SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY

A NEW APPROACH FOR RANKING OF FUZZY NUMBERS BY LEVELS

Supervisor

Prof. Saeid Abbasbandy

Dr. Hamid Reza Maleki

Advisers

Prof. Esmail Babolian

By

Tayebeh Hajjari

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY(Ph.D.)
IN APPLIED MATHEMATICS (NUMERICAL ANALYSIS)
APRIL 2009

SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY
DEPARTMENT OF MATHEMATICS

The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**A New Approach For Ranking Of Fuzzy Numbers By Levels**" by **Tayebeh Hajjari** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

Dated: April 2009

Research Supervisor: Prof. Saeid Abbasbandy
Dr. Hamid Reza Maleki

External Examiner: Dr. M.A. Fariborzi Araghi
Dr. A.R Vahidi

Examing Committee: Prof. Gh.R Jahanshahloo

Prof. F. Hosseinzadeh

SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY [1]

Date: **April 2009**

Author: **Tayebeh Hajjari**

Title: **A New Approach For Ranking Of Fuzzy Numbers By Levels**

Department: **Mathematics**

Degree: **Ph.D.** Convocation: ***** Year: **2009**

Signature of Author

[1]Poonak, Tehran

To My Dears:

My Honorable Parents,

My Husband

and

My Son "Behrad"

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Acknowledgments

Many people have made this thesis possible, none more than my supervisor Prof. Saeid Abbasbandy has. He managed to provide me guidance in the right direction at the right time. I am deeply grateful for his patience in reading draft manuscripts and for all invaluable helpful suggestions about how to improve the quality of the final work. I will always remain indebted to him for his support and guidance throughout the course of my PhD and hope to continue to benefit from his experience in the future. I am also grateful to Dr. Hamid-Reza Maleki (Shiraz University of Technology) for providing his comments and recommendations as the co-supervisor of the thesis. I would like to express my cordial acknowledgement to Prof. Esmail Babolian the advisor of my thesis for providing his invaluable comments and recommendations.

I am also thankful to Prof. Gh.R Jahanshahloo, Prof. F. Hosseinzadeh, Dr. A. Tehranian, Dr. T. Allahviranloo, Dr. M. Rostemi, Dr. M.A. Fariborzi Araghi and Dr. A.R. Vahidi. Many thanks also go to Firuz Kuh branch of Islamic Azad University for scholarship, which made research like this possible.

Now, I wish to thank my husband and my son "Behrad" for their patience during the period of my long studies particularly, my PhD research. I especially want to thank my dear parents-it is your love and support that has made me the person I am today! To them, this thesis is dedicated.

Science and Research Branch, Islamic Azad University, Tehran
Tayebeh Hajjari
March 2009

Abstract

Ranking fuzzy of numbers plays a very important role in linguistic decision making and some other fuzzy application systems. So far, several strategies have been proposed for ranking of fuzzy numbers. Each of these techniques has been shown to produce non-intuitive results in certain cases. Defuzzification, approximation and fuzzy distance are the subjects, which are related to ranking fuzzy numbers.

In this thesis, it will be introduced two new approaches (defuzzification) for ranking of trapezoidal fuzzy numbers. The first one is an improvement on based-distance method and the second one is based on the left and the right spreads at some α-levels of trapezoidal fuzzy numbers, which is easy to handle and has a natural interpretation. Furthermore, some properties of these methods will be investigated.

Meanwhile, we suggest a weighted trapezoidal approximation of an arbitrary fuzzy number, which preserves its core. We then discuss properties of the approximation strategy including translation invariance, scale invariance identity and continuity. The advantage is that the presented method is simpler than other methods computationally.

Fuzzy numbers and more generally linguistic values are approximate assessments, it is reasonable to define a fuzzy distance between fuzzy objects. To achieve this aim, in this study, the researchers describe the concepts of positive fuzzy number, negative fuzzy number, fuzzy zero and fuzzy absolute of a fuzzy number. Then these concepts are utilized to produce a fuzzy distance between two fuzzy numbers. The fuzzy distance is presented as a trapezoidal fuzzy number. In addition, it is shown that the fuzzy distance is a fuzzy metric. The metric properties are also studied.

The presented method has two advantages in comparing with existing methods. The first advantage is that the presented method is simpler than other methods computationally. The second advantage is that the ranking order of authors' approach is more consistent with our intuitions than other methods. The aforementioned fuzzy distance has all of the properties of normal metrics.

Originality

The following chapters are proposed in this work:

Chapter 3: In this chapter, we have introduced two new approaches for ranking fuzzy numbers. The first one is an improvement on Cheng's distance method. Meanwhile, the improved method overcome the shortcoming in Cheng's distance method, Chu and Tsao's formulae and new revised method by Wang and Lee. Then we pointed out the shortcoming of "Distance minimization" and in order to solve the problem we have presented the second approaches, which is a simple ranking method for trapezoidal fuzzy numbers. The proposed method can effectively rank various fuzzy numbers and their images.

Chapter 4: In this chapter, we used ordinary distance between two fuzzy numbers to investigate a trapezoidal approximation of an arbitrary fuzzy number. The proposed operator, called the weighted trapezoidal approximation operator-preserving core. Since the aforementioned operator is easy to implement, computational inexpensive and convenience interpretation, it is a satisfactory approximation operator.

Chapter 5: In this chapter, we have extended the absolute value of a real number to fuzzy absolute value. Then, we have used them to introduce a fuzzy distance between two fuzzy numbers as a trapezoidal fuzzy number. The mentioned fuzzy distance has all of the properties of normal metrics.

Articles

1. A new approach for ranking of trapezoidal fuzzy numbers, *Comput. Math. Appl.* 57 (2009) 413-419.

2. Fuzzy Euclidean Distance for Fuzzy Data, *Proceedings of the* 8^{th} *Iranian Conference on Fuzzy Systems*, Tehran, Iran (2008).

3. Comparison of Fuzzy Numbers by Modified Centroid Point Method, *The Third International Conference in Mathematical Sciences*, Al Ain, Dubai (2008) 1139-1145.

4. Ranking Fuzzy Numbers by Sign Length, *Proceedings of the* 7^{th} *Iranian Conference on Fuzzy Systems*, Meshed, Iran (2007) 297-301.

Chapter 1

Introduction to Fuzzy Mathematics

1.1 Introduction

Most of our traditional tools for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. By crisp we mean dichotomous, that is, yes-or-no type rather than more-or-less type. In conventional dual logic, for instance, a statement can be true or false and nothing in between. In set theory, an element can either belong to a set or not; and in optimization, a solution is either feasible or not. Precision assumes that the parameters of a model represent exactly either our perception of the phenomenon modeled or the features of the real system that has been modeled. Generally, precision also implies that the model is unequivocal, that is, that it contains no ambiguities.

Certainty eventually indicates that we assume the structures and parameters of the model to be definitely known, and that there are no doubts about their values or their occurrence. If the model under consideration is a formal model [86], that is, if it does not pretend to model reality adequately, then the model assumptions are in a sense arbitrary, that is the model builder can freely decide which model characteristics he chooses. If, however, the model or theory asserts factuality that is, if conclusions drawn from these models have bearing on reality and are supposed to model reality adequately, then the modeling language has to be suited to model the characteristics of the situation under study appropriately.

The usefulness of the mathematical language for modeling purposes is undisputed. For factual models or modeling languages, two major complications arise:

1. Real situations are very often not crisp and deterministic, and they cannot be described precisely.

2. The complete description of a real system often would require far more detailed data than a human being could ever recognize simultaneously, process, and understand.

Fuzzy sets were introduced in 1965 by Lotfi Zadeh with a view to reconcile mathematical modeling and human knowledge in the engineering science. Since then, a considerable body of literature has blossomed around the concept of fuzzy sets in an incredibly wide range of areas, from mathematics and logics to traditional and advanced engineering methodologies (from civil engineering to computational intelligence). Applications are found in many contexts, from medicine to finance, from human factors to consumer products, from vehicle control to computational linguistics; and so on. Fuzzy logic is now currently used in the industrial practice of advanced information technology. L. Zadeh referred to the second point when he wrote, "As the complexity of a system increases, our ability to make precise and yet significant (or relevance) become almost mutually exclusive characteristics". Let us consider characteristic features of real-world systems again: Real situations are very often uncertain or vague in a number of ways. Due to lack of information, the future state of a system might not be known completely.

Sugeno [65] has used the word fuzziness in a radically different context. Fuzziness can be found in many areas of daily life, such as in engineering [12], medicine [67], meteorology [15] and others. It is particularly frequent, however, in all areas of decision-making, reasoning, learning, and so on. Some reasons for this fuzziness have already been mentioned. Others are that most of our daily communication uses "natural languages", and a good part of our thinking is done in it. In these natural languages, the meaning of words are very often vague.

Fuzzy set theory is composed of an organized body of mathematical tools particularly well studied for handling incomplete information, the unsharpness of classes of objects or situations, or the gradualness of preference profiles, in a flexible way. It offers a unifying framework for modeling various types of information ranging from precise numerical, interval-valued data, to symbolic and linguistic knowledge, with a stress on semantic rather than syntax (some misunderstanding with logicians).

The present chapter is meant to provide to the historical emergence of fuzzy sets and the main components of fuzzy set theory, as it stands now. Most basic concepts and formal notations are briefly introduced.

1.2 The Historical Emergence of Fuzzy Sets

About a hundred years ago, the American philosopher Charles Peirce was one of the first scholars in the modern age to point out, and to regret, that "Logicians have too much neglected the study of vagueness, not suspecting the important part it plays in mathematical thought [34]". Bertrand Russel (1923) also expressed this point of view some time later. Discussion on the links between logic and vagueness are not usual in the philosophical literature in the first half of the century (Copilowish, 1939; Hempel, 1939). Even Wittgenstein (1953) pointed out that concepts in natural language do not possess a clear collection of properties defining them, but have extendable boundaries, and that there are central and less central members in a category. In spite of considerable interest for multiple-valued logics raised in the 1930s by Jan Luckasiewicz (1910a, b; 1920, 1930) and his school who developed logics with intermediary truth value(s) it was the American philosopher.

Max Black (1937) who first proposed so-called "consistency profiles" (the ancestors of fuzzy member ship functions) in order to "characterize vague symbols". H. Weyl

(1940), who explicitly replaces it by a continuous characteristic function, has first considered the generalization of the traditional characteristic function. Kaplan and Scott (1951) further proposed the same kind of generalization in 1951. They suggested Caculi for generalized characteristic functions of vague predicates, and the basic fuzzy set connectives already appeared in this thesis. Karl Menger (1951a), who, in 1951, was first to use the term "ensemble flou" (the French counterpart set of "fuzzy set") in the title of a paper of his. The notation of a fuzzy set stems from the observation made by Zadeh (1965a) that "more often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership". This observation emphasizes the gap existing between mental representation of reality and usual mathematical representations thereof, which are based on binary logic, precise numbers, differential equations and the like.

Classes of objects referred to in Zadeh's citation exist only through such mental representations through natural language term such as high temperature, young man, big size, etc., and with nouns such as bird, chair, etc. Classical logic is too rigid to account for such categories where that membership is a gradual notation rather than all-or-nothing matter. The power of expressivity of real numbers is far beyond the limited level of precision found in mental representations. The latter are meaningful summaries of perceptive phenomena that account for complexity of the world. Analytical representations of physical phenomena can be faithful as models of reality, but are sometimes difficult to understand because they do not explain much by themselves, and may remain opaque to the non-specialist. Mental representations make more sense but are pervaded with vagueness, which encompasses at the same time the lack of specificity of linguistic term, and the lack of well-defined boundaries of the class of objects they refer to.

In the literature of fuzzy sets, the word fuzzy often stands for the word vague. Some comments on the links between vagueness and fuzziness are useful. In common use, there is a property of objects called "fuzziness"; from the Oxford English Dictionary we read that "fuzzy" means either not firm or sound in substance, or fringed into loose fibers. Fuzzy means also covered by fuzz, i.e., with loose volatile matter. "Something is fuzzy". For example, "a bear is fuzzy". It may sound strange to say, "bald is fuzzy", or that "young is fuzzy". Words (adjectives in this case) bald and young are vague (but not fuzzy in the material sense) because their meanings are not fixed by sharp boundaries. Similarly, objects are not vague. Here however, the word "fuzzy" is applied to words, especially predicates, and is supposed to refer to the gradual nature of some of these words, which causes them to appear as vague. However, the term "vagueness" designates a much larger kind of ill definition for words (including ambiguity), generally.

The specificity of fuzzy sets is to capture the idea of partial membership. The characteristic function of a fuzzy set, often called membership function, is a function whose range is an ordered membership set containing more that two (often a continuum of) values (typically, the unit interval). Therefore, a fuzzy set is often understood as a function. This has been a source of criticism from mathematicians (Arbib, 1977) as functions are already well known, and a theory of functions already exists. However, the novelty of fuzzy set theory, as first proposed by Zadeh, is to treat functions as if they were subsets of their domains, since such functions are used to represent gradual categories. It means that the classical set-theoretic notations like intersection, union, complement, inclusion, etc. are extended to combine functions ranging on an ordered membership set. In elementary fuzzy set theory, the set-union of functions is performed by taking their point wise maximum, their intersection by their point wise minimum, their complementation by means of an order reversing automorphism of the membership scale, and set-inclusion by the point wise inequality between functions. Mathematicians had not envisaged this point of view earlier, if we except some pioneers, mainly logician.

Fuzzy set theory is indeed closely connected to many-value logics that appeared in the thirties, if degrees of membership are understood as degrees of truth, intersection as conjunction, union as disjunction, complementation as negation and set-inclusion as implication. This chapter is meant to account for history of how the notation of fuzzy sets could become known, and it presents a catalogue of basic notations, which are presented in details in the other chapters of this thesis. For more details we refer the reader to [20, 51, 87].

1.3 Fuzzy Sets-Basic definitions

A *classical* (crisp) set is normally defined as a collection of elements or objects $x \in X$ that can be finite, countable, or overcountable. Each single element can either belong to or not belong to a set A, $A \subseteq X$. In the former case, the statement "x belong to A" is true, whereas in the latter case this statement is false. Such a classical set can be described in different ways: one can either enumerate (list) the elements that belong to the set analytically, for instance, by stating conditions for membership; or define the elements by using the characteristic function, in which 1 indicates membership function and 0 non-membership function. For a fuzzy set, the characteristic function allows various degrees of membership for the elements of a given set.

Definition 1.3.1. (Membership function) If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \tilde{A}(x))|x \in X\}, \tag{1.3.1}$$

$\tilde{A}(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\tilde{A}$ that maps X to the membership space M

(when M contains only the two points 0 and 1, $\tilde{A}$ is non-fuzzy and $\tilde{A}(x)$ is identical to the characteristic function of a non-fuzzy set). The range of the membership function is a subset of the non-negative real numbers whose supremum is finite. Elements with a zero degree of membership are normally not listed. It should be noted membership functions enable the notation of a class to be extended to categories that have no clear-cut boundaries, as often encountered in linguistic information.

A more convenient notation was proposed by Zadeh [83]. When X is a finite set $\{x_1, ..., x_n\}$ a fuzzy set on X is expressed as

$$\tilde{A} = \frac{\tilde{A}(x_1)}{x_1} + \cdots + \frac{\tilde{A}(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\tilde{A}(x_i)}{x_i}. \tag{1.3.2}$$

When X is not finite, we write

$$\tilde{A} = \int_x A(x).$$

1.3.1 Characteristic of fuzzy sets

Fuzzy sets are characterized in more detail by referring to the concept of support, core, normality, convexity, etc. Let $f(X)$ denote the set of all possible fuzzy sets defined on the universe of discourse X. For a fuzzy set $\tilde{A} \in F(X)$ we can give the following definitions.

Definition 1.3.2. (Support) The support of a fuzzy set $\tilde{A}$ is the ordinary subset of X:

$$\text{supp } \tilde{A} = \{x \in X, \ \tilde{A}(x) > 0\}. \tag{1.3.3}$$

Definition 1.3.3. (Core) The core of a fuzzy set $\tilde{A}$ is the set of all points with the membership degree one in $\tilde{A}$:

$$core\tilde{A} = \{x \in X|\ \tilde{A}(x) = 1\}. \tag{1.3.4}$$

Definition 1.3.4. (Height of a fuzzy set) The height of $\tilde{A}$ is the least upper bound of $\tilde{A}(x)$ i.e.,

$$\text{hgt}(\tilde{A}) = \sup_{x \in X} \tilde{A}(x). \tag{1.3.5}$$

Definition 1.3.5. (Universal fuzzy set) $\tilde{A}$ is a universal fuzzy set if $\tilde{A}(x) = 1,\ \forall\ x \in X$, that is if $core(\tilde{A}) = X$.

