ii).

If $D_y g(x) < 0,\ \forall x \in \prod_{i=1}^{n}(x_0 - \delta, x_0 + \delta)\backslash\{x_0\}$, it follows $g(x_0 + hy) - g(x_0) = \alpha_1(x_0, y, h) \leq 0$, $g(x_0) - g(x_0 - hy) = \alpha_2(x_0, y, h) \leq 0$ and F is differentiable in the direction y at x_0 according to Definition 4.2.1(i).

Proving of the cases (iv)-(viii) are like proving the above cases (i), (ii), (iii). The proof of the last case (ix) is like the case (iii), where finishes the proof of theorem. □

Theorem 4.2.4. *Let $F : M(\subset \mathbb{R}^n) \to E$ be a fuzzy function.*

(1) If F is (i)-differentiable in the sense of Definition 4.2.1, then it is continuous.

(2) If F is (ii)-differentiable in the sense of Definition 4.2.1, then it is continuous.

Proof. In the same way with extension of Theorem 5.4 in [30], we can prove the case (1).

To prove the case (2), let $x, x + hy \in M$ with $h > 0$. Then by properties metric D we have

$$D(F(x), F(x + hy)) = D(F(x) -_H F(x + hy), \tilde{0}) = hD(\tfrac{F(x) -_H F(x+hy)}{(-h)}, \tilde{0})$$

$$\leq hD(\tfrac{F(x) -_H F(x+hy)}{(-h)}, D_y F(x)) + hD(D_y F(x), \tilde{0}),$$

where h is sufficiently small such that H-difference $F(x) -_H F(x + hy)$ exists. Using (ii)-differentiability, the right-hand side goes to zero as $h \to 0^+$ and hence F is right continuous. The left continuous is proved similarly. □

Theorem 4.2.5. *Let $F : M(\subset \mathbb{R}^n) \to E$ be a fuzzy function and denotes $F(x)[\gamma] = [F^-(x, \gamma), F^+(x, \gamma)]$ for any $\gamma \in [0, 1]$.*

(1) If F is (i)-differentiable in the sense of Definition 4.2.1, then $F^-(x, \gamma)$ and $F^+(x, \gamma)$ are differentiable functions and $D_y F(x)[\gamma] = [D_y F^-(x, \gamma), D_y F^+(x, \gamma)]$.

(2) If F is (ii)-differentiable in the sense of Definition 4.2.1, then $F^-(x, \gamma)$ and $F^+(x, \gamma)$ are differentiable functions and $D_y F(x)[\gamma] = [D_y F^+(x, \gamma), D_y F^-(x, \gamma)]$.

Proof. In the same way with extension of Theorem 5.2 in [30], we can prove the case (1).

To prove the case (2), for $x, x + hy \in M$, $\gamma \in [0, 1]$ and $h > 0$ we have

$$(F(x) -_H F(x + hy))[\gamma] = [F^-(x, \gamma) - F^-(x + hy, \gamma), F^+(x, \gamma) - F^+(x + hy, \gamma)],$$

and with multiply by $\frac{1}{(-h)}$ we obtain

$$\frac{(F(x)-_H F(x+hy))[\gamma]}{(-h)} = \frac{1}{(-h)}[F^-(x,\gamma) - F^-(x+hy,\gamma), F^+(x,\gamma) - F^+(x+hy,\gamma)]$$

$$= [\frac{F^+(x,\gamma)-F^+(x+hy,\gamma)}{(-h)}, \frac{F^-(x,\gamma)-F^-(x+hy,\gamma)}{(-h)}].$$

analogously, we have

$$\frac{(F(x-hy) -_H F(x))[\gamma]}{(-h)} = [\frac{F^+(x-hy,\gamma) - F^+(x,\gamma)}{(-h)}, \frac{F^-(x-hy,\gamma) - F^-(x,\gamma)}{(-h)}].$$

passing to the limit gives

$$D_y F(x)[\gamma] = [D_y F^+(x,\gamma), D_y F^-(x,\gamma)],$$

and this finishes the proof of theorem. □

The concept in Definition 4.2.1 can be generalized by the following, which is the extension of Definition 11 in [12].