Definition 1.3.6. (Empty fuzzy set) $\tilde{A}$ is an empty fuzzy set if $\tilde{A}(x) = 0,\ \forall\ x \in X$, that is if $supp(\tilde{A}) = \phi$.

Definition 1.3.7. (Fuzzy singleton) Let $\tilde{A}$ be a fuzzy set. If $\text{supp}(\tilde{A}) = \{x_0\}$ then $\tilde{A}$ is called a fuzzy point and we use the notation $\tilde{A} = \overline{x_0}$.

Definition 1.3.8. (Normal fuzzy set) A is said to be normal if and only if $\exists\ x \in X,\ \tilde{A}(x) = 1$; this definition implies $\text{hgt}(\tilde{A}) = 1$.

A more general and even more useful notion is that of an $\alpha-$cut or $\alpha-$level set.

Definition 1.3.9. ($\alpha-$Levels) The set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree $\alpha \in (0, 1]$ is called the $\alpha-$cut or $\alpha-$level set:

$$[\tilde{A}]_\alpha = \{x \in X\ :\ \tilde{A}(x) \geq \alpha\}. \tag{1.3.6}$$

If non-equality is hold strictly then $[\tilde{A}]_\alpha$ is called "strong $\alpha-$level set", also for $\alpha = 0$ we have

$$[\tilde{A}]_0 = \overline{\bigcup_{\alpha\in(0,1]} [\tilde{A}]_\alpha}.$$

For simplicity we consider $[\tilde{A}]_\alpha = \tilde{A}_\alpha$

Definition 1.3.10. The m-th power of a fuzzy set $\tilde{A}$ is a fuzzy set with membership function

$$\tilde{A}^m(x) = [\tilde{A}(x)]^m \quad , \quad \forall x \in X \quad , \quad \forall m \in \mathbb{R}^+. \tag{1.3.7}$$

Convexity also plays a role in fuzzy set theory. By contrast to classical set theory, however, convexity conditions are defined with reference to the membership function rather than the support of a fuzzy set.

The membership function of a fuzzy set $\tilde{A}$ can be expressed in terms of the characteristic functions of its $\alpha-$cut according to the formula [32]

$$\tilde{A}(x) = \sup_{\alpha\in]0,1]} \min(\alpha, \tilde{A}_\alpha(x)),$$

where

$$\tilde{A}_\alpha(x) = \begin{cases} 1, & x \in \tilde{A}_\alpha, \\ 0, & otherwise. \end{cases}$$

It is easily checked that the following properties hold:

$$(\tilde{A} \cup \tilde{B})_\alpha = \tilde{A}_\alpha \cup \tilde{B}_\alpha, \qquad (\tilde{A} \cap \tilde{B})_\alpha = \tilde{A}_\alpha \cap \tilde{B}_\alpha.$$

Definition 1.3.11. (Convexity) A fuzzy set $\tilde{A}$ of X is called convex if $\tilde{A}_\alpha$ is a convex subset of X for each $\alpha \in [0,1]$. We now state a useful theorem that provides us with an alternative formulation of convexity of fuzzy sets. For the sake of simplicity, we restrict the theorem to fuzzy sets on $\mathbb{R}$, which are of primary interest in this text.

Theorem 1.3.1. *A fuzzy set $\tilde{A}$ on $\mathbb{R}$ is convex if and only if*

$$\tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\tilde{A}(x_1), \tilde{A}(x_2)\} \tag{1.3.8}$$

for all $x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0,1]$, where min *denotes the minimum operator.*

Proof. Assume that $\tilde{A}$ is convex and let $\alpha = A(x_1) \leq A(x_2)$. From the convexity of $\tilde{A}$ then, $\lambda x_1 + (1-\lambda)x_2 \in \tilde{A}_\alpha$ for any $\lambda \in [0,1]$. Consequently,

$$\tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = \tilde{A}(x_1) = \min\{\tilde{A}(x_1), \tilde{A}(x_2)\}.$$

Conversely, assume that $\tilde{A}$ satisfies (1.3.8). We need to prove that for any $\alpha \in [0,1]$, $\tilde{A}_\alpha$ is convex. Now, for any $x_1, x_2 \in \tilde{A}_\alpha$(*i.e.*, $\tilde{A}(x_1) \geq \alpha$, $\tilde{A}(x_2) \geq \alpha$), and for any $\lambda \in [0,1]$, by (1.3.8)

$$\tilde{A}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\tilde{A}(x_1), \tilde{A}(x_2)\} \geq \min(\alpha, \alpha) = \alpha,$$

i.e., $\lambda x_1 + (1-\lambda)x_2 \in \tilde{A}_\alpha$. Therefore, $\tilde{A}_\alpha$ is convex for any $\alpha \in [0,1]$. Hence, $\tilde{A}$ is convex. □

1.3.2 Basic relationships between fuzzy sets

As in set theory, we can define generic relations between two fuzzy sets, such as inclusion and equality. Let $\tilde{A} \in F(X)$ and $\tilde{B} \in F(X)$ be fuzzy sets.

Definition 1.3.12. (Inclusion) We say that $\tilde{A}$ is included in $\tilde{B}$, denoted by $\tilde{A} \subseteq \tilde{B}$, iff $\tilde{A}(x) \leq \tilde{B}(x),\ \forall x \in X$.

Definition 1.3.13. (Equality) $\tilde{A}$ and $\tilde{B}$, are said to be equal, denoted by $\tilde{A} = \tilde{B}$, iff $\tilde{A} \subseteq \tilde{B}$, and $\tilde{B} \subseteq \tilde{A}$.

1.3.3 Fuzzy number

Definition 1.3.14. (Fuzzy number) A fuzzy subset $\tilde{A}$ of the real line $\mathbb{R}$ with membership function $\tilde{A}(x),\ \tilde{A} : \mathbb{R} \to [0, 1]$, is called a fuzzy number if

(a) $\tilde{A}$ is normal, i.e., there exists an element x_0 such that $\tilde{A}(x_0) = 1$,

(b) $\tilde{A}$ is convex, i.e., $\tilde{A}(\lambda x_1 + (1 - \lambda)x_2) \geq \tilde{A}(x_1) \wedge \tilde{A}(x_2)$,

(c) $\tilde{A}(x)$ is upper semi-continuous,

(d) $supp(\tilde{A})$ is bounded, where $supp(\tilde{A}) = cl\{x \in \mathbb{R} : \tilde{A}(x) > 0\}$, and cl is the closure operator.

A fuzzy number $\tilde{A}$ is called *positive* (*negative*) if its membership function is such that $\tilde{A}(x) = 0,\ \forall x < 0\ (\forall x > 0)$.

Dubois and Prade [32] suggest a special type of representation for fuzzy numbers of the following type: They call L (and R), which map $\mathbb{R}^+ \longrightarrow [0,1]$, and are decreasing, *shape functions* or *reference functions* if $L(0)=1$, $L(x)<1$ for $x>0$; $L(x)>0$ for $x<1$; $L(1)=0$ or $[L(x)>0, \forall x$ and $L(+\infty)=0]$. The most useful type of fuzzy numbers is LR type as follows :

A fuzzy number is a convex fuzzy subset like of the real line is completely defined by its membership function. Let $\tilde{A}$ be a fuzzy number, whose membership function $\tilde{A}(x)$ can generally be defined as [31]

$$\tilde{A}(x) = \begin{cases} L_{\tilde{A}}(x) & a \leq x \leq b, \\ \omega & b \leq x \leq c, \\ R_{\tilde{A}}(x) & c \leq x \leq d, \\ 0 & otherwise, \end{cases} \tag{1.3.9}$$

where $0 < \omega \leq 1$ is a constant, $L_{\tilde{A}}(x) : [a,b] \to [0,\omega]$ and $R_{\tilde{A}}(x) : [c,d] \to [0,\omega]$ are two strictly monotonic and continuous mapping from $\mathbb{R}$ to closed interval $[0,\omega]$. If $\omega = 1$, then $\tilde{A}$ is a normal fuzzy number; otherwise, it is trapezoidal fuzzy number and is usually denoted by $\tilde{A} = (a,b,c,d;\omega)$ or $\tilde{A} = (a,b,c,d)$ if $\omega = 1$. In particular, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $\tilde{A} = (a,b,d;\omega)$ or $\tilde{A} = (a,b,d)$ if $\omega = 1$. So triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers.

Since $L_{\tilde{A}}(x)$ and $R_{\tilde{A}}(x)$ are both strictly monotonic and continuous functions, their inverse functions exist and should also be continuous and strictly monotonic. Let $\tilde{A}_L : [0, \omega] \to [a, b]$ and $\tilde{A}_R : [0, \omega] \to [c, d]$ be the inverse functions of $L_{\tilde{A}}(x)$ and $R_{\tilde{A}}(x)$, respectively. Then $\tilde{A}_L(y)$ and $\tilde{A}_R(y)$ should be integrable on the close interval $[0, \omega]$. In other words, both $\int_0^{\omega} \tilde{A}_L(y)dy$ and $\int_0^{\omega} \tilde{A}_R(y)dy$ should exist. In the case of trapezoidal fuzzy number, the inverse functions $\tilde{A}_L(y)$ and $\tilde{A}_R(y)dy$ can be analytically expressed as

$$\tilde{A}_L(y) = a + (b - a)y/\omega \qquad 0 \leq y \leq \omega, \tag{1.3.10}$$

$$\tilde{A}_R(y) = d - (d - c)y/\omega \qquad 0 \leq y \leq \omega. \tag{1.3.11}$$

The functions $L_{\tilde{A}}(x)$ and $R_{\tilde{A}}(x)$ are also called the left and right side of the fuzzy number $\tilde{A}$, respectively [31, 32].

In this thesis, we assume that

$$\int_{-\infty}^{+\infty} \tilde{A}(x)dx < +\infty.$$

According to the definition of a fuzzy number, it is seen at once that every $\alpha-$cut of a fuzzy number is a closed interval. Hence, for a fuzzy number $\tilde{A}$, we have

$$\tilde{A}_\alpha = [\tilde{A}_L(\alpha), \tilde{A}_U(\alpha)], \tag{1.3.12}$$

where

$$\tilde{A}_L(\alpha) = \inf\{x \in \mathbb{R} : \tilde{A}(x) \geq \alpha\},$$

$$\tilde{A}_U(\alpha) = \sup\{x \in \mathbb{R} : \tilde{A}(x) \geq \alpha\}. \tag{1.3.13}$$

If the left and right sides of the fuzzy number $\tilde{A}$ are strictly monotone, obviously, $\tilde{A}_L$ and $\tilde{A}_U$ are inverse functions of $L_{\tilde{A}}(x)$ and $R_{\tilde{A}}(x)$, respectively.

An important kind of fuzzy numbers was introduced in [13] as follows: Let $a, b, c, d \in \mathbb{R}$ such that $a < b \leq c < d$. A fuzzy number $\tilde{A}$ defined as

$$\tilde{A}(x) = \begin{cases} 0 & x \leq a, \\ (\frac{x-a}{b-a})^r & a \leq x \leq b \\ 1 & b \leq x \leq c, \\ (\frac{d-x}{d-c})^r & c < x \leq d, \\ 0 & d < x, \end{cases} \tag{1.3.14}$$

where $r > 0$, is denoted by $\tilde{A} = (a, b, c, d)_r$. If $\tilde{A} = (a, b, c, d)_r$ then

$$\tilde{A}_\alpha = [\tilde{A}_L(\alpha), \tilde{A}_U(\alpha)] = [a + (b-a)\alpha^{1/r},\ d - (d-c)\alpha^{1/r}], \qquad \alpha \in [0, 1].$$

If $r = 1$, we obtain trapezoidal fuzzy number. When $r = 1,\ b = c$ we obtain triangular fuzzy number, and it is denoted by $\tilde{A} = (a, b, d)$.

In case that $b - a = d - b,\ \tilde{A}$ is a symmetrical triangular fuzzy number.

Since the trapezoidal fuzzy number is completely characterized by $r = 1$ and four real numbers $a \leq b \leq c \leq d$, it is often denoted in brief as $\tilde{A} = (a, b, c, d)$.

The conditions $r = 1,\ a = b$ and $c = d$ imply the close interval and in case $r = 1,\ a = b = c = d = t$ we have the crisp number t.

Note 1: A normal trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is also denoted by (b, c, α, β), where α and β are left and right spread, respectively. Moreover, $\alpha = b - a,\ \beta = d - c$.

Note 2: In this thesis, a family of fuzzy numbers and trapezoidal fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$ and $\mathbb{F}^T(\mathbb{R})$ respectively.

Definition 1.3.15. (Quasi fuzzy number) A quasi fuzzy number $\tilde{A}$ is a fuzzy set of the real line with a normal, convex and continuous membership function satisfying the limit conditions

$$\lim_{t \to +\infty} \tilde{A}(t) = 0, \qquad \lim_{t \to -\infty} \tilde{A}(t) = 0.$$

Remark 1.3.1. Let $\tilde{A}$ be a fuzzy number. Then $\tilde{A}_\alpha$ is a closed convex (compact) subset of $\mathbb{R}$ for all $\alpha \in [0, 1]$.

Definition 1.3.16. We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements [39]:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$,

2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$,

3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A crisp number λ is simply represented by $\underline{u}(r) = \overline{u}(r) = \lambda, 0 \leq r \leq 1$.

Definition 1.3.17. Let v and w be fuzzy numbers and s be a real number. Then for $0 \leq r \leq 1$

$$u = v \text{ if and only if } \underline{u}(r) = \underline{v}(r) \text{ and } \overline{u}(r) = \overline{v}(r),$$

$$v + w = (\underline{v}(r) + \underline{w}(r), \overline{v}(r) + \overline{w}(r)),$$

$$v - w = (\underline{v}(r) - \overline{w}(r), \overline{v}(r) - \underline{w}(r)),$$

$$v.w = (\min\{\underline{v}(r).\underline{w}(r), \underline{v}(r).\overline{w}(r), \overline{v}(r).\underline{w}(r), \overline{v}(r).\overline{w}(r)\},$$

$$\max\{\underline{v}(r).\underline{w}(r), \underline{v}(r).\overline{w}(r), \overline{v}(r).\underline{w}(r), \overline{v}(r).\overline{w}(r)\}),$$

See [64].

Definition 1.3.18. The space $\mathbb{F}^n(\mathbb{R})$ is all of fuzzy subsets U of $\mathbb{R}^n$ which satisfy the following conditions:

1. U is normal,
2. U is convex,
3. U is upper semi-continuous,
4. $[U]_0$ is a bounded subset of $\mathbb{R}^n$,

when $n = 1$, elements of $\mathbb{F}^1(\mathbb{R})$ are Fuzzy numbers.

1.4 The extension principle

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. In this elementary form, it was already implied in Zadeh's first contribution [1965]. In the meantime, modifications have been suggested [Zadeh 1973a; Zadeh et al. 1975; Jain 1976]. Following Zadeh [1973a] and Dubois and Prade [1980a], we define the extension principle as follows:

Definition 1.4.1. [87] (extension principle) Assume X and Y are crisp sets and let f be a mapping from X to Y,

$$f : X \to Y,$$

such that for each $x \in X$, $f(x) = y \ \in \ Y$. Assume $\tilde{A}$ is a fuzzy subset of X, using extension principle, we can define $f(\tilde{A})$ as a fuzzy subset of Y such that

$$f(\tilde{A})(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \tilde{A}(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Definition 1.4.2. (*sup-min extension n-place functions*) Let X_1, X_2,...,X_n and Y be a family of sets. Assume f is a mapping from the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ into Y. Let $\tilde{A}_1$, $\tilde{A}_2$,...,$\tilde{A}_n$ be fuzzy subsets of X_1, X_2,...,X_n, respectively, then we use the extension principle for the evaluation of $f(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$. $f(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ is a fuzzy set such that

$$f(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)(y) = \begin{cases} \sup\{\min\{\tilde{A}_1(x_1), \tilde{A}_2(x_2), \ldots, \tilde{A}_n(x_n)\} \mid x \in f^{-1}(y)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad f^{-1}(y) \neq \emptyset, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

where $x = (x_1, x_2, ..., x_n)$.

Example 1.4.1. *Let $f : X \times X \to X$ be defined as*

$$f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2, \ \lambda_1, \ \lambda_2 \in \mathbb{R}.$$

Suppose $\tilde{A}_1$ and $\tilde{A}_2$ are fuzzy subsets of X. Then using the extension principle we get

$$f(\tilde{A}_1, \tilde{A}_2)(y) = \sup_{\lambda_1 x_1 + \lambda_2 x_2 = y} \min\{\tilde{A}_1(x_1), \tilde{A}_2(x_2)\}$$

and we use the notation $f(\tilde{A}_1, \tilde{A}_2) = \lambda_1 \tilde{A}_1 + \lambda_2 \tilde{A}_2$.

By the extension principle we can define fuzzy distance in the following.

Definition 1.4.3. The fuzzy distance function on $\mathbb{F}(\mathbb{R})$, $\delta : \mathbb{F}(\mathbb{R}) \times \mathbb{F}(\mathbb{R}) \to \mathbb{F}(\mathbb{R})$, is defined by

$$\delta(\tilde{A}, \tilde{B})(z) = \sup_{|x-y|=z} \min(\tilde{A}(x), \tilde{B}(y)). \tag{1.4.1}$$

Definition 1.4.4. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be a function from $\mathbb{F}(X)$ to $\mathbb{F}(Y)$. Then f is called a fuzzy function (or mapping) and we use the notation

$$f : \mathbb{F}(X) \to \mathbb{F}(Y).$$

It should be noted, however, that a fuzzy function is not necessarily defined by Zadeh's extension principle. It can be any function which maps a fuzzy set $A \in \mathbb{F}(X)$ into a fuzzy set $B := f(A) \in \mathbb{F}(Y)$.

Definition 1.4.5. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets . A fuzzy mapping $f : \mathbb{F}(X) \to \mathbb{F}(Y)$ is said to be monotonic increasing if $A, A' \in \mathbb{F}(X)$ and $A \subset A'$ imply that $f(A) \subset f(A')$.

Theorem 1.4.1. *Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets. Then every fuzzy mapping $f : \mathbb{F}(X) \to \mathbb{F}(Y)$ defined by the extension principle is monotonic increasing.*

Proof. Let $A, A' \in \mathbb{F}(X)$ such that $A \subset A'$. Then using the definition of sup-min extension principle we get

$$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \leq \sup_{x \in f^{-1}(y)} A'(x) = f(A')(y)$$

for all $y \in Y$. $\square$

Let $\tilde{A} = (\underline{a}, \overline{a}, \alpha, \beta)_{LR}$ and $\tilde{B} = (\underline{b}, \overline{b}, \sigma, \gamma)_{LR}$ be fuzzy numbers of $LR-type$. Using the (sup-min) extension principle, we can verify the following rules for addition and subtraction of fuzzy numbers of $LR-type$:

$$\tilde{A} + \tilde{B} = (\underline{a} + \underline{b}, \overline{a} + \overline{b}, \alpha + \sigma, \beta + \gamma)_{LR},$$

$$\tilde{A} - \tilde{B} = (\underline{a} - \underline{b}, \overline{a} - \overline{b}, \alpha + \gamma, \beta + \sigma)_{LR},$$

furthermore, if $\lambda \in \mathbb{R}$ is a real number then $\lambda\tilde{A}$ can be represented as

$$\lambda\tilde{A} = \begin{cases} (\lambda\underline{a}, \lambda\overline{a}, \lambda\alpha, \lambda\beta)_{LR} & \text{if } \lambda \geq 0, \\ (\lambda\overline{a}, \lambda\underline{a}, \mid \lambda \mid \beta, \mid \lambda \mid \alpha)_{LR} & \text{if } \lambda < 0. \end{cases}$$

In particular, if $\tilde{A} = (\underline{a}, \overline{a}, \alpha, \beta)$ and $\tilde{B} = (\underline{b}, \overline{b}, \sigma, \gamma)$ are fuzzy numbers of trapezoidal form, then

$$\tilde{A} + \tilde{B} = (\underline{a} + \underline{b}, \overline{a} + \overline{b}, \alpha + \sigma, \beta + \gamma)$$

$$\tilde{A} - \tilde{B} = (\underline{a} - \underline{b}, \overline{a} - \overline{b}, \alpha + \gamma, \beta + \sigma).$$

If $\tilde{A} = (a, \alpha, \beta)$ and $\tilde{B} = (b, \sigma, \gamma)$ are fuzzy numbers of triangular form, then

$$\tilde{A} + \tilde{B} = (a + b, \alpha + \sigma, \beta + \gamma)$$

$$\tilde{A} - \tilde{B} = (a - b, \alpha + \gamma, \beta + \sigma),$$

and if $\tilde{A} = (a, \alpha)$ and $\tilde{B} = (b, \beta)$ are fuzzy numbers of symmetrical triangular form, then

$$\tilde{A} + \tilde{B} = (a + b, \alpha + \beta)$$

$$\tilde{A} - \tilde{B} = (a - b, \alpha + \beta),$$

$$\lambda\tilde{A} = (\lambda a, \mid \lambda \mid \alpha).$$

The above results can be generalized to linear combination of fuzzy numbers.

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers with $[\tilde{A}]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ and $[\tilde{B}]_\alpha = [\underline{b}(\alpha), \overline{b}(\alpha)]$, $0 \leq \alpha \leq 1$. Then it can easily be shown that

$$[\tilde{A} + \tilde{B}]_\alpha = [\underline{a}(\alpha) + \underline{b}(\alpha), \overline{a}(\alpha) + \overline{b}(\alpha)],$$

$$[-\tilde{A}]_\alpha = [-\overline{a}(\alpha), -\underline{a}(\alpha)],$$

$$[\tilde{A} - \tilde{B}]_\alpha = [\underline{a}(\alpha) - \overline{b}(\alpha), \overline{a}(\alpha) - \underline{b}(\alpha)],$$

$$[\lambda\tilde{A}]_\alpha = [\lambda\underline{a}(\alpha), \lambda\overline{a}(\alpha)] \quad if \quad \lambda \geq 0,$$

$$[\lambda\tilde{A}]_\alpha = [\lambda\overline{a}(\alpha), \lambda\underline{a}(\alpha)] \quad if \quad \lambda < 0,$$

for all $\alpha \in [0, 1]$, i.e. any α-level set of the extended sum of two fuzzy numbers is equal to the sum of their α-level sets. The following two theorems show that this property is valid for any continuous function.

Theorem 1.4.2. *[60] Let $f : X \to X$ be a continuous function and let $\tilde{A}$ be a fuzzy number. Then,*

$$[f(\tilde{A})]_\alpha = f([\tilde{A}]_\alpha),$$

where $f(\tilde{A})$ is defined by the extension principle and

$$f([\tilde{A}]_\alpha) = \{f(x) \mid x \in [\tilde{A}]_\alpha\}.$$

If $[\tilde{A}]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$ and f is continuous and monotone increasing then from the above theorem we get

$$[f(\tilde{A})]_\alpha = f([\tilde{A}]_\alpha) = f([\underline{a}(\alpha), \overline{a}(\alpha)]) = [f(\underline{a}(\alpha)), f(\overline{a}(\alpha))].$$

Theorem 1.4.3. *[60] Let $f : X \times X \to X$ be a continuous function and let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers. Then*

$$[f(\tilde{A}, \tilde{B})]_\alpha = f([\tilde{A}]_\alpha, [\tilde{B}]_\alpha)$$

where,

$$f([\tilde{A}]_\alpha, [\tilde{B}]_\alpha) = \{f(x_1, x_2) \mid x_1 \in [\tilde{A}]_\alpha, x_2 \in [\tilde{B}]_\alpha\}.$$

Let $f(x, y) = xy$ and let $[\tilde{A}]_\alpha = [\underline{a}(\alpha), \overline{a}(\alpha)]$, $[\tilde{B}]_\alpha = [\underline{b}(\alpha), \overline{b}(\alpha)]$ be the $\alpha-$level sets of two fuzzy numbers $\tilde{A}$ and $\tilde{B}$. Applying above theorem we get

$$[f(\tilde{A}, \tilde{B})]_\alpha = f([\tilde{A}]_\alpha, [\tilde{B}]_\alpha) = [\tilde{A}]_\alpha[\tilde{B}]_\alpha.$$

The equation

$$[\tilde{A}\tilde{B}]_\alpha = [\tilde{A}]_\alpha[\tilde{B}]_\alpha = [\underline{a}(\alpha)\underline{b}(\alpha), \overline{a}(\alpha)\overline{b}(\alpha)]$$

holds if and only if $\tilde{A}$ and $\tilde{B}$ are both nonnegative, i.e. $\tilde{A}(x) = \tilde{B}(x) = 0$ for $x \leq 0$. If $\tilde{B}$ is nonnegative then we have

$$[\tilde{A}]_\alpha[\tilde{B}]_\alpha = [\min\{\underline{a}(\alpha)\underline{b}(\alpha), \underline{a}(\alpha)\overline{b}(\alpha)\}, \max\{\overline{a}(\alpha)\underline{b}(\alpha), \overline{a}(\alpha)\overline{b}(\alpha)\}].$$

In general, we obtain a very complicated expression for the α-level sets of the product $\tilde{A}\tilde{B}$

$$[\tilde{A}]_\alpha[\tilde{B}]_\alpha = [\min\{\underline{a}(\alpha)\underline{b}(\alpha), \underline{a}(\alpha)\overline{b}(\alpha), \overline{a}(\alpha)\underline{b}(\alpha), \overline{a}(\alpha)\overline{b}(\alpha)\},$$

$$\max\{\underline{a}(\alpha)\underline{b}(\alpha), \underline{a}(\alpha)\overline{b}(\alpha), \overline{a}(\alpha)\underline{b}(\alpha), \overline{a}(\alpha)\overline{b}(\alpha)\}].$$

1.5 Operation on fuzzy numbers

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to *add*, *subtract*, *multiply* and *divide* with fuzzy quantities. The process of doing these operations is called *fuzzy arithmetic*. Some previous works related to operations on fuzzy numbers are as follows:

Jain [48], Nahmias [58], Mizumoto and Tanaka [57] and [56] Baas and Kwakernaak [9].

1.5.1 Addition and Multiplication

Addition: Addition is an increasing operation. Hence, the extended addition ($\oplus$) of fuzzy numbers gives a fuzzy number. Note that $-(\tilde{A} \oplus \tilde{B}) = (-\tilde{A}) \oplus (-\tilde{B})$. The operation ($\oplus$) is commutative and associative but has no group structure. The identity of ($\oplus$) is the non-fuzzy number 0. But $\tilde{A}$ has no symmetrical element in the sense of a group structure. In particular, $\tilde{A} \oplus (-\tilde{A}) \neq 0, \quad \forall \tilde{A} \in \mathbb{F}(\mathbb{R}) - \mathbb{R}$.

Multiplication: Multiplication is an increasing operation on $\mathbb{R}^+$ and a decreasing operation on $\mathbb{R}^-$. Hence, the product of fuzzy numbers ($\odot$) that are all either positive or negative gives a positive fuzzy number. Note that $-(\tilde{A}) \odot \tilde{B} = -(\tilde{A} \odot \tilde{B})$, so that the factors can have different signs. The operation ($\odot$) is commutative and associative. The set of positive fuzzy numbers is not a group for ($\odot$): although $\forall \tilde{A}, \quad \tilde{A} \odot 1 = \tilde{A}$, the product $\tilde{A} \odot \tilde{A}^{-1} \neq 1$ as soon as $\tilde{A}$ is not a real number. $\tilde{A}$ has no inverse in the sense of group structure.

1.5.2 Subtraction

Subtraction is neither increasing nor decreasing. However, it is easy to check that $\tilde{A} \ominus \tilde{B} = \tilde{A} \oplus (-\tilde{B}), \quad \forall(\tilde{A}, \tilde{B}) \in \mathbb{F}(\mathbb{R})^2$ so that $\tilde{A} \ominus \tilde{B}$ is a fuzzy number whenever $\tilde{A}$ and $\tilde{B}$ are.

1.5.3 Division

Division is neither increasing nor decreasing. But, since $\tilde{A} \oslash \tilde{B} = \tilde{A} \odot (\tilde{B}^{-1}), \forall \tilde{A} \, \forall \tilde{B}, \tilde{A} \in \mathbb{F}(\mathbb{R}), \tilde{B} \in \mathbb{F}(\mathbb{R} - \{0\})$, $\tilde{A} \oslash \tilde{B}$ is a fuzzy number when $\tilde{A}$ and $\tilde{B}$ are both positive or both negative fuzzy numbers. The division of ordinary fuzzy numbers can be performed similar to multiplication, by decomposition.

1.6 Hausdorff metric

Denote by κ^n the set of all nonempty compact subsets of $\mathbb{R}^n$ and by κ_c^n the subset of κ^n consisting of nonempty convex compact sets.

In this section, we define a metric space by Hausdorff separation. Recall that

$$\rho(x, A) = \min_{a \in A} \|x - a\| \tag{1.6.1}$$

is the distance of a point $x \in \mathbb{R}^n$ from $A \in \kappa^n$ and that the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \kappa^n$ is defined as

$$\rho(A, B) = \max_{a \in A} \rho(a, B). \tag{1.6.2}$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$.

An open ϵ-neighborhood of $\mathrm{A} \in \kappa^n$ is the set

$$\mathrm{N}(\mathrm{A}, \epsilon) = \{x \in \mathbb{R}^n \ : \ \rho(x, \mathrm{A}) < \epsilon\} = \mathrm{A} + \epsilon \mathrm{B}^n, \tag{1.6.3}$$

where B^n is the open unit ball in $\mathbb{R}^n$, [29].

Definition 1.6.1. A mapping $F : \mathbb{R}^n \to \kappa^n$ is *upper semi-continuous* (usc) at x_0 if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, x_0)$ such that

$$F(x) \subset N(F(x_0), \epsilon) = F(x_0) + \epsilon B^n, \tag{1.6.4}$$

for all $x \in N(x_0, \delta)$.

Definition 1.6.2. The *Hausdorff metric* d_H on κ^n is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}. \tag{1.6.5}$$

The space (κ^n, d_H) is a complete metric space. Let D^n denote the set of usc normal fuzzy sets on $\mathbb{R}^n$ with compact support. That is, $\tilde{u} \in D^n$, then $\tilde{u} : \mathbb{R}^n \to [0, 1]$ is usc, $\mathrm{supp}(\tilde{u})$ is compact and there exists at least one $\xi \in \mathrm{supp}(\tilde{u})$ for which $\tilde{u}(\xi) = 1$. The β-level set of $\tilde{u}$, $0 < \beta \leq 1$ is

$$[\tilde{u}]_\beta = \{x \in \mathbb{R}^n \ : \ \mu_{\tilde{u}}(x) \geq \beta\}. \tag{1.6.6}$$

Clearly, for $\alpha \leq \beta, [\tilde{u}]_\alpha \supseteq [\tilde{u}]_\beta$. The level sets are nonempty from normality and compact by usc and compact support. The metric d_∞ is defined on D^n as

$$d_\infty(\tilde{u}, \tilde{v}) = \sup\{d_H([\tilde{u}]_\alpha, [\tilde{v}]_\alpha) \ : \ 0 \leq \alpha \leq 1\}, \quad \tilde{u}, \tilde{v} \in D^n \tag{1.6.7}$$

and (D^n, d_∞) is a complete metric space. $\mathbb{F}^n(\mathbb{R})$ is the subset of fuzzy convex elements of D^n. The metric space $(\mathbb{F}^n(\mathbb{R}), d_\infty)$ is also complete, [28].