Definition 4.2.2. Let $F : M(\subset \mathbb{R}^n) \to E$ and $x_0 \in M$. The function F is weakly generalized directional differentiable in the direction y at x_0, if for any sequence $h_n \to 0^+$, there exists $n_0 \in \mathbb{N}$ such that $A_{n_0}^{(1)} \cup A_{n_0}^{(2)} \cup A_{n_0}^{(3)} \cup A_{n_0}^{(4)} = \{n \in \mathbb{N} : n \geq n_0\}$ when

$$A_{n_0}^{(1)} = \{n \geq n_0 : \exists B_n^{(1)} := F(x_0 + h_n y) -_H F(x_0)\},$$

$$A_{n_0}^{(2)} = \{n \geq n_0 : \exists B_n^{(2)} := F(x_0) -_H F(x_0 + h_n y)\},$$

$$A_{n_0}^{(3)} = \{n \geq n_0 : \exists B_n^{(3)} := F(x_0) -_H F(x_0 - h_n y)\},$$

$$A_{n_0}^{(4)} = \{n \geq n_0 : \exists B_n^{(4)} := F(x_0 - h_n y) -_H F(x_0)\}.$$

Moreover, there exists $D_yF(x_0) \in E$ such that if for some $j \in \{1, 2, 3, 4\}$ we have

$$\lim_{n\to+\infty,\ n\in A_{n_0}^{(j)}} D(\frac{B_n^{(j)}}{(-1)^{j+1}h_n}, D_yF(x_0)) = 0.$$

Remark 4.2.2. In the same way of Remark 12 in [12], we can extend this remark for Definition 4.2.2.

Theorem 4.2.6. *Let $\bar{C} \in E$ and $g : M(\subset \mathbb{R}^n) \to \mathbb{R}$. If g is differentiable in the direction y at x_0 (in usual sense), then the function $F : M(\subset \mathbb{R}^n) \to E$ defined by $F(x) = \bar{C} \odot g(x)$, is weakly generalized directional differentiable in the direction y at x_0 and we have $D_yF(x_0) = \bar{C} \odot D_yg(x_0)$.*

Proof. We can exhibit the same as reasoning by generalization of Theorem 13 in [12] and Definition 4.2.2. □

Denote $e_i = (a_1, a_2, \ldots, a_i, \ldots, a_n)$ with

$$a_j = \begin{cases} 1 & if\ \ j = i, \\ 0 & if\ \ j \neq i. \end{cases}$$

We consider Definition 4.2.1 and define strongly generalized partial derivative of fuzzy function $F : M(\subset \mathbb{R}^n) \to (E, D)$.

Definition 4.2.3. Let $F : M(\subset \mathbb{R}^n) \to (E, D)$ be a fuzzy function, $x \in M$. If F is the strongly generalized directional differentiable in the direction e_i at x, then we say that F is generalization of partially differentiable at x with respect to the i-th component, call $F_{x_i}(x)$ i.e. strongly generalized partial derivative of F at x with respect to the i-th component.

Finally, we define the strongly generalized partial derivative for fuzzy function F from $M(\subset \mathbb{R}^2)$ into E.

Definition 4.2.4. Let $F : M(\subset \mathbb{R}^2) \to E$. We say that F is the strongly generalized partial differentiable at (t_0, x_0) with respect to t, if there exists an element $F_t(t_0, x_0) \in E$ such that

(I) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F(t_0 + h, x_0) -_H F(t_0, x_0)$, $F(t_0, x_0) -_H F(t_0 - h, x_0)$ and the limits (in the metric D)

$$\lim_{h\to 0^+} \frac{F(t_0 + h, x_0) -_H F(t_0, x_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0, x_0) -_H F(t_0 - h, x_0)}{h} = F_t(t_0, x_0),$$

or

(II) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F(t_0, x_0) -_H F(t_0 + h, x_0)$, $F(t_0 - h, x_0) -_H F(t_0, x_0)$ and the limits

$$\lim_{h\to 0^+} \frac{F(t_0, x_0) -_H F(t_0 + h, x_0)}{(-h)} = \lim_{h\to 0^+} \frac{F(t_0 - h, x_0) -_H F(t_0, x_0)}{(-h)} = F_t(t_0, x_0),$$

or

(III) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F(t_0 + h, x_0) -_H F(t_0, x_0)$, $F(t_0 - h, x_0) -_H F(t_0, x_0)$ and the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h, x_0) -_H F(t_0, x_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0 - h, x_0) -_H F(t_0, x_0)}{(-h)} = F_t(t_0, x_0),$$

or

(IV) for all $h \in (0, \delta)$ sufficiently small, there exist $(t_0 + h, x_0), (t_0 - h, x_0) \in M$, $F(t_0, x_0) -_H F(t_0 + h, x_0)$, $F(t_0, x_0) -_H F(t_0 - h, x_0)$ and the limits

$$\lim_{h \to 0^+} \frac{F(t_0, x_0) -_H F(t_0 + h, x_0)}{(-h)} = \lim_{h \to 0^+} \frac{F(t_0, x_0) -_H F(t_0 - h, x_0)}{h} = F_t(t_0, x_0).$$

(h and $(-h)$ at denominators mean $\frac{1}{h}\odot$ and $-\frac{1}{h}\odot$, respectively.)

Corollary 4.2.7. *Let $g : M(\subset \mathbb{R}^2) \to \mathbb{R}$ and $\bar{C} \in E$. Define $F : M(\subset \mathbb{R}^2) \to E$ by $F(t, x) = \bar{C} \odot g(t, x)$, for all $(t, x) \in M$. If g is differentiable on M with respect to t and $g_t(t, x)$ is differentiable on $(t_0, x_0) \in M$ with respect to t, then $F_{tt}(t, x) = \bar{C} \odot g_{tt}(t, x)$.*

Proof. It is an immediately conclusion of Theorem 4.2.6. □

Remark 4.2.3. In general, if g is $(n-1)$ times differentiable with respect to t and $\frac{\partial^{n-1} g(t,x)}{\partial t^{n-1}}$ is differentiable at (t_0, x_0) with respect to t, then $\frac{\partial^n F(t,x)}{\partial t^n} = \bar{C} \odot \frac{\partial^n g(t,x)}{\partial t^n}$.

Remark 4.2.4. In the same way, we can apply Definition 4.2.4, Corollary 4.2.7 and Remark 4.2.3 with respect to x.

4.3 Application of fuzzy partial differential equations

In this section we discuss about existence and convergence of solution of a fuzzy partial differential equation by the successive iterations of Adomian decomposition method. In this section $\sum$ means the sum of fuzzy numbers for each (t, x) in domain.

Remark 4.3.1. Suppose $u(t, x)$ is continuous on $[a, b] \times [c, d]$, in here the Riemann integral $\int_a^b u(t, x)dt$ means that

$$\lim_{v(d_n)\to 0} D(\int_a^b u(t, x)dt, \sum_{k=0}^{n-1}(t_{k+1} - t_k) \odot u(\varepsilon_k, x)) = 0 \ \ for\, each\ x \in [c, d],$$

for all divisions of (a, b), $d_n : t_0 = a < t_1 < \cdots < t_k < t_{k+1} < \cdots < b = t_n$ and all $\varepsilon_k \in (t_k, t_{k+1}),\ k = 0, \ldots, n-1$, when $v(d_n)$ shows the norm of division d_n.

Consider the fuzzy partial differential equation

$$u_t(t, x) = \rho(t, x, L_x)u(t, x), \tag{4.3.1}$$

subject to

$$u(t_0, x) = f(x), \tag{4.3.2}$$

where $L_x = \frac{\partial}{\partial x}$ and $(t, x) \in M = [t_0, +\infty) \times \mathbb{R}$ with $t_0 \geq 0$. The operator $\rho(t, x, L_x)$ will be a polynomial, with continuous variable coefficient respect to t and x on M, in L_x where $L_x(L_x) = L_{xx}$ and denotes the partial derivative with respect to x. Also $f : \mathbb{R} \to E$ and $u : M \to E$ are continuous fuzzy functions where f is the strongly generalized differentiable in the sense Definition 1.4.4 and u is the strongly generalized partial differentiable in the sense Definition 4.2.4.