If $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, then, according to Zadeh extension principle, we can extend f to $\mathbb{F}^n(\mathbb{R}) \times \mathbb{F}^n(\mathbb{R}) \to \mathbb{F}^n(\mathbb{R})$ by the equation

$$f(u, v)(z) = \sup_{z=f(x,y)} \min\{\mu_u(x), \mu_v(y)\}. \tag{1.6.8}$$

Chapter 2

Fuzzy Ranking

2.1 Introduction

The problem of ordering fuzzy quantities has been addressed by many researches. In the literature, more than 100 indices are presented. However, there exists a lot of debates over this issue and one author may disagree with another. So far, none of them is commonly accepted.

2.2 Difference of opinion over the ranking methodology

In fuzzy decision analysis, fuzzy quantities are frequently employed to describe the performance of alternatives in modeling a real-world problem. The ranking or choice of alternatives eventually leads to comparisons of fuzzy quantities. Unlike in the case of real numbers, fuzzy quantities have no natural order. A straightforward idea with the ordering of fuzzy quantities is to convert a fuzzy quantity into a real number and base the comparison of fuzzy quantities on that numbers. Each individual conversion way, however, pays attention to a special aspect of a fuzzy quantity. As a consequence, each approach suffer from, some defects if only one real number is associated with each fuzzy quantity as Freeling pointed out " by reducing the whole of our analysis to a single number, we are loosing much of the information we have purposely been keeping throughout our calculations" [38]. Precisely, the linguistic approach is motivated based on this argument. Unfortunately, the linguistic approach at the moment is just an interpretation of fuzzy quantities rather than the ranking of them in our opinion.

To avoid the partially in an ordering procedure, another alternative to the linguistic approach is the use of multiple indices as Dubois and Prade did. They proposed four indices (so-called complete indices) to locate the relative position of two fuzzy numbers in comparison. Certainly, four indices may not induce the consistent ranking in some cases. The final choice is left to the decision maker. Therefore, Yaun argued "this somehow defeats the purpose of the ranking method which is supposed to derive a conclusion for the decision maker" [80]. These opposite standpoints show that a compromise should be made between the reductions of information and the indecision.

2.3 Some aspects for ordering fuzzy numbers

In this section, we bring some principle definitions which are used to rank fuzzy numbers.

Definition 2.3.1. (Reducing function) [68] A function $s : [0,1] \longrightarrow [0,1]$ is a reducing function if s is increasing and $s(0) = 0$ and $s(1) = 1$. We say that s is a regular function if $\int_0^1 s(r)\mathrm{d}r = \frac{1}{2}$.

Basically, a reducing function allow the decision maker to decrease the influence of the lower $\alpha-$levels of a fuzzy number and to adjust the significance of the upper levels as is deemed appropriate.

The notations of value and ambiguity of a fuzzy number are given in the following. This value essentially assesses a fuzzy number by assigning a number to the ill-defined magnitude represented by the fuzzy number. Last parameter, ambiguity, which basically measures how much vagueness is present in the ill-defined magnitude of the fuzzy number.

Definition 2.3.2. (Value of fuzzy number) [68] If $\tilde{A}$ is a fuzzy number with $\alpha-$cut representation, $(\tilde{A}_L(\alpha), \tilde{A}_R(\alpha))$, and s is a reducing function then the value of $\tilde{A}$ (with respect to s) is defined by

$$Val(\tilde{A}) = \int_0^1 s(\alpha)\big[\tilde{A}_L(\alpha) + \tilde{A}_R(\alpha)\big]\mathrm{d}\alpha. \tag{2.3.1}$$

Definition 2.3.3. (Ambiguity of fuzzy number) [68] If $\tilde{A}$ is a fuzzy number with $\alpha-$cut representation, $(\tilde{A}_L(\alpha), \tilde{A}_R(\alpha))$, and s is a reducing function then the ambiguity of $\tilde{A}$ (with respect to s) is defined by

$$Amb(\tilde{A}) = \int_0^1 s(\alpha)\big[\tilde{A}_R(\alpha) - \tilde{A}_L(\alpha)\big]\mathrm{d}\alpha. \qquad (2.3.2)$$

2.3.1 A set of reasonable ordering properties

Wang and Kerre [70] present a set of reasonable ordering properties as follows:

Let M be an ordering method, $\mathbb{F}(\mathbb{R})$ the set of fuzzy quantities for which the method M can be applied and Γ a finite sub-set of $\mathbb{F}(\mathbb{R})$. The statement "two elements $\tilde{A}_i$ and $\tilde{A}_j$ in $\mathbb{F}(\mathbb{R})$ satisfy that $\tilde{A}_i$ has a higher ranking than $\tilde{A}_j$ when M is applied to the fuzzy quantities in Γ" will be written as $\tilde{A}_i \succ \tilde{A}_j$ by M on Γ. $\tilde{A}_i \sim \tilde{A}_j$ by M on Γ and $\tilde{A}_i \succeq \tilde{A}_j$ by M on Γ are similarly interpreted. In addition, we assume the following:

1. The fuzzy quantities satisfy the conditions for the application of the ranking method when method is investigated.

2. When a ranking method is applied one the set $\mathbb{F}(\mathbb{R})$ of fuzzy quantities on of the following is true for every $(\tilde{A}_i, \tilde{A}_j) \in \mathbb{F}(\mathbb{R})^2$:

   $\tilde{A}_i \succ \tilde{A}_j, \quad \tilde{A}_i \succeq \tilde{A}_j, \quad \tilde{A}_i \sim A_j$

The following axioms are reasonable properties of ordering fuzzy quantities for an ordering approach M.

(i) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $\tilde{A}_1 \in \Gamma, \tilde{A}_1 \succeq \tilde{A}_1$ by M on Γ.

(ii) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}_1, \tilde{A}_2) \in \Gamma^2, \tilde{A}_1 \succeq \tilde{A}_2$ and $\tilde{A}_2 \succeq \tilde{A}_1$, we should have $\tilde{A}_1 \sim \tilde{A}_2$ by M on Γ.

(iii) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3) \in \Gamma^3, \tilde{A}_1 \succeq \tilde{A}_2$ and $\tilde{A}_2 \succeq \tilde{A}_3$, we should have $\tilde{A}_1 \succeq \tilde{A}_3$ by M on Γ.

(iv) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}_1, \tilde{A}_2) \in \Gamma^2$, $\inf\{\text{supp}(\tilde{A}_1)\} > \sup\{\text{supp}(\tilde{A}_2)\}$, we should have $\tilde{A}_1 \succeq \tilde{A}_2$ by M on Γ.

This property means that if two fuzzy quantities have separate supports then the fuzzy quantity with the support on the right is at least as good as the one with the support on the left. One stronger version of this axiom is as follows:

(iv') For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}_1, \tilde{A}_2) \in \Gamma^2$, $\inf\{\text{supp}(\tilde{A}_1)\} > \sup\{\text{supp}(\tilde{A}_2)\}$, we should have $\tilde{A}_1 \succ \tilde{A}_2$ by M on Γ.

(v) Let Γ and Γ' be two arbitrary finite sub-sets of fuzzy quantities in which M can be applied and $\tilde{A}_1$ and $\tilde{A}_2$ are in $\Gamma \cap \Gamma'$. We obtain the ranking order $\tilde{A}_1 \succeq \tilde{A}_2$ by M on Γ' iff $\tilde{A}_1 \succeq \tilde{A}_2$ by M on Γ.

(vi) Let $\tilde{A}_1, \tilde{A}_2$, $\tilde{A}_1 + \tilde{A}_3$ and $\tilde{A}_2 + \tilde{A}_3$ be elements of $\mathbb{F}(\mathbb{R})$. If $\tilde{A}_1 \succeq \tilde{A}_2$ by M on $\{\tilde{A}_1, \tilde{A}_2\}$, then $\tilde{A}_1 + \tilde{A}_3 \succeq \tilde{A}_2 + \tilde{A}_3$ by M on $\{\tilde{A}_1 + \tilde{A}_3, \tilde{A}_2 + \tilde{A}_3\}$.

The axiom (vi) indicates that "+" is compatible with $\succeq$ defined by the ordering approach M. Concerning " $\succ$ ", a similar axiom (vi) is as follows:

(vi') If $\tilde{A}_1 \succ \tilde{A}_2$ by M on $\{\tilde{A}_1, \tilde{A}_2\}$, then $\tilde{A}_1+\tilde{A}_3 \succ \tilde{A}_2+\tilde{A}_3$ by M on $\{\tilde{A}_1+\tilde{A}_3, \tilde{A}_2+\tilde{A}_3\}$ when $\tilde{A}_3 \neq \tilde{0}$.

(vii) Let $\tilde{A}_1, \tilde{A}_2$, $\tilde{A}_1\tilde{A}_3$ and $\tilde{A}_2\tilde{A}_3$ be elements of $\mathbb{F}(\mathbb{R})$ and $\tilde{A}_3 \geq 0$, $\tilde{A}_1 \succeq \tilde{A}_2$ by M on $\{\tilde{A}_1, \tilde{A}_2\}$ implies $\tilde{A}_1\tilde{A}_3 \succeq \tilde{A}_2\tilde{A}_3$ by M on $\{\tilde{A}_1\tilde{A}_3, \tilde{A}_2\tilde{A}_3\}$.

2.3.2 Some existing defuzzification method

In this part briefly, we review some existing defuzzification methods: COG, FOM, MOM, LOM, BADD, centroid point, respectively. Some commonly used defuzzification methods are listed [30, 52, 61, 74].

***(i)* Center of gravity/area (COG)**

The best known defuzzification operator is the center of gravity defuzzification method, which computes the center of gravity of the area under the membership function, i.e.

$$COG(\tilde{A}) = \frac{\int_a^d x\tilde{A}(x)\mathrm{d}x}{\int_a^d \tilde{A}(x)\mathrm{d}x}. \tag{2.3.3}$$

***(ii)* First of maxima (FOM)**

The FOM method selects the smallest element of core$(\tilde{A})$ as the defuzzification value, i.e.

$$FOM(\tilde{A}) = \min core(\tilde{A}). \tag{2.3.4}$$

(*iii*) Last of maxima (LOM)

The LOM method selects the greatest element of core($\tilde{A}$) as the defuzzification value, i.e.

$$LOM(\tilde{A}) = \max core(\tilde{A}). \tag{2.3.5}$$

(*iv*) Middle of maxima (MOM)

The MOM method is very similar to the FOM or LOM methods. Instead of taking the first or the last values from core($\tilde{A}$), it takes the average of these two values, i.e.

$$MOM(\tilde{A}) = \frac{\min core(\tilde{A}) + \max core(\tilde{A})}{2}. \tag{2.3.6}$$

(*v*) Basic defuzzification distribution (BADD)

An extension of both the COG and MOM method is the BADD defuzzification approach [35]. It basically suggests that the final decision may depend non-linearly on the membership value of the fuzzy set. The BADD defuzzification value of $\tilde{A}$ is

$$BADD(\tilde{A}) = \frac{\int_a^d x\tilde{A}_\gamma(x)\mathrm{d}x}{\int_a^d \tilde{A}_\gamma(x)\mathrm{d}x}, \tag{2.3.7}$$

where γ is a free parameter in $[0, \infty)$. The parameter γ can be used to adjust the method to the special cases: When $\gamma = 0$, $BADD(\tilde{A}) = MOS(\tilde{A}) = \frac{a+d}{2}$, where MOS=mean of support; $\gamma = 1$, $BADD(\tilde{A}) = MOS(\tilde{A})$; when $\gamma \longrightarrow \infty$, $BADD(\tilde{A}) \longrightarrow MOS(\tilde{A})$.

(*vi*) Centroid point.

In order to the determine the centroid point $(x_0(\tilde{A}), y_0(\tilde{A}))$ of a fuzzy number

$\tilde{A}$, which is defined by relation (1.3.9), Cheng [22] provided the following centroid formulae:

$$\bar{x}_0(\tilde{A}) = \frac{\int_a^b \big(xL_{\tilde{A}}(x)\big)\mathrm{d}x + \int_b^c (x\omega)dx + \int_c^d \big(xR_{\tilde{A}}(x)\big)\mathrm{d}x}{\int_a^b \big(L_{\tilde{A}}(x)\big)\mathrm{d}x + \int_b^c (\omega)dx + \int_c^d \big(R_{\tilde{A}}(x)\big)\mathrm{d}x}, \tag{2.3.8}$$

$$\bar{y}_0(\tilde{A}) = \frac{\int_0^\omega \alpha(\tilde{A}_R(\alpha) + \tilde{A}_L(\alpha))\mathrm{d}\alpha}{\int_0^\omega (\tilde{A}_R(\alpha) + \tilde{A}_L(\alpha))\mathrm{d}\alpha}. \tag{2.3.9}$$

In 2008, Wang et al. [73] found that the centroid formulae provided by Cheng are incorrect and have led to some misapplications. The incorrect formulae appeared in many papers such as by Chu and Tsao [24] and Pan and Yeh. To avoid possible more misapplications or spread in the future, they presented the correct formulae as follows:

$$\bar{x}_0(\tilde{A}) = \frac{\int_a^b \big(xL_{\tilde{A}}(x)\big)\mathrm{d}x + \int_b^c (x\omega)\mathrm{d}x + \int_c^d \big(xR_{\tilde{A}}(x)\big)\mathrm{d}x}{\int_a^b \big(L_{\tilde{A}}(x)\big)\mathrm{d}x + \int_b^c (\omega)\mathrm{d}x + \int_c^d (R_{\tilde{A}}(x))\mathrm{d}x}, \tag{2.3.10}$$

$$\bar{y}_0(\tilde{A}) = \frac{\int_0^\omega \alpha\big(\tilde{A}_R(\alpha) - \tilde{A}_L(\alpha)\big)\mathrm{d}\alpha}{\int_0^\omega \big(\tilde{A}_R(\alpha) - \tilde{A}_L(\alpha)\big)\mathrm{d}\alpha}. \tag{2.3.11}$$

For trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; \omega)$, formulas (2.3.10) and (2.3.11) can be simplified as

$$\bar{x}_0(\tilde{A}) = \frac{1}{3}\Big[a + b + c + d - \frac{cd - ab}{(c+d) - (a+b)}\Big], \tag{2.3.12}$$

$$\bar{y}_0(\tilde{A}) = \frac{1}{3}\omega\Big[1 + \frac{c-b}{(c+d)-(a+b)}\Big]. \qquad (2.3.13)$$

For more details we refer the reader to [73].

2.4 A review of some ordering indices

The ordering indices are so diverse that is necessary to organize them into several lines to investigate them more efficiently.

The ordering indices are classified into three categories by Wang and Kerre [70]. In the first class, each index is associated with a mapping F from the set of fuzzy quantities to the real line R in order to transform the involved fuzzy quantities into real numbers. Fuzzy quantities are then compared according to the corresponding real numbers. In the second class, reference set(s) is (are) set up and all the fuzzy quantities to be ranked are compared with the reference set(s). In the last class, A fuzzy relation is constructed to make pairwise comparisons serve as a basis to obtain the final ranking orders.

In the following we review some ordering indices based on Wang and Kerre idea.

2.4.1 A brief introduction to the first class of ordering approaches

Assume $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ are the fuzzy quantities to be ranked and $a_{i\alpha}^{-} = \inf A_{i\alpha}$ and $a_{i\alpha}^{+} = \sup A_{i\alpha}$, Where $A_{i\alpha}$ denotes the $\alpha-$cut of $\tilde{A}_i$. In the following we recall the ordering indices in the first, second and third class.

1. Adamo's approach (1980) [6]

Given α in $[0, 1]$, Adamo simply evaluates a fuzzy quantity based on utmost right point of the $\alpha-$cut. Therefore, his ordering index is

$$AD^{\alpha}(\tilde{A}_i) = a_{i\alpha}^{+}. \tag{2.4.1}$$

2. Yager's approaches (1978-1981) [76, 77, 78]

Yager proposed four indices which may be employed for the purpose of ordering fuzzy quantities in $[0, 1]$.

$$Y_1(\tilde{A}_i) = \frac{\int_0^1 g(x)\tilde{A}_i(x)\mathrm{d}x}{\int_0^1 \tilde{A}_i(x)\mathrm{d}x} \tag{2.4.2}$$

where $g(x)$ measures the importance of value x.

$$Y_2(\tilde{A}_i) = \int_0^{hgt} M(A_{i\alpha})\mathrm{d}\alpha, \tag{2.4.3}$$

where $M(\tilde{A_{i\alpha}})$ is the mean value of the elements of $\tilde{A_{i\alpha}}$.

$$Y_3(\tilde{A}_i) = \int_0^1 |x - A_i\tilde{(}x)|\mathrm{d}x, \tag{2.4.4}$$

$$Y_4(\tilde{A}_i) = \sup_{x \in [0,1]} \min\big(x, (\tilde{A}_i(x)\big). \tag{2.4.5}$$

3. Chang's approach (1981) [18]

Chang's index is simply defined by the integral

$$C(\tilde{A}_i) = \int_{x \in supp\tilde{A}_i} x\tilde{A}_i(x)\mathrm{d}x. \tag{2.4.6}$$

4. Campos and Munoz's approach (1989) [16]

A family of ordering indices called average index is proposed by Campos and Munoz to rank fuzzy numbers. It is of the form

$$CM(\tilde{A}_i) = \int_Y f_{\tilde{A}_i}(\alpha)\mathrm{d}P(\alpha), \tag{2.4.7}$$

where Y is a sub-set of the unit interval, P a probability measure on Y and $f_{\tilde{A}_i} : Y \longrightarrow \mathbb{R}$ represents the position of the $\alpha-$cut $\tilde{A_{i\alpha}}$ for any $\alpha \in Y$. The suggested expression by Campos and Munoz is $f_{\tilde{A}_i}(\alpha) = \lambda a^+_{i\alpha} + (1-\lambda)a^-_{i\alpha}$ with $\lambda \in [0,1]$ an optimism-pessimism indicator.

5. Liou and Wang's approach (1992) [54]

Let $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ be continuous fuzzy numbers. Suppose the left spread l_i of $\tilde{A}_i$ is strictly increasing function and the right spread r_i of $\tilde{A}_i$ a strictly decreasing function. The inverse functions of l_i and r_i are denoted by l_i^{-1} and r_i^{-1}, respectively. The ordering index proposed by Liou and Wang is defined by

$$LW^\lambda(\tilde{A}_i) = \lambda \int_0^1 r_i^{-1}(y)\mathrm{d}y + (1-\lambda)\int_0^1 l_i^{-1}(y)\mathrm{d}y, \tag{2.4.8}$$

where $\lambda \in [0, 1]$ is the optimism index reflecting the optimism degree of a decision maker. The larger λ is, the more optimistic the decision maker is. The two extreme cases are: $\lambda = 0$, the decision maker completely is pessimistic; and $\lambda = 1$, the decision maker completely is optimistic. The case $\lambda = 1/2$, reflects a linear decision attitude.

6. Choobineh and Li's approach (1993) [23]

Let a and d be two real numbers satisfying $a \leq \inf\{x|x \in \bigcup_{i=1}^{n} supp\tilde{A}_i\}$ and $d \geq \sup\{x|x \in \bigcup_{i=1}^{n} supp\tilde{A}_i\}$. Choobine and Li evaluate $\tilde{A}_i$ by

$$CL(\tilde{A}_i) = \frac{1}{2}\Big(hgt(\tilde{A}_i) - \frac{1}{d-a}\big(\int_0^{hgt(\tilde{A}_i)} (d - a_{i\alpha}^{+})\mathrm{d}\alpha - \int_0^{hgt(\tilde{A}_i)} (a_{i\alpha}^{-} - a)\mathrm{d}\alpha\big)\Big). \quad (2.4.9)$$

7. Fortemps and Roubens' approach (1996) [37]

Let $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ be fuzzy quantities. Fortemps and Roubens suggested the following evaluation of $\tilde{A}_i$:

$$FR(\tilde{A}_i) = \frac{1}{2hgt(\tilde{A}_i)} \int_0^{hgt(\tilde{A}_i)} (a_{i\alpha}^+ + a_{i\alpha}^-) \mathrm{d}\alpha. \tag{2.4.10}$$

For the first class of approaches, it is easy to determine the resulting ranking orders among $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ as soon as the mapping F is recognized. Generally, it will be formulated by

$\tilde{A}_i \succ \tilde{A}_j$ by $F \Longleftrightarrow F(\tilde{A}_i) \succ F(\tilde{A}_j)$

$\tilde{A}_i \sim \tilde{A}_j$ by $F \Longleftrightarrow F(\tilde{A}_i) = F(\tilde{A}_j)$

$\tilde{A}_i \succeq \tilde{A}_j$ by $F \Longleftrightarrow \tilde{A}_i \succ \tilde{A}_j$ by F or $\tilde{A}_i \sim \tilde{A}_j$ by F.

Clearly,

$\tilde{A}_i \sim \tilde{A}_j$ by $F \Longleftrightarrow F(\tilde{A}_i) \geq F(\tilde{A}_j)$.

An expectation is Yager's Y_3 index for which

$\tilde{A}_i \succ \tilde{A}_j$ by $Y_3 \Longleftrightarrow Y_3(\tilde{A}_i) < Y_3(\tilde{A}_j)$

$\tilde{A}_i \sim \tilde{A}_j$ by $Y_3 \Longleftrightarrow Y_3(\tilde{A}_i) = Y_3(\tilde{A}_j)$

8. Cheng's approach (1998) [22]

In 1998, Cheng presented a new approach for ranking fuzzy numbers by distance method, which is calculated as follows:

$$R(\tilde{A}_i) = \sqrt{\bar{x}_0(\tilde{A}_i)^2 + \bar{y}_0(\tilde{A}_i)^2}, \tag{2.4.11}$$

where $\bar{x}_0(\tilde{A}_i)$ and $\bar{y}_0(\tilde{A}_i)$ are the centroid points of fuzzy number $\tilde{A}_i$, which may computed by relations 2.3.10 and 2.3.11 respectively.

9. Chu and Tsao's approach (2002) [24]

Chu and Tsao as pointed out the drawback of Cheng's approach, they suggested a method, which could overcome the shortcoming of Cheng's method. They computed the area between the centroid point and original point. This area were utilized to rank fuzzy numbers as:

$$S(\tilde{A}_i) = \bar{x}_0(\tilde{A}_i).\bar{y}_0(\tilde{A}_i), \tag{2.4.12}$$

where $\big(\bar{x}_0(\tilde{A}_i), \bar{y}_0(\tilde{A}_i)\big)$ is the coordinate of centroid point of a fuzzy number $\tilde{A}_i$.

10. Abbasbandy and Asady's approach (2006) [3]

Abbasbany and Asady's approach is called sign distance method. It is denoted by $d_p(\tilde{A}_i, \tilde{A}_0)$, which is computed as follows:

$$d_p(\tilde{A}_i, \tilde{A}_0) = \gamma(\tilde{A}_i) D_p(\tilde{A}_i, \tilde{A}_0), \tag{2.4.13}$$

such that $\gamma : \mathbb{F}(\mathbb{R}) \longrightarrow \{-1, 1\}$ and

$$\forall \tilde{A}_i \in \mathbb{F}(\mathbb{R}) : \gamma(\tilde{A}_i) = sign\left[\int_0^1 \big(\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha)\big)\mathrm{d}\alpha\right],$$

where

$$\gamma(\tilde{A}_i) = \begin{cases} 1, & if \quad \int_0^1 \big(\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha)\big)\mathrm{d}\alpha \geq 0, \\ \\ -1, & if \quad \int_0^1 \big(\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha)\big)\mathrm{d}\alpha < 0. \end{cases}$$

Also,

$$\tilde{A}_0 = \begin{cases} 1, & if \quad x = 0, \\ \\ 0, & if \quad x \neq 0, \end{cases}$$

and

$$D_p(\tilde{A}_i, \tilde{A}_0) = \Big(\int_0^1 (|\tilde{A}_L(\alpha)|^P + |\tilde{A}_U(\alpha)|^P) \mathrm{d}\alpha \Big)^{\frac{1}{P}}.$$

11. Asady and Zendehnam's approach (2008) [8]

Asady and Zendehnam's method is called distance minimization method, which means the nearest point of a fuzzy number. That is

$$M(\tilde{A}) = \frac{1}{2} \int_0^1 (\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha)) \mathrm{d}\alpha. \tag{2.4.14}$$

12. Wang and Lee's approach (2008) [72]

In 2008, Wang and Lee have done a revised method for ranking fuzzy numbers. The revised method is based on the Chu and Tsao's method [24]. In short, they ranked fuzzy numbers based on their $\bar{x}_0$'s values if they are different. In case that they are equal, they further compare with their $\bar{y}_0$'s values to form their ranks. Consequently, for two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ if $\bar{y}_0(\tilde{A}) \geq \bar{y}_0(\tilde{B})$ based on $\bar{x}_0(\tilde{A}) = \bar{x}_0(\tilde{B})$, then $\tilde{A} \geq \tilde{B}$.

2.4.2 A brief introduction to the second class of ordering approaches

1. Jain's approach (1976) [46, 47]

Jain's method is based on the following fuzzy maximizing set for any $x \in \mathbb{R}$:

$$\tilde{A}_{max}(x) = \big(\frac{x}{x_{max}}\big)^k,$$

where $k > 0$ is a real number and $x_{max} = \sup \bigcup_{i=1}^{n} \mathrm{supp}\tilde{A}_i$. The fuzzy quantity is evaluated by the index

$$J^k(\tilde{A}_i) = \sup_{x \in \mathbb{R}} \min\big(\tilde{A}_{max}(x), \tilde{A}_i(x)\big). \tag{2.4.15}$$

A larger $J^k(\tilde{A}_i)$ implies a higher ranking for $\tilde{A}_i$.

2. Kerre's approach (1982) [46, 47]

Kerre proposed an ordering method by calculating the Hamming distance between $\tilde{A}_i$ and $\widetilde{max}(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$.

$$K(\tilde{A}_i) = \int_S |\tilde{A}_i(x) - \widetilde{max}(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)(x)| \mathrm{d}x, \tag{2.4.16}$$

where $S = \bigcup_{i=1}^{n} \mathrm{supp}\tilde{A}_i$.

3. Chen's approach (1985) [19]

Chen realized that Jain's method is not applicable if $\mathrm{supp}\tilde{A}_i \cap]-\infty, 0[\neq \emptyset$ for some $i (1 \leq i \leq n)$. So, he redefined the fuzzy maximizing set $\tilde{A}_{max}$ for any $x \in \mathbb{R}$ as

$$\tilde{A}_{max}(x) = \big(\frac{x - x_{min}}{x_{max} - x_{min}}\big)^k,$$

where x_{max} and k are the same as in Jain's index and $x_{min} = \inf \bigcup_{i=1}^{n} supp\tilde{A}_i$. Meanwhile, the fuzzy minimizing set $\tilde{A}_{min}$, the dual concept of $\tilde{A}_{max}$, is introduced and defined as

$$\tilde{A}_{min}(x) = (\frac{x_{max} - x}{x_{max} - x_{min}})^k.$$

Then the left utility $L(\tilde{A}_i)$ and right utility $R(\tilde{A}_i)$ of the fuzzy quantity are, respectively, defined by

$$L(\tilde{A}_i) = \sup_{x \in \mathbb{R}} \min\big(\tilde{A}_{min}(x), \tilde{A}_i(x)\big)$$

and

$$R(\tilde{A}_i) = \sup_{x \in \mathbb{R}} \min\big(\tilde{A}_{max}(x), \tilde{A}_i(x)\big).$$

Finally, the total utility is calculated as

$$CH^k(\tilde{A}_i) = \frac{1}{2}\big(R(\tilde{A}_i) + 1 - L(\tilde{A}_i)\big). \tag{2.4.17}$$

For some particular types of fuzzy numbers, the concentrate calculation formulas are given by Chen with $k = \frac{1}{2}, 1$ and 2.

4. ***Wang's approach*** (1997) [69]

Let $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n$ be fuzzy numbers. Inspired by Kerre's method, Wang generally utilized the closeness of $\tilde{A}_i$ to $\widetilde{\max}(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ to evaluate $\tilde{A}_i$. The closer $\tilde{A}_i$ and $\widetilde{max}(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ are, the better $\tilde{A}_i$ is ranked. Besides Hamming distance, Wang considered three other closeness measures. Here, he mentioned the following one.

$$W(\tilde{A}_i) = \frac{\int_R \min\big(\tilde{A}_i(x).\widetilde{max}(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)(x)\big)\mathrm{d}x}{\int_R \max\big(\tilde{A}_i(x).\widetilde{max}(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)(x)\big)\mathrm{d}x} \tag{2.4.18}$$

5. Kim and Park's approach (1990) [50]

Similar to Chen's method, Kim and Park also introduced the fuzzy maximizing set and the fuzzy minimizing set as reference sets. They are, respectively, defined as for any $x \in \mathbb{R}$:

$$\tilde{A}_{max}(x) = \frac{x - x_{min}}{x_{max} - x_{min}}$$

and

$$\tilde{A}_{min}(x) = \frac{x_{max} - x}{x_{max} - x_{min}},$$

with parameter $k \in [0, 1]$ reflecting the DM's risk attitude, the ordering index is defined by

$$KP^k(\tilde{A}_i) = khgt(\tilde{A}_i \cap \tilde{A}_{max}) + (1 - k)(1 - hgt(\tilde{A}_i \cap \tilde{A}_{min}), \tag{2.4.19}$$

when $k = \frac{1}{2}$, the DM is risk-neutral. When $k > \frac{1}{2}$ or $k < \frac{1}{2}$, the DM tends to be a risk-taker or a risk-aveser. Clearly, $KP^{1/2} = CH^1$.

2.4.3 A brief introduction to the third class of ordering approaches

In the third class, an ordering approach is based on a fuzzy relation to make pairwise comparisons. We will mention the fuzzy relations employed for the purpose of ordering fuzzy quantities in the following review. Assume that $A_1, A_2, A_3, ...A_n$ are fuzzy quantities to be ranked.

1. Bass and Kwakernaak's approach (1977) [9]

In 1977, the issue of ordering fuzzy quantities is encountered by Bass and Kwakernaak [9] in modeling the multiple attribute decision making under fuzziness. They introduced the following conditional fuzzy set O/R:

$$\widetilde{O/R}(i|x_1, x_2, ..., x_n) = \begin{cases} 1, & if \quad x_i \geq x_j \ (\forall j \neq i) \\ 0, & otherwise. \end{cases}$$

Then, the ordering index is defined by

$$\begin{aligned} BK(A_i) &= \sup_{x_1, x_2, \ldots, x_n} \min\left(\widetilde{O/R}(i|x_1, x_2, ..., x_n), \min(A_1(x_1), ..., A_n(x_n)\right) \\ &= \sup_{x_1, x_2, \ldots, x_n} \min\left(A_1(x_1), ..., A_n(x_n)\right) \\ &= \min_{j \neq i} \sup_{x_i \geq x_j} \min\left(A_i(x_i), A_j(x_j)\right). \end{aligned}$$

Let $P_{BK}(A_i, A_j) = \sup_{x_i \geq x_j} \min(A_i(x_i), A_j(x_j)$. We write BK as

$$BK(A_i) = \min_{j \neq i} P_{BK}(A_i, A_j).$$

Since $P_{BK}(A_i, A_j) = hgtA_i \geq (P_{BK}(A_i, A_j)$, We may write

$$BK(A_i) = \min_{j} P_{BK}(A_i, A_j). \tag{2.4.20}$$

2. Baldwin Guild's approach (1979) [10]

Similar to Bass and Kwakernaak's conditional fuzzy set, Baldwin and Guild defined a binary fuzzy relation R_{ij} on $\mathbb{R}$ with the value $R_{ij}(x_i, x_j)$ indicating how much

$x_i$is greater than x_i. The ordering index is accordingly modified into

$$BG(A_i) = \min_{j \neq i} \sup_{x_i, x_j} \min \big(A_i(x_i), A_j(x_j), R_{ij}(x_i, x_j)\big).$$

The determination of R_{ij} depends on the decision maker's (DM's) risk attitude. Particularly, they suggested

$R_{ij}(x_i, x_j) = x_i - x_j$ for a neutral DM;

$R_{ij}(x_i, x_j) = x_i^2 - x_j^2$ for a risk taker;

$R_{ij}(x_i, x_j) = \sqrt{x_i} - \sqrt{x_j}$ for a risk-averse DM.