Theorem 4.3.1. *Let us suppose the following conditions hold:*

(a) $f : \mathbb{R} \to E$ *be a continuous and bounded function.*

(b) There exist $\gamma > 0$, $\beta > 1$ *and* $e^{-\beta t_0}\gamma \leq 1$ *such that*

$$D(\rho(t,x,L_x)u(t,x), \rho(t,x,L_x)v(t,x)) \leq \gamma e^{-\beta t} D(u,v), \tag{4.3.3}$$

and $\rho(t,x,L_x)u(t,x)$ *and* $\rho(t,x,L_x)v(t,x)$ *are continuous.*

Then the fuzzy partial differential equation (4.3.1) with the fuzzy initial condition (4.3.2) has two the type of solutions (one differentiable as in Definition 4.2.4(I) and the other differentiable as in Definition 4.2.4(II)) $u, u^* : M \to E$ *with respect to* t *and the successive iterations*

$$\begin{aligned} &\varphi_0(t,x) = f(x), \\ &\varphi_{n+1}(t,x) = f(x) \oplus \textstyle\sum_{i=1}^{n+1} \int_{t_0}^{t} \rho(s,x,L_x)u_{i-1}(s,x)ds, \quad (n \geq 0), \end{aligned} \tag{4.3.4}$$

and

$$\begin{aligned} &\varphi_0^*(t,x) = f(x), \\ &\varphi_{n+1}^*(t,x) = f(x) -_H (-1) \odot \textstyle\sum_{i=1}^{n+1} \int_{t_0}^{t} \rho(s,x,L_x)u_{i-1}^*(s,x)ds, \quad (n \geq 0), \end{aligned} \tag{4.3.5}$$

uniformly convergent to these two the type of solutions, respectively.

Proof. The case (I)-differentiable is obtained as the case (II)-differentiable and is omitted. To prove the case (II)-differentiable, by the invariance to translation of distance D and the hypotheses for uniform convergence of the sequence $\{\varphi_n^*(t,x)\}$ we have

$D(\varphi_{n+1}^*(t,x),\varphi_n^*(t,x)) = D(f(x) -_H \varphi_{n+1}^*(t,x), f(x) -_H \varphi_n^*(t,x))$

$= D((-1)\odot\sum_{i=1}^{n+1}\int_{t_0}^t \rho(s,x,L_x)u_{i-1}^*(s,x)ds,$

$(-1)\odot\sum_{i=1}^{n}\int_{t_0}^t \rho(s,x,L_x)u_{i-1}^*(s,x)ds)$

$= D((-1)\odot\sum_{i=1}^{n}\int_{t_0}^t \rho(s,x,L_x)u_{i-1}^*(s,x)ds\oplus$

$(-1)\odot\int_{t_0}^t \rho(s,x,L_x)u_n^*(s,x)ds, (-1)\odot\sum_{i=1}^{n}\int_{t_0}^t \rho(s,x,L_x)u_{i-1}^*(s,x)ds)$

$= D((-1)\odot\int_{t_0}^t \rho(s,x,L_x)u_n^*(s,x)ds,\tilde{0})$

$\leq \int_{t_0}^t D(\rho(s,x,L_x)u_n^*(s,x),\tilde{0})ds$

$\leq \gamma\sup_{(t,x)\in M} D(u_n^*(t,x),\tilde{0})\int_{t_0}^t e^{-\beta s}ds$

$\leq \frac{\gamma e^{-\beta t_0}}{\beta}\sup_{(t,x)\in M} D(u_n^*(t,x),\tilde{0}) \leq \frac{1}{\beta}\sup_{(t,x)\in M} D(u_n^*(t,x),\tilde{0}),$

in result, we have

$$D(\varphi_{n+1}^*(t,x),\varphi_n^*(t,x)) \leq \frac{1}{\beta}\sup_{(t,x)\in M} D(u_n^*(t,x),\tilde{0}). \tag{4.3.6}$$

On the other hand, from the ADM we can obtain for $n \geq 1$,

$D(u_n^*(t,x),\tilde{0}) = D((-1)\odot\int_{t_0}^t \varphi(s,x,L_x)u_{n-1}^*(s,x)ds,\tilde{0})$

$\leq \int_{t_0}^t D(\varphi(s,x,L_x)u_{n-1}^*(s,x),\tilde{0})ds \leq \frac{\gamma e^{-\beta t_0}}{\beta}\sup_{(t,x)\in M} D(u_{n-1}^*(t,x),\tilde{0})$

$\leq \frac{1}{\beta} \sup_{(t,x)\in M} D(u^*_{n-1}(t,x), \tilde{0})$

$\vdots$

$\leq \frac{1}{\beta^n} \sup_{(t,x)\in M} D(u^*_0(t,x), \tilde{0}) = \frac{1}{\beta^n} \sup_{x\in\mathbb{R}} D(f(x), \tilde{0}).$