Let $P_{BG}(A_i, A_j) = \sup_{x_i, x_j} \min \big(A_i(x_i), A_j(x_j), R_{ij}(x_i, x_j)\big)$. Then

$$BG(A_i) = \min_{j \neq i} P_{BG}(A_i, A_j). \tag{2.4.21}$$

3. Dubois and Prade's approach (1983) [33]

Based on possibility and necessity measures, Dubois and Prade proposed four fuzzy relations to compare normal convex fuzzy quantities A_i and A_j.

$PD(A_i, A_j) = \sup_{x_i, x_j \ x_i \geq x_j} \min \big(A_i(x_i), A_j(x_j)\big)$,

$PSD(A_i, A_j) = \sup_{x_i} \inf_{x_j \ x_i \leq x_j} \min \big(A_i(x_i), 1 - A_j(x_j)\big)$,

$ND(A_i, A_j) = \inf_{x_i} \sup_{x_j \ x_i \geq x_j} \min \big(A_i(x_i), 1 - A_j(x_j)\big)$,

$NSD(A_i, A_j) = 1 - \sup_{x_i, x_j \ x_i \leq x_j} \min \big(A_i(x_i), A_j(x_j)\big)$.

4. Nakamura's approach (1986) [59]

Assume $A_1, A_2, A_3, ...A_n$ are fuzzy numbers. Nakumara used Hamming distance to compare $\underline{A_i}$ with $\widetilde{\min}(\underline{A_i}, \underline{A_j})$ and $\overline{A_i}$ with $\widetilde{\min}(\overline{A_i}, \overline{A_{ij}})$ respectively. A parameter $\lambda(\lambda \in [0, 1])$ is introduced to describe the DM's attitude. Thus this fuzzy relation is:

$$P_{N^\lambda}(A_i, A_j) = \frac{\lambda d_H\big(\underline{A_i}, \widetilde{\min}(\underline{A_i}, \underline{A_j})\big) + (1-\lambda) d_H\big(\overline{A_i}, \widetilde{\min}(\overline{A_i}, \overline{A_j})\big)}{\lambda d_H(\underline{A_i}, \underline{A_j}) + (1-\lambda) d_H(\overline{A_i}, \overline{A_j})}. \tag{2.4.22}$$

In the case of $\lambda d_H(\underline{A_i}, \underline{A_j}) + (1-\lambda) d_H(\overline{A_i}, \overline{A_j}) = 0$, assume $P_{N^\lambda}(A_i, A_j) = \frac{1}{2}$.

5. Kolodziejczyk's approach (1990) [53]

Let $A_1, A_2, A_3, ...A_n$ be fuzzy numbers. Using fuzzy maximum and Hamming distance, Kolodziejczyk constructed the following fuzzy relations:

$$P_{K1}(A_i, A_j) = \frac{d_H\big(\underline{A_i}, \widetilde{\max}(\underline{A_i}, \underline{A_j})\big) + d_H\big(\overline{A_i}, \widetilde{\max}(\overline{A_i}, \overline{A_i})\big)}{d_H(\underline{A_i}, \underline{A_j}) + d_H(\overline{A_i}, \overline{A_j})}, \tag{2.4.23}$$

$$P_{K2}(A_i, A_j) = \frac{d_H\big(\underline{A_i}, \widetilde{\max}(\underline{A_i}, \underline{A_j})\big) + d_H\big(\overline{A_i}, \widetilde{\max}(\overline{A_i}, \overline{A_i})\big) + d_H(A_i \cap A_j, \emptyset)}{d_H(\underline{A_i}, \underline{A_j}) + d_H(\overline{A_i}, \overline{A_j}) + 2d_H(A_i \cap A_j, \emptyset)}. \tag{2.4.24}$$

6. Delgado et al's approach (1988) [25]

Delgado et al. suggested two very general fuzzy relations by the use of concepts 'greater than $F(\geq F)$' and "less than $F(\leq F)$' for a given fuzzy number F.

Assume M and N are fuzzy numbers and T is a $t-$norm. Two fuzzy relations α_T and γ_T are defined by

$$\alpha_T(M, N) = \sup_{x \in \mathbb{R}} T\big(\geq N(x), M(x)\big),$$

$$\gamma_T(M, N) = \sup_{x \in \mathbb{R}} T\big(\leq N(x), M(x)\big).$$

Use them to construct the fuzzy relations

$$\beta_T(M,N) = 1 - \alpha_T(M,N)$$

and

$$\delta_T(M,N) = 1 - \gamma_T(M,N).$$

For the practical comparison of fuzzy quantities, they choose

$$
\begin{aligned}
{}_{\geq}N(x) &= \frac{\underline{N}(x)(1+\lambda)}{1+\lambda\underline{N}(x)} \quad (\lambda \in]0,1]),\\
{}_{\leq}N(x) &= \frac{\overline{N}(x)}{1+\lambda(1-\overline{N}(x))} \quad (\lambda \in [0,+\infty]).
\end{aligned}
$$

Unless otherwise specified, we will take these expressions for ${}_{\geq}N$ and ${}_{\leq}N$.

Now, assume $A_1, A_2, A_3, ...A_n$ are fuzzy numbers. For each pair $(A_i, A_j), \beta_T(A_i, A_j)$ and $\delta_T(A_i, A_j)$ are calculated.

7. Yuan's approach (1991) [80]

Yuan proposed a fuzzy relation which compares the right spread of one fuzzy number with the left spread of another fuzzy number. It is defined as

$$P_Y(A_i, A_j) = \frac{\Delta_{ij}}{\Delta_{ij} + \Delta_{ji}}. \tag{2.4.25}$$

8. Saade and Schwarzlander's approach (1991) [63]

Let $A_1, A_2, A_3, ...A_n$ be normal convex fuzzy quantities. Saade and Schwarzlander constructed the following index to make the comparison between A_i and A_j.

$$P_{SS}(A_i, A_j) = d_H\big(\underline{A_i}, \widetilde{\max}(\underline{A_i}, \underline{A_j})\big) + d_H\big(\overline{A_i}, \widetilde{\max}(\overline{A_i}, \overline{A_j})\big) \tag{2.4.26}$$

2.5 Concluding Remarks

Toward the end of this section we would like to point out the following:

In Table 2.1, we listed all the results on the fulfillment of the axioms for the first and second class of ordering approaches under the condition that the involved fuzzy quantities are fuzzy numbers. From Table 2.1 we know that AD^{α} for any $\alpha \in]0,1]$ satisfies all the axioms and Y_2, CL, FR, $CM_1^{\lambda}, CM_2^{\lambda}, LW^{\lambda}$ for any $\lambda \in [0,1]$ satisfy all the axiom expect A_7. Table 2.2 shows that$P_N, P_{K1}, P_{K2}, P_{SS}$ and P_Y satisfy all the axioms except A_7. So, the ordering procedures associated with these indices are reasonable ones based on Wang and Kerre's axioms. For more details we refer the reader to [70, 71].

Table 2.1

Fulfillment of axioms for the ordering approaches in the first and second class

Index	A_1	A_2	A_3	A_4	A'_4	A_5	A_6	A'_6	A_7
Y_1	Y	Y	Y	Y	Y	Y	N	N	N
Y_2	Y	Y	Y	Y	Y	Y	Y	Y	N
Y_3	Y	Y	Y	N	N	Y	N	N	N
Y_4	Y	Y	Y	Y	Y	Y	N	N	N
C	Y	Y	Y	N	N	Y	N	N	N
FR	Y	Y	Y	Y	Y	Y	Y	Y	N
CL	Y	Y	Y	Y	Y	Y	Y	Y	N
LW^{λ}	Y	Y	Y	Y	Y	Y	Y	Y	N
CM_1^{λ}	Y	Y	Y	Y	Y	Y	Y	Y	N
CM_2^{λ}	Y	Y	Y	Y	Y	Y	Y	Y	N
AD^{λ}	Y	Y	Y	Y	Y	Y	Y	Y	Y
K	Y	Y	Y	N	N	N	N	N	N
W	Y	Y	Y	Y	N	N	N	N	N
J^k	Y	Y	Y	Y	Y	N	N	N	N
CH^k	Y	Y	Y	Y	Y	N	N	N	N
KP^k	Y	Y	Y	Y	Y	N	N	N	N
d_p	Y	Y	Y	Y	Y	Y	N	N	N
M	Y	Y	Y	Y	Y	Y	Y	Y	Y

Table 2.2

Fulfillment of axioms for the ordering approaches in the third class

Index	A_1	A_2	A_3	A_4	A'_4	A_5	A_6	A'_6	A_7
BK	Y	Y	Y	Y	N	N	Y	N	Y
PD	Y	Y	Y	Y	N	N	Y	N	Y
PSD	Y	Y	Y	Y	N	N	Y	N	Y
ND	Y	Y	Y	Y	N	N	Y	N	Y
NSD	Y	Y	Y	Y	N	N	Y	N	Y
P_N^λ	Y	Y	Y	Y	Y	Y	Y	Y	N
BG	Y	Y	Y	Y	N	N	N	N	N
P_{SS}	Y	Y	Y	Y	Y	Y	Y	Y	N
β_T	Y	Y	Y	Y	Y	N	Y	N	Y
δ_T	Y	Y	Y	Y	Y	N	Y	N	Y
P_Y	Y	Y	Y	Y	Y	Y	Y	Y	N
P_{K1}	Y	Y	Y	Y	Y	Y	Y	Y	N
P_{K2}	Y	Y	Y	Y	Y	Y	Y	Y	N

Chapter 3

New Approaches for Ranking of Fuzzy Numbers

3.1 Introduction

Ranking fuzzy numbers plays an important role in linguistic decision making and some other fuzzy application systems. As we mentioned, several strategies have been proposed for ranking of fuzzy numbers. Each of these techniques have been shown to produce non-intuitive results in certain cases. In this chapter, we will introduce two new approaches for ranking of fuzzy numbers.

First, we will bring an example from [14] to interpret a controversial case. Then the new approaches will be introduced. The first one is an improvement in Cheng's method (centroid based distance method) and the second approach, which applies for trapezoidal fuzzy numbers is based on the left and the right spreads at some α-levels of trapezoidal fuzzy numbers.

3.2 Inconsistency of the ranking orders among different approaches

Let us extract an example from [14] to interpret a controversial case. As you observe Fig. 3.1, the fuzzy numbers A, B and C are the same on the right of the modal values and differ from each other on the left. Using the extension principle, we have $\widetilde{\max}(A, B) = A$ and $\widetilde{\min}(A, B) = B$. Clearly, $A \neq B$. Therefore, A should have a higher ranking than B based on the extension principle.

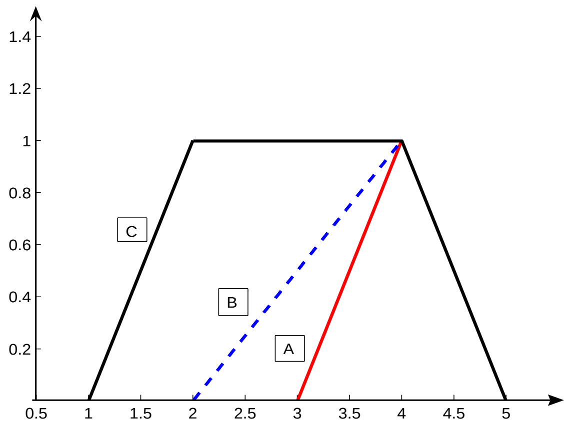

Figure 3.1: A controversial case.

In the same sense, B has a higher ranking than C. As a matter of fact, many ordering approaches such as Yager indices [77, 78], Chen's index [19], etc. derive these ranking orders on $\{A, B, C\}$. This is also in accordance with Baldwin and Guild's intuition [10] although their method derives the conclusion that A has a higher ranking than B and B and C have the same ranking. However, Chang [18] argued that C is intuitively preferable to B while the application of approaches by Bass and Kwakemaak [9], Jain [46, 47] and Adamo [6] derives the same ranking for all three fuzzy numbers. In referring to Chen's intuition that A should have a higher ranking than B [19], Saade and Schwarlander [63] argued that the ranking order of A and B should depend on DM's subjective attitude A and B have the same ranking for an optimistic DM. Hence, a lot of arguments and some conflicting ranking orders exist around the fuzzy numbers in this example.

3.3 An improvement in centroid point method for ranking fuzzy numbers

In a paper by Cheng [22], a centroid-based distance method was suggested for ranking fuzzy numbers, both normal and non-normal. It is found that the mentioned method could not rank fuzzy number correctly. For instance it cannot rank fuzzy numbers when they have the same centroid point. Some other researchers tried to overcome the shortcoming of the inconsistency of Cheng's method but next methods still have some drawbacks.

In this section, we want to indicate these problems of Cheng's distance, Chu and Tsao's area and Wang and Lee's revised method, and then propose an improvement method which can avoid these problems for ranking fuzzy numbers. The improved method can effectively rank various fuzzy numbers and their images. Some numerical examples demonstrate the advantage of the improved method.

We utilize Chen's distance method with corrected formulae and connect it to a sign function which will be introduced to improve the mentioned method.

Since in Cheng's method non of fuzzy number is considered as fuzzy zero, therefore, it would be a drawback. In point of our view every fuzzy number may lead to zero, positive or negative real number. To overcome with this subject, we first introduce the sign function as follows:

Definition 3.3.1. Let $\mathbb{F}(\mathbb{R})$ stands the set of fuzzy numbers, $\tilde{A}$ is a fuzzy number with $\alpha-$cut $\tilde{A}_\alpha = \left[\tilde{A}_L(\alpha), \tilde{A}_U(\alpha)\right]$ and ω be a constant provided that $0 < \omega \leq 1$. Moreover, $\gamma : \mathbb{F}(\mathbb{R}) \longrightarrow \{-1, 0, 1\}$ be a function that is defined as:

$$\forall\ \tilde{A} \in \mathbb{F}(\mathbb{R}) : \gamma(\tilde{A}) = sign\left[\int_0^\omega \big(\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha)\big)\mathrm{d}\alpha\right],$$

i.e.,

$$\gamma(\tilde{A}) = \begin{cases} 1 \quad if \quad \int_0^\omega \big(\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha)\big)\mathrm{d}\alpha > 0, \\ 0 \quad if \quad \int_0^\omega \big(\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha)\big)\mathrm{d}\alpha = 0, \\ -1 \quad if \quad \int_0^\omega \big(\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha)\big)\mathrm{d}\alpha < 0. \end{cases} \tag{3.3.1}$$

It is clear for normal fuzzy numbers $\omega = 1$.

Remark 3.3.1. If $\inf \operatorname{supp}(\tilde{A}) \geq 0$ then $\gamma(\tilde{A}) = 1$.

Remark 3.3.2. If $\sup \operatorname{supp}(\tilde{A}) < 0$ then $\gamma(\tilde{A}) = -1$.

Remark 3.3.3. If $\tilde{A} = (a, b, c, d; \omega)$ be a symmetric trapezoidal fuzzy number such that $b + c = 0$ then $\gamma(\tilde{A}) = 0$.

Let $\tilde{A}$ a fuzzy number and its centroid point be $\big(\bar{x_0}(\tilde{A}), \bar{y_0}(\tilde{A})\big)$, which can be calculated by (2.3.10) and (2.3.11). By connecting the sign function, that introduced in (3.3.1) with Cheng's distance method [22], we have the following:

$$IR(\tilde{A}) = \gamma(\tilde{A})R(\tilde{A}). \tag{3.3.2}$$

In other words,

$$IR(\tilde{A}) = \gamma(\tilde{A})\sqrt{\bar{x}_0(\tilde{A})^2 + \bar{y}_0(\tilde{A})^2}. \tag{3.3.3}$$

The aforementioned method is called improved Cheng's distance method. Here, the improved method $IR(\tilde{A})$ is utilized to rank fuzzy numbers.

we define the ranking of $\tilde{A}_1$ and $\tilde{A}_2$ by the $IR(.)$ on $\mathbb{F}(\mathbb{R})$ as follows:

1. $IR(\tilde{A}_1) > IR(\tilde{A}_2)$ if and only if $\tilde{A}_1 \succ \tilde{A}_2$.

2. $IR(\tilde{A}_1) < IR(\tilde{A}_2)$ if and only if $\tilde{A}_1 \prec \tilde{A}_2$.

3. $IR(\tilde{A}_1) = IR(\tilde{A}_2)$ if and only if $\tilde{A}_1 \sim \tilde{A}_2$.