Therefore we have

$$\sup_{(t,x)\in M} D(u^*_n(t,x), \tilde{0}) \leq \frac{Q}{\beta^n}, \tag{4.3.7}$$

where $Q = \sup_{x\in\mathbb{R}} D(f(x), \tilde{0})$. In result, from (4.3.6) and (4.3.7) we get

$$\sup_{(t,x)\in M} D(\varphi^*_{n+1}(t,x), \varphi^*_n(t,x)) \leq \frac{Q}{\beta^{n+1}},$$

which denotes because the series $\frac{Q}{\beta} \sum_{n=0}^{+\infty} \frac{1}{\beta^n}$ is convergent hence the series

$$\sum_{n=0}^{+\infty} D(\varphi^*_{n+1}(t,x), \varphi^*_n(t,x)),$$

(The two preceding summations are the usual summations.)

is uniformly convergent on M. If we show $u^*(t,x) = \lim_{n\to+\infty} \varphi^*_n(t,x)$, then $u^*(t,x)$ satisfies (4.3.1).

To prove the uniqueness of solution by $\varphi^*_n(t,x)$, assume $u^*(t,x)$ and $v^*(t,x)$ be two solutions of (4.3.1) on M, then

$$0 \leq D(u^*(t,x), v^*(t,x)) = D(u^*(t,x) \oplus \varphi^*_n(t,x), v^*(t,x) \oplus \varphi^*_n(t,x))$$

$$\leq D(u^*(t,x), \varphi^*_n(t,x)) + D(v^*(t,x), \varphi^*_n(t,x)),$$

and because $\varphi^*_n(t,x)$ is convergent to solution of (4.3.1) thus

$$D(u^*(t,x), \varphi^*_n(t,x)) \to 0,$$

$$D(v^*(t,x),\varphi_n^*(t,x)) \to 0,$$

when $n \to +\infty$, then $D(u^*(t,x), v^*(t,x)) = 0$ i.e. $u^*(t,x) = v^*(t,x)$.

Let $t_0 \le t < t+h < +\infty$, we observe that

$$\varphi_{n+1}^*(t,x) -_H \varphi_{n+1}^*(t+h,x) = (-1) \odot \sum_{i=1}^{n+1} \int_t^{t+h} \rho(s,x,L_x) u_{i-1}^*(s,x)ds. \qquad (4.3.8)$$

Indeed, we have by direct computation

$$\varphi_{n+1}^*(t+h,x) \oplus (-1) \odot \sum_{i=1}^{n+1} \int_t^{t+h} \rho(s,x,L_x) u_{i-1}^*(s,x)ds$$

$$= f(x) -_H (-1) \odot \sum_{i=1}^{n+1} \int_{t_0}^{t+h} \rho(s,x,L_x) u_{i-1}^*(s,x)ds \oplus$$

$$(-1) \odot \sum_{i=1}^{n+1} \int_t^{t+h} \rho(s,x,L_x) u_{i-1}^*(s,x)ds$$

$$= f(x) -_H (-1) \odot \sum_{i=1}^{n+1} \int_{t_0}^{t+h} \rho(s,x,L_x) u_{i-1}^*(s,x)ds \oplus$$

$$(-1) \odot \sum_{i=1}^{n+1} \int_{t_0}^{t+h} \rho(s,x,L_x) u_{i-1}^*(s,x)ds$$

$$-_H (-1) \odot \sum_{i=1}^{n+1} \int_{t_0}^{t} \rho(s,x,L_x) u_{i-1}^*(s,x)ds$$

$$= \varphi_{n+1}^*(t,x).$$

With multiply $\frac{1}{(-h)}$ and passing to limit with $h \to 0^+$ we have by Definition 4.2.4,

$$\lim_{h\to 0^+} \frac{\varphi_{n+1}^*(t,x) -_H \varphi_{n+1}^*(t+h,x)}{(-h)} = \lim_{h\to 0^+} \frac{1}{h} \odot \sum_{i=1}^{n+1} \int_t^{t+h} \rho(s,x,L_x) u_{i-1}^*(s,x)ds$$

$$= \lim_{h\to 0^+} \frac{1}{h} \odot \sum_{i=0}^{n} \int_t^{t+h} \rho(s,x,L_x) u_i^*(s,x)ds.$$