Then we formulate the order $\succeq$ and $\preceq$ as $\tilde{A}_1 \succeq \tilde{A}_2$ if and only if $\tilde{A}_1 \succ \tilde{A}_2$ or $\tilde{A}_1 \sim \tilde{A}_2$, $\tilde{A}_1 \preceq \tilde{A}_2$ if and only if $\tilde{A}_1 \prec \tilde{A}_2$ or $\tilde{A}_1 \sim \tilde{A}_2$.

We consider the following reasonable properties for the ordering approaches, see [70].

(i) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $\tilde{A}_1 \in \Gamma, \tilde{A}_1 \succeq \tilde{A}_1$.

(ii) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}_1, \tilde{A}_2) \in \Gamma^2, \tilde{A}_1 \succeq \tilde{A}_2$ and $\tilde{A}_2 \succeq \tilde{A}_1$, we should have $\tilde{A}_1 \sim \tilde{A}_2$.

(iii) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3) \in \Gamma^3, \tilde{A}_1 \succeq \tilde{A}_2$ and $\tilde{A}_2 \succeq \tilde{A}_3$, we should have $\tilde{A}_1 \succeq \tilde{A}_3$.

(iv) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}_1, \tilde{A}_2) \in \Gamma^2$, $\inf\{\text{supp}(\tilde{A}_1)\} > \sup\{\text{supp}(\tilde{A}_2)\}$, we should have $\tilde{A}_1 \succeq \tilde{A}_2$.

(v) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}_1, \tilde{A}_2) \in \Gamma^2$, $\inf\{\text{supp}(\tilde{A}_1)\} > \sup\{\text{supp}(\tilde{A}_2)\}$, we should have $\tilde{A}_1 \succ \tilde{A}_2$.

(vi) Let $\tilde{A}_1, \tilde{A}_2$, $\tilde{A}_1 + \tilde{A}_3$ and $\tilde{A}_2 + \tilde{A}_3$ be elements of $\mathbb{F}(\mathbb{R})$. If $\tilde{A}_1 \succeq \tilde{A}_2$, then $\tilde{A}_1 + \tilde{A}_3 \succeq \tilde{A}_2 + \tilde{A}_3$.

Remark 3.3.4. The function $IR(.)$ has the properties $(i) - (vi)$.

3.3.1 Numerical examples

Example 3.3.1. *Consider two fuzzy numbers* $\tilde{A} = (0, 0.0484, 0.0968, 0.9)$ *and* $\tilde{B} = (-0.03, 0, 0.03)$ *which are indicated in Figure 3.2.*

Intuitively, the ranking order is $\tilde{B} \prec \tilde{A}$. Table 3.1 shows the result obtained by improvement centroid method, Chu and Tsao's formula [24], revised centroid and Cheng's method [22], respectively. It is very clear that Cheng's distance method lead to an incorrect ranking order $\tilde{B} \succ \tilde{A}$, which is contrary to the ranking order $\tilde{B} \prec \tilde{A}$ obtained by using our proposed method and also some other methods which utilize the centroid points. This shows the fact that Cheng's distance method can lead to wrong ranking orders.

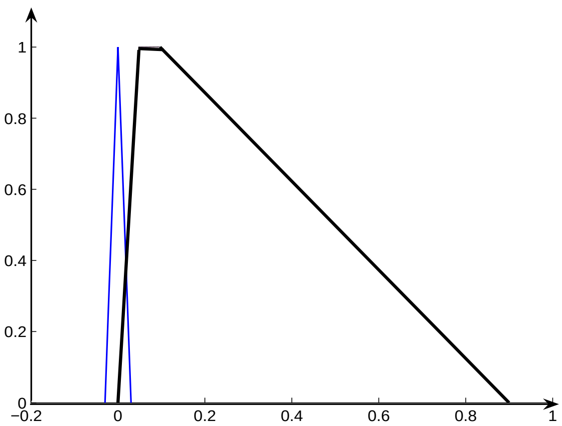

Figure. 3.2: Fuzzy numbers, $\tilde{A} = (0, 0.0484, 0.0968, 0.9)$ and $\tilde{B} = (-0.03, 0, 0.03)$.

Example 3.3.2. *Consider two fuzzy numbers* $\tilde{A} = (-5/2, -1/2, 0, 1)$ *and* $\tilde{B} = (-9/4, -1/4, 3/4)$ *shown in Fig. 3.3.*

Intuitively, both numbers are negative and the ranking order is $\tilde{A} \prec \tilde{B}$. However, by Cheng's distance method [22], the ranking order is $\tilde{A} \succ \tilde{B}$ which is unreasonable result. On the other hand the images of these two fuzzy numbers are

$-\tilde{A} = (-1, 0, 1/2, 5/2)$ and $-\tilde{B} = (-3/4, 1/4, 9/4)$, respectively. By our method producing ranking order $-\tilde{A} \succ -\tilde{B}$. Clearly, similar to Chu and Tsao's area method [24] and revised centroid [72] as you see in Table 3.2. On the contrary, the result of Cheng's distance and is $-\tilde{A} \prec -\tilde{B}$ which is unreasonable. The proposed method can also overcome the shortcoming of Cheng's based distance method in ranking fuzzy numbers and their images. The results are shown in Table 3.2.

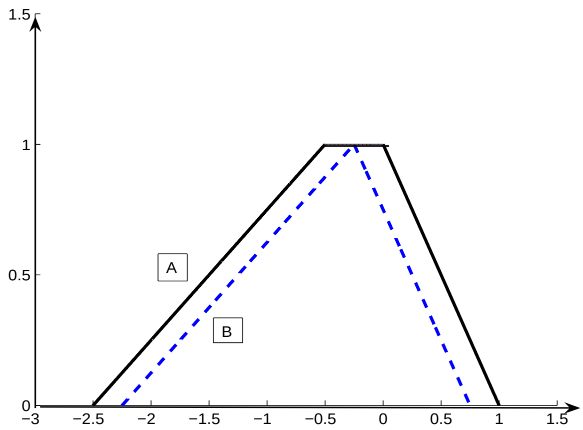

Figure. 3.3: Fuzzy numbers, $\tilde{A} = (-5/2, -1/2, 0, 1)4$ and $\tilde{B} = (-9/4, -1/4, 3/4)$.

Example 3.3.3. *Consider the three triangular fuzzy numbers* $\tilde{A} = (-3, 0, 3), \tilde{B} = (-3, -2, 5)$ *and* $\tilde{C} = (-3, 1, 2)$, *which are indicated in Fig. 3.4, and their membership* *functions are respectively defined as*

$$\tilde{A}(x) = \begin{cases} \frac{1}{3}(x+3) & -3 \leq x < 0, \\ 1 & x = 0, \\ \frac{-1}{3}(x-3) & 0 \leq x < 3, \\ 0 & otherwise, \end{cases}$$

$$\tilde{B}(x) = \begin{cases} x+3 & -3 \leq x < -2, \\ 1 & x = -2, \\ \frac{-1}{7}(x-5) & -2 \leq x < 5, \\ 0 & otherwise, \end{cases}$$

$$\tilde{C}(x) = \begin{cases} \frac{1}{4}(x+3) & -3 \leq x < 1, \\ 1 & x = 1, \\ -x+2 & 1 \leq x < 2, \\ 0 & otherwise. \end{cases}$$

The corrected centroid point formulae (2.3.10) and (2.3.11) for mentioned numbers yield the same results, i.e., $x_0(\tilde{A}) = 0$ and $y_0(\tilde{A}) = \frac{1}{3}$. Then by using Cheng's centroid-based distance method $R(\tilde{A}) = R(\tilde{B}) = R(\tilde{C}) = \frac{1}{3}$. On the other hand, by applying Tsao's method $S(\tilde{A}) = S(\tilde{B}) = S(\tilde{C}) = 0$. Therefore, both the distance method and area method producing the ranking order $\tilde{A} \sim \tilde{B} \sim \tilde{C}$. This shows the fact that the mentioned methods can lead to wrong ranking order. According to formulae 3.3.1 $\gamma(\tilde{A}) = 0, \gamma(\tilde{B}) = -1$ and $\gamma(\tilde{C}) = +1$. Also, by using our method it will be obtained

$IR(\tilde{A}) = 0, IR(\tilde{B}) = -\frac{1}{3}$ and $IR(\tilde{C}) = \frac{1}{3}$. Hence, the ranking order is $\tilde{B} \prec \tilde{A} \prec \tilde{C}$. So, our improved method can efficiently deal with the fuzzy ranking problems when different generalized fuzzy numbers have the same centroid point.

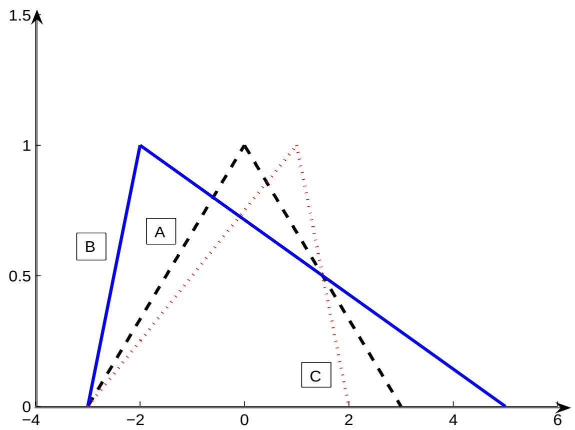

Figure. 3.4: Fuzzy numbers, $\tilde{A} = (-3, 0, 3), \tilde{B} = (-3, -2, 5)$, and $\tilde{C} = (-3, 1, 2)$.

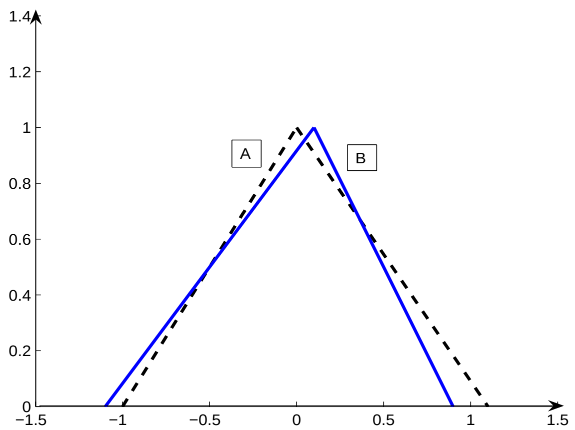

Figure. 3.5: Fuzzy numbers, $\tilde{A} = (-1, 0, 1.1)$ and $\tilde{B} = (-1.1, 0.1, 0.9)$.

Example 3.3.4. *The two triangular fuzzy numbers* $\tilde{A} = (-1, 0, 1.1)$ *and*

$\tilde{B} = (-1.1, 0.1, 0.9)$ *which are shown in Figure 3.5, are ranked by our method. The ranking order is* $IR(\tilde{A}) = IR(\tilde{B}) = 0.4714$.

To compare with other methods we refer the reader to Table 3.3.

3.3.2 Discussion

In this section, we introduced a new approach for ranking fuzzy numbers by improving Cheng's distance method. The improved method overcome the shortcoming in Cheng's distance method, Chu and Tsao's formulae and new revised method by Wang and Lee. In this work, those methods are considered, which utilized the centroid point. The improved method can effectively rank various fuzzy numbers and their images. Thus the method is superior to Cheng's distance, Chu and Tsao's area and Wang and Lee's revised method. The improved method includes all situations of mentioned method. So, it improves the distance method of Cheng as well.

3.4 A new approach for ranking of trapezoidal fuzzy numbers by levels

A new method based on "Distance Minimization" is introduced by Assady and Zendehnam [8]. This method has some drawbacks, i.e., for all triangular fuzzy numbers $\tilde{A} = (x_0, \sigma, \beta)$ where $x_0 = \frac{\sigma - \beta}{4}$ and also trapezoidal fuzzy numbers $\tilde{A} = (x_0, y_0, \sigma, \beta)$, such that $x_0 + y_0 = \frac{\sigma-\beta}{2}$, gives the same results. However, it is clear that these fuzzy numbers do not place in an equivalence class. Some numerical examples are illustrated to show this fact.

In present section, we introduce a new approach which is easy to handle and has a natural interpretation. This new approach applies for ranking of trapezoidal fuzzy numbers. Moreover, we investigate some properties of this method and use some numerical examples to show the advantage of the proposed method and an illogical condition of "Distance Minimization" method.

For an arbitrary trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \sigma, \beta)$, with parametric form $\tilde{A} = \big(\tilde{A}_L(\alpha), \tilde{A}_U(\alpha)\big)$, we define the *magnitude* of the trapezoidal fuzzy number $\tilde{A}$ as

$$Mag_f(\tilde{A}) = \frac{1}{2}\Big(\int_0^1 (\tilde{A}_L(\alpha) + \tilde{A}_U(\alpha) + x_0 + y_0\big)f(\alpha)\mathrm{d}\alpha, \tag{3.4.1}$$

where the function $f(\alpha)$ is a non-negative and increasing function on $[0,1]$ with $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(\alpha)\mathrm{d}\alpha = \frac{1}{2}$. Clearly, function $f(\alpha)$ can be considered as a weighting function. In actual applications, function $f(\alpha)$ can be chosen according to the actual situation. In this section, we use $f(\alpha) = \alpha$. Obviously, the magnitude of a trapezoidal fuzzy number $\tilde{A}$ which is defined by (3.4.1), synthetically reflects the information on every membership degree, and meaning of this magnitude is visual and natural. The resulting scalar value is used to rank the fuzzy numbers. In the other words $Mag_f(\tilde{A})$ is used to rank fuzzy numbers. The larger $Mag_f(\tilde{A})$, the larger fuzzy number. Therefore, for any two trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B} \in \mathbb{F}(\mathbb{R})$, we define the ranking of $\tilde{A}$ and $\tilde{B}$ by the $Mag_f(.)$ on $\mathbb{F}(\mathbb{R})$ as follows:

1. $Mag_f(\tilde{A}) > Mag_f(\tilde{B})$ if and only if $\tilde{A} \succ \tilde{B}$,

2. $Mag_f(\tilde{A}) < Mag_f(\tilde{B})$ if and only if $\tilde{A} \prec \tilde{B}$,

3. $Mag_f(\tilde{A}) = Mag_f(\tilde{B})$ if and only if $\tilde{A} \sim \tilde{B}$.

Then we formulate the order $\succeq$ and $\preceq$ as $\tilde{A} \succeq \tilde{B}$ if and only if $\tilde{A} \succ \tilde{B}$ or $\tilde{A} \sim \tilde{B}$, $\tilde{A} \preceq \tilde{B}$ if and only if $\tilde{A} \prec \tilde{B}$ or $\tilde{A} \sim \tilde{B}$. In the other words, this method is placed in the first class of Kerre's categories [70].

Remark 3.4.1. If $\inf \operatorname{supp}(\tilde{A}) \geq 0$ then $Mag_f(\tilde{A}) \geq 0$.

Remark 3.4.2. If $\sup \operatorname{supp}(\tilde{A}) \leq 0$ then $Mag_f(\tilde{A}) \leq 0$.

Remark 3.4.3. For two arbitrary trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, we have

$$Mag_f(\tilde{A} + \tilde{B}) = Mag_f(\tilde{A}) + Mag_f(\tilde{B}).$$

Remark 3.4.4. For all symmetric trapezoidal fuzzy numbers $\tilde{A} = (-x_0, x_0, \sigma, \sigma)$,

$$Mag_f(\tilde{A}) = 0.$$

Remark 3.4.5. For any two symmetric trapezoidal fuzzy numbers $\tilde{A} = (x_0, y_0, \sigma, \sigma)$ and $\tilde{B} = (x_0, y_0, \beta, \beta)$,

$$Mag_f(\tilde{A}) = Mag_f(\tilde{B}).$$

We recall the following reasonable properties for the ordering approaches, see [70].

(i) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $\tilde{A} \in \Gamma, \tilde{A} \succeq \tilde{A}$.

(ii) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}, \tilde{B}) \in \Gamma^2, \tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{A}$, we should have $\tilde{A} \sim \tilde{B}$.

(iii) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}, \tilde{B}, \tilde{C}) \in \Gamma^3, \tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{C}$, we should have $\tilde{A} \succeq \tilde{C}$.