By $\varphi_n^*(t,x) = \sum_{i=0}^n u_i^*(t,x)$ we observe that

$$D(\tfrac{1}{h} \odot \sum_{i=0}^{n} \int_t^{t+h} \rho(s,x,L_x) u_i^*(s,x)ds, \rho(t,x,L_x)\varphi_n^*(t,x))$$

$= D(\frac{1}{h} \odot \int_t^{t+h} \rho(s,x,L_x)\varphi_n^*(s,x)ds, \rho(t,x,L_x)\varphi_n^*(t,x))$

$= D(\frac{1}{h} \odot \int_t^{t+h} \rho(s,x,L_x)\varphi_n^*(s,x)ds, \frac{1}{h} \odot \int_t^{t+h} \rho(t,x,L_x)\varphi_n^*(t,x)ds)$

$\leq \frac{1}{h} \odot \int_t^{t+h} D(\rho(s,x,L_x)\varphi_n^*(s,x), \rho(t,x,L_x)\varphi_n^*(t,x))ds$

$\leq \sup_{|s-t|\leq h} D(\rho(s,x,L_x)\varphi_n^*(s,x), \rho(t,x,L_x)\varphi_n^*(t,x)),$

and thus for $h \to 0^+$ the last term$\searrow 0^+$ where means that

$$\lim_{h\to 0^+} \frac{\varphi_{n+1}^*(t,x) -_H \varphi_{n+1}^*(t+h,x)}{(-h)} = \rho(t,x,L_x)\varphi_n^*(t,x).$$

Analogous (4.3.8) we can obtain

$$\varphi_{n+1}^*(t-h,x) -_H \varphi_{n+1}^*(t,x) = (-1) \odot \sum_{i=1}^{n+1} \int_{t-h}^{t} \rho(s,x,L_x)u_{i-1}^*(s,x)ds,$$

where by similar reasonings leads to

$$\lim_{h\to 0^+} \frac{\varphi_{n+1}^*(t-h,x) -_H \varphi_{n+1}^*(t,x)}{(-h)} = \rho(t,x,L_x)\varphi_n^*(t,x).$$

Finally, it follows that $\varphi_{n+1}^*(t,x)$ is (II)-differentiable with respect to t and

$$(\varphi_{n+1}^*(t,x))_t = \rho(t,x,L_x)\varphi_n^*(t,x),\ \forall (t,x) \in M,$$

where this finishes the proof of theorem. □

Lemma 4.3.2. *If the conditions of Theorem 4.3.1 hold and*

$u_0^*(t,x) = f(x),$

$u_n^*(t,x) = (-1) \odot \int_{t_0}^{t} \rho(s,x,L_x)u_{n-1}^*(s,x)ds,\ (n \geq 1),$

then

(a) $u_n^*(t,x)$ *is bounded on* M*,*

(b) $u_n^*(t,x)$ *is continuous on* M*.*

Proof. (a) By the hypothesis, $u_0^*(t,x) = f(x)$ is bounded. Assume $u_{n-1}^*(t,x)$ is bounded. By Theorem 4.3.1(b) we observe that

$$D(u_n^*(t,x),\tilde{0}) = D((-1)\odot \int_{t_0}^{t} \rho(s,x,L_x)u_{n-1}^*(s,x)ds,\tilde{0})$$

$$\leq \int_{t_0}^{t} D(\rho(s,x,L_x)u_{n-1}^*(s,x),\tilde{0})ds$$

$$\leq \tfrac{1}{\beta}\sup_{(t,x)\in M} D(u_{n-1}^*(t,x),\tilde{0}),$$

and by induction $u_n^*(t,x)$ is bounded on M.