(iv) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}, \tilde{B}) \in \Gamma^2$, $\inf \operatorname{supp}(\tilde{A}) > \sup \operatorname{supp}(\tilde{B})$, we should have $\tilde{A} \succeq \tilde{B}$.

(iv') For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $(\tilde{A}, \tilde{B}) \in \Gamma^2$, $\inf \operatorname{supp}(\tilde{A}) > \sup \operatorname{supp}(\tilde{B})$, we should have $\tilde{A} \succ \tilde{B}$.

(v) Let Γ and Γ' be two arbitrary finite subsets of $\mathbb{F}(\mathbb{R})$ also $\tilde{A}$ and $\tilde{B}$ are in $\Gamma \cap \Gamma'$. We obtain the ranking order $\tilde{A} \succ \tilde{B}$ by $Mag_f(.)$ on Γ' if and only if $\tilde{A} \succ \tilde{B}$ by $Mag_f(.)$ on Γ.

(vi) Let $\tilde{A}, \tilde{B}$, $\tilde{A}+\tilde{C}$ and $\tilde{B}+\tilde{C}$ be elements of $\mathbb{F}(\mathbb{R})$. If $\tilde{A} \succeq \tilde{B}$, then $\tilde{A}+\tilde{C} \succeq \tilde{B}+\tilde{C}$.

(vi') Let $\tilde{A}$, $\tilde{B}$, $\tilde{A}+\tilde{C}$ and $\tilde{B}+\tilde{C}$ be elements of $\mathbb{F}(\mathbb{R})$. If $\tilde{A} \succ \tilde{B}$, then $\tilde{A}+\tilde{C} \succ \tilde{B}+\tilde{C}$, when $\tilde{C} \neq 0$.

(vii) For an arbitrary finite subset Γ of $\mathbb{F}(\mathbb{R})$ and $\tilde{A} \in \Gamma$, the $Mag_f(\tilde{A})$ must belong to its support.

Remark 3.4.6. The function $Mag_f(.)$ has the properties $(i) - (vi)$.

Proof. It is easy to verify that the properties $(i)-(v)$ are hold. For the proof of (vi) we consider the trapezoidal fuzzy numbers $\tilde{A}=(x_0,y_0,\sigma_0,\beta_0)$, $\tilde{B}=(x_1,y_1,\sigma_1,\beta_1)$ and $\tilde{C}=(x_2,y_2,\sigma_2,\beta_2)$. Let $\tilde{A}\succeq\tilde{B}$, from the relation (5.2.1) we have

$$Mag_f(\tilde{A})\geq Mag_f(\tilde{B}),$$

by adding $Mag_f(\tilde{C})$

$$Mag_f(\tilde{A})+Mag_f(\tilde{C})\geq Mag_f(\tilde{B})+Mag_f(\tilde{C}),$$

and by Remark 3.4.3

$$Mag_f(\tilde{A}+\tilde{C})\geq Mag_f(\tilde{B}+\tilde{C}).$$

Therefore,

$$\tilde{A}+\tilde{C}\succeq\tilde{B}+\tilde{C},$$

which the proof is completed. Similarly (vi') is hold. □

3.4.1 Numerical Examples

Example 3.4.1. *Consider the two triangular fuzzy numbers* $\tilde{A}=(0,1,1)$ *and* $\tilde{B}=(1,5,1)$*, which are indicated in Fig. 3.6.*

Intuitively, the ranking order is $\tilde{A}\prec\tilde{B}$. However, by Distance minimization, the ranking order is $\tilde{A}\sim\tilde{B}$, which is an unreasonable result. By the proposed method, $Mag_f(\tilde{A})=0.0000$ and $Mag_f(\tilde{B})=1.3333$. Therefore, the ranking order is $\tilde{A}\prec\tilde{B}$.

So, our method can overcome the shortcoming of "Distance minimization" method [8].

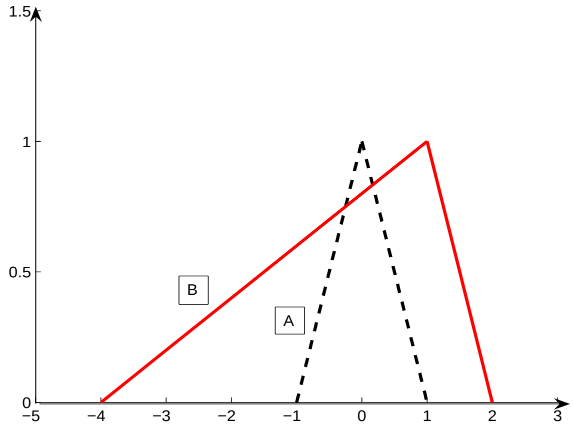

Figure 3.6: Fuzzy numbers $\tilde{A} = (0, 1, 1)$, $\tilde{B} = (1, 5, 1)$.

Example 3.4.2. *Consider the three fuzzy numbers* $\tilde{A} = (1, 5, 1)$, $\tilde{B} = (\frac{1}{4}, 2, 1)$ *and* $\tilde{C} = (2, 9, 1)$, *Fig. 3.7.*

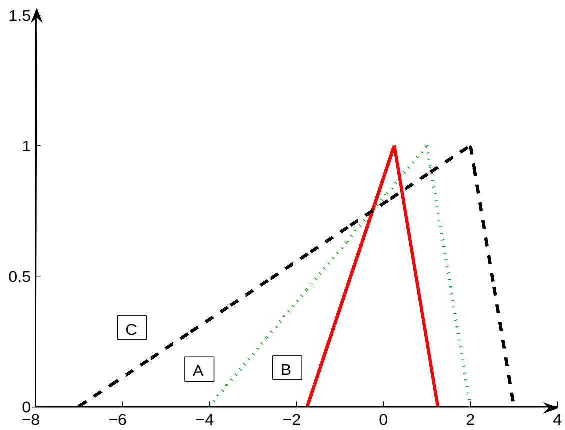

Figure 3.7: Fuzzy numbers, $\tilde{A} = (1, 5, 1)$, $\tilde{B} = (\frac{1}{4}, 2, 1)$ and $\tilde{C} = (2, 9, 1)$

Intuitively, the ranking order is $\tilde{B} \prec \tilde{A} \prec \tilde{C}$. By using our new approach $Mag_f(\tilde{A}) = 0.6666$, $Mag_f(\tilde{B}) = 0.1666$, and $Mag_f(\tilde{C}) = 1.3334$. Hence, the ranking order is $\tilde{B} \prec \tilde{A} \prec \tilde{C}$ too. Obviously, the results obtained by "Distance minimization" method is $\tilde{A} \sim \tilde{B} \sim \tilde{C}$, which is unreasonable. To compare with some of other methods, the reader can refer to Table 3.4.

Example 3.4.3. *The three triangular fuzzy numbers* $\tilde{A} = (0.3, 0.1, 0.2)$, $\tilde{B} = (0.32, 0.15, 0.26)$ *and* $\tilde{C} = (0.4, 0.15, 0.3)$ *taken from paper [22, 24], are ranked by our method (See Fig. 3.8).*

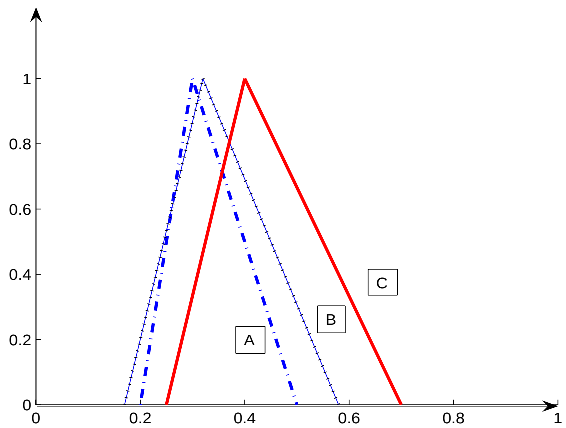

Figure 3.8: Fuzzy numbers $\tilde{A} = (0.3, 0.1, 0.2)$, $\tilde{B} = (0.32, 0.15, 0.26)$ and $\tilde{C} = (0.4, 0.15, 0.3)$.

From our new method $Mag_f(\tilde{A}) = 0.3084$, $Mag_f(\tilde{B}) = 0.3292$ and $Mag_f(C) = 0.4126$, producing the ranking order $\tilde{A} \prec \tilde{B} \prec \tilde{C}$. Furthermore, $Mag_f(-\tilde{A}) =$

-0.3084, $Mag_f(-B) = -0.3292$ and $Mag_f(-C) = -0.4126$, consequently the ranking order of the images of three fuzzy number is $-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$. Clearly, our method has consistency in ranking fuzzy numbers and their images, which could not be guaranteed by Cheng's CV index.

Example 3.4.4. *Consider the following sets, see Yao and Wu [79], which are indicated in Figure 3.9 to 3.12.*

Set 1: $\tilde{A} = (0.5, 0.1, 0.5)$, $\tilde{B} = (0.7, 0.3, 0.3)$, $\tilde{C} = (0.9, 0.5, 0.1)$, *(Fig. 3.9).*

Set 2: $\tilde{A} = (0.4, 0.7, 0.1, 0.2)$, $\tilde{B} = (0.7, 0.4, 0.2)$, $\tilde{C} = (0.7, 0.2, 0.2)$, *(Fig. 3.10).*

Set 3: $\tilde{A} = (0.5, 0.2, 0.2)$, $\tilde{B} = (0.5, 0.8, 0.2, 0.1)$, $\tilde{C} = (0.5, 0.2, 0.4)$, *(Fig. 3.11).*

Set 4: $\tilde{A} = (0.4, 0.7, 0.4, 0.1)$, $\tilde{B} = (0.5, 0.3, 0.4)$, $\tilde{C} = (0.6, 0.5, 0.2)$, *(Fig. 3.12).*

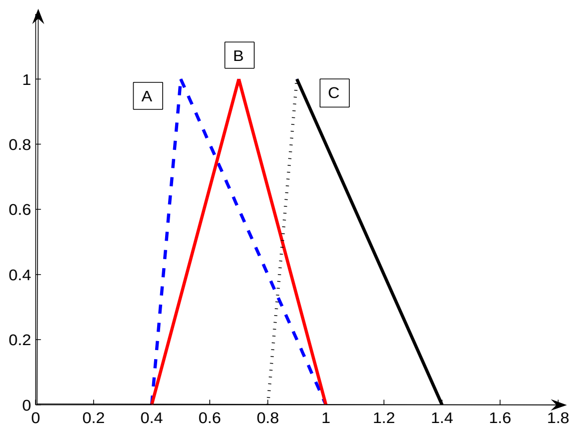

Figure. 3.9: Set 1.

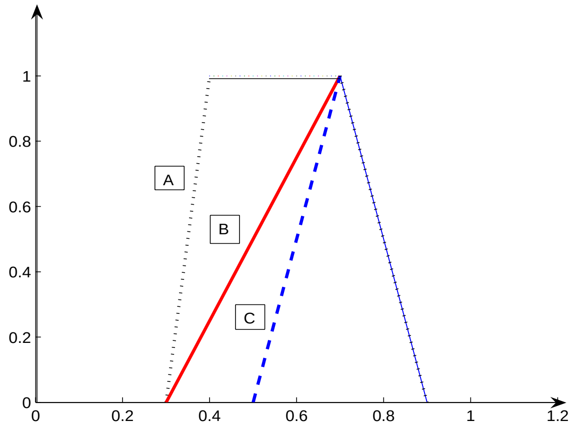

Figure. 3.10: Set 2.

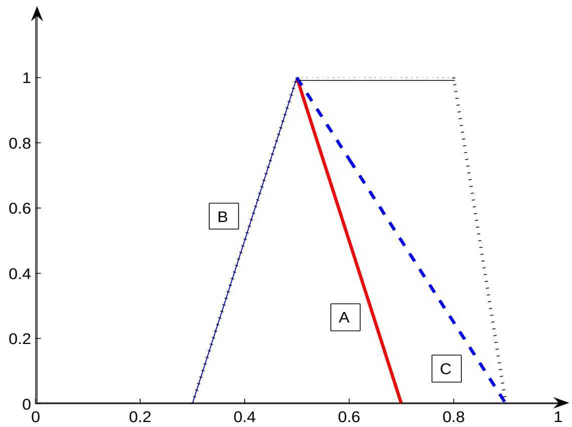

Figure. 3.11: Set 3.

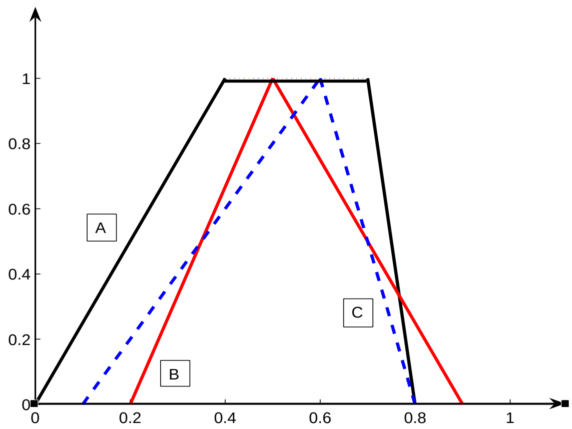

Figure. 3.12: Set 4.

To compare with other method we refer the reader to Table 3.5 (see [3]).

Example 3.4.5. *Let the three triangular fuzzy numbers,* $\tilde{A} = (6, 1, 1)$, $\tilde{B} = (6, 0.1, 1)$ *and* $\tilde{C} = (6, 0, 1)$.

As you consider they have the same right side an intuitively the ranking order is $\tilde{A} \prec \tilde{B} \prec \tilde{C}$. By using our method $Mag_f(\tilde{A}) = 6.0000$, $Mag_f(\tilde{B}) = 6.0750$ and $Mag_f(\tilde{C}) = 6.0834$. Thus the ranking order is $\tilde{A} \prec \tilde{B} \prec \tilde{C}$. As you see in Table 3.6, the results of Chu-Tsao method and Cheng CV index are unreasonable. The results of sign distance method [3] and Cheng distance method, are the same as our new approach.

All the above numerical examples show that the results of proposed method, similar to sign distance method, can overcome the drawbacks of Distance Minimization and CV index.

3.5 Concluding Remarks

In spite of many ranking methods, no one can rank fuzzy numbers with human intuition consistently in all cases. In this chapter, we introduced two new approaches for ranking fuzzy numbers. The first one is an improving on Cheng's distance method. The improved method overcome the shortcoming in Cheng's distance method , Chu and Tsao's formulae and new revised method by Wang and Lee. Then we pointed out the shortcoming of "Distance minimization" and in order to solve the problem we have presented a simple ranking method for trapezoidal fuzzy numbers. The proposed method can effectively rank various fuzzy numbers and their images. Our new ranking method has some mathematical properties. It does not imply much computational effort and does not require a priori knowledge of the set of all alternatives. We also used comparative examples to illustrate the advantage of the proposed method.

Table 3.1 : Comparative results of Example 3.3.1

Approach	$\tilde{A}$	$\tilde{B}$	Result
Centroid point method $(\bar{x}_0\ \bar{y}_0)$	(0.0000, 0.3333)	(0.0484, 0.3000)	
Improved method $IR = \gamma(.)\sqrt{\bar{x}_0^2 + \bar{y}_0^2}$	0.0000	0.3039	$\tilde{A} \prec \tilde{B}$
Chu and Tsao's method $S = \bar{x}_0\bar{y}_0$	0.0000	0.0145	$\tilde{A} \prec \tilde{B}$
Wang and Lee's Revised $\bar{x}_0$	0.0000	0.0484	$\tilde{A} \prec \tilde{B}$
Cheng's distance method $R = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}$	0.3333	0.3039	$\tilde{A} \succ \tilde{B}$

Table 3.2 : Comparative results of Example 3.3.2

Approach	$\tilde{A}$	$\tilde{B}$	$-\tilde{A}$	$-\tilde{B}$	Result
Centroid point ($\bar{x}_0$) method ($\bar{y}_0$)	-0.5625 0.3750	-0.5833 0.3333	0.5625 0.3750	0.5833 0.3333	
Improved method $IR = \gamma(.)\sqrt{\bar{x}_0^2 + \bar{y}_0^2}$	-0.6760	-0.6719	0.6760	0.6719	$\tilde{A} \prec \tilde{B}$ $-\tilde{A} \succ -\tilde{B}$
Chu and Tsao's method $S = \