(b) Suppose $t_0 < t \leq \hat{t} < +\infty$ and $-\infty < x \leq \hat{x} < +\infty$, we have

$$D(u_n^*(t,x),u_n^*(\hat{t},\hat{x}))$$

$$= D((-1)\odot \int_{t_0}^{t} \rho(s,x,L_x)u_{n-1}^*(s,x)ds, (-1)\odot \int_{t_0}^{\hat{t}} \rho(s,\hat{x},L_{\hat{x}})u_{n-1}^*(s,\hat{x})ds)$$

$$= D(\int_{t_0}^{t} \rho(s,x,L_x)u_{n-1}^*(s,x)ds, \int_{t_0}^{t} \rho(s,\hat{x},L_{\hat{x}})u_{n-1}^*(s,\hat{x})ds$$

$$\oplus \int_{t}^{\hat{t}} \rho(s,\hat{x},L_{\hat{x}})u_{n-1}^*(s,\hat{x})ds)$$

$$\leq D(\int_{t_0}^{t} \rho(s,x,L_x)u_{n-1}^*(s,x)ds, \int_{t_0}^{t} \rho(s,\hat{x},L_{\hat{x}})u_{n-1}^*(s,\hat{x})ds)$$

$$+D(\int_{t}^{\hat{t}} \rho(s,\hat{x},L_{\hat{x}})u_{n-1}^*(s,\hat{x})ds,\tilde{0})$$

$$\leq \int_{t_0}^{t} D(\rho(s,x,L_x)u_{n-1}^*(s,x), \rho(s,\hat{x},L_{\hat{x}})u_{n-1}^*(s,\hat{x}))ds$$

$$+ \int_{t}^{\hat{t}} D(\rho(s,\hat{x},L_{\hat{x}})u_{n-1}^*(s,\hat{x}),\tilde{0})ds$$

$$\leq (t-t_0)\sup_{x,\hat{x}\in\mathbb{R},t\in[t_0,+\infty)} D(\rho(t,x,L_x)u^*_{n-1}(t,x),\rho(t,\hat{x},L_{\hat{x}})u^*_{n-1}(t,\hat{x}))$$

$$+\gamma\sup_{\hat{x}\in\mathbb{R},t\in[t_0,+\infty)} D(u^*_{n-1}(t,\hat{x}),\tilde{0})\int_t^{\hat{t}} e^{-\beta s}ds,$$

consequently, we obtain

$$D(u^*_n(t,x),u^*(\hat{t},\hat{x}))\to 0 \quad as \quad (t,x)\to(\hat{t},\hat{x}),$$

i.e. $u^*_n(t,x)$ is continuous on M. □

Lemma 4.3.3. *If the conditions of Theorem 4.3.1 hold and*

$$u_0(t,x)=f(x),$$

$$u_n(t,x)=\int_{t_0}^t \rho(s,x,L_x)u_{n-1}(s,x)ds,\ (n\geq 1),$$

then

(a) $u_n(t,x)$ *is bounded on* M*,*

(b) $u_n(t,x)$ *is continuous on* M*.*

Proof. It is an immediately consequence of Lemma 4.3.2. □

Theorem 4.3.4. *If the conditions of Theorem 4.3.1 hold, then* $\varphi^*_{n+1}(t,x)$ *and* $\varphi_{n+1}(t,x)$ *are bounded and continuous on* M*.*

Proof. It is an immediately consequence of Lemma 4.3.2 and Lemma 4.3.3, respectively. □

Theorem 4.3.5. *If the conditions of Theorem 4.3.1 hold, then* $u(t,x), u^*(t,x) \in E$ *for each* $(t,x) \in M$.

Proof. In the same way Theorem 3.4.4, we can prove $u(t,x), u^*(t,x) \in E$ for each $(t,x) \in M$. □

Theorem 4.3.6. *If the conditions of Theorem 4.3.1 hold, then the largest interval of existence of any fuzzy solution* $u(t,x)$ *of (4.3.1) is* M *and the limit*

$$\lim_{x\to\pm\infty} \lim_{t\to+\infty} u(t,x) = \xi \in E,$$

exists. In the same way, we can also present for $u^*(t,x)$.

Proof. By Theorem 4.3.4 $u(t,x)$ is continuous and bounded on M, then the limits

$$[\xi^-(\gamma), \xi^+(\gamma)] = [\lim_{x\to\pm\infty} \lim_{t\to+\infty} u^-(t,x,\gamma), \lim_{x\to\pm\infty} \lim_{t\to+\infty} u^+(t,x,\gamma)],$$

exist. On the other hand by Theorem 4.3.5, we can obtain that the intervals

$$[\xi^-(\gamma), \xi^+(\gamma)],\ 0 < \gamma \leq 1,$$

define a fuzzy number $\xi \in E$. $u^*(t,x)$ can similarly be proved. □

Theorem 4.3.7. *Assume the following conditions hold:*

(a) $f : \mathbb{R} \to E$ *be a continuous and bounded function.*

(b) There exist $\gamma > 0$, $0 < \beta \leq 1$ *and* $e^{-\beta t_0}\gamma \leq \frac{\beta^2}{2}$ *such that*

$$D(\rho(t, x, L_x)u(t, x), \rho(t, x, L_x)v(t, x)) \leq \gamma e^{-\beta t} D(u, v), \tag{4.3.9}$$

and $\rho(t, x, L_x)u(t, x)$ *and* $\rho(t, x, L_x)v(t, x)$ *are continuous.*

Then the fuzzy partial differential equation (4.3.1) with the fuzzy initial condition (4.3.2) has two the type of solutions (one differentiable as in Definition 4.2.4(I) and the other differentiable as in Definition 4.2.4(II)) $u, u^* : M \to E$ *with respect to* t *and the successive iterations (4.3.4) and (4.3.5) uniformly convergent to these two the type of solutions, respectively.*

Proof. Proving of theorem is similar to proving of Theorem 4.3.1. □

Remark 4.3.2. If we consider the conditions of Theorem 4.3.7 instead of Theorem 4.3.1, we obtain that the results of Lemma 4.3.2, Lemma 4.3.3, Theorem 4.3.4, Theorem 4.3.5 and Theorem 4.3.6 are hold.

In this part we denote application by two examples of fuzzy patrial differential equations using fuzzy input data (fuzzy coefficients) of fuzzy numbers with real-valued functions.

Example 4.3.1. *Let us consider the fuzzy partial differential equation*

$$u_t(t, x) = (-1) \odot u(t, x),$$

$$u(t_0, x) = f(x), \;\; x \in \mathbb{R}, \, t \geq t_0, \, t_0 \geq 0,$$

where $u : [t_0, +\infty) \times \mathbb{R} \to E$*, the* $f : \mathbb{R} \to E$ *is the strongly generalized differentiable (see [12]) and bounded.*

If we denote $u(t, x) = e^{-(t-t_0)} \odot f(x)$, the u is the strongly generalized partial differentiable with respect to all t, $u_t(t, x) = (-1)e^{-(t-t_0)} \odot f(x)$ (see Theorem 4.2.3) and $u(t, x)$ satisfies the above fuzzy partial differential equation.

On the other hand, because $e^{-(t-t_0)} > 0$ and $(e^{-(t-t_0)})' < 0$, $\forall t \in (t_0, +\infty)$, we cannot say nothing about the existence of $u_t(t, x)$ in the H-differentiability sense, since the H-difference $u(t + h, x) -_H u(t, x)$ does not exist, as $u(t, x) \in E \backslash \mathbb{R}$.

This denotes the advantage of the strongly generalized partial differentiability with respect to the usual differentiability.

Example 4.3.2. *Let us consider the fuzzy partial differential equation*

$$u_t(t, x) = \alpha \odot u_x(t, x),$$

$$u(t_0, x) = \bar{C} \odot f(x), \;\; x \in \mathbb{R}, \, t \geq t_0, \, t_0 \geq 0,$$

where $\bar{C} \in E$*,* $\alpha \in (0, +\infty)$*, the* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and bounded on* $\mathbb{R}$*.*

We have

$$u(t, x) = \bar{C} \odot f(x + \alpha(t - t_0)),$$

by Theorem 4.2.6 we easily obtain that $u(t, x)$ satisfies the fuzzy partial differential equation, for all $x \in \mathbb{R}$, $t \geq t_0$.

Conclusion and Further works

In this work, we introduced three point of views for solving fuzzy partial differential equations under Buckley-Feuring method, Hukuhara derivative and generalized directional derivative.

It seems the third point of view is better than other two point of views, because its solutions may have a decreasing length of their support, which is an important property in order to reflect the rich behaviour of solutions of crisp partial differential equations. We can have in this case an asymptotic behaviour of the solutions similar to the classical case or even richer!

In the third point of view, fuzzy partial differential equation may have several solutions. The advantage of the existence of these solutions is that we can choose the solution that reflects better behaviour of the modelled real-world system.

For further research we suggest the study of other fuzzy partial differential equations using the strongly generalized differentiability concept.

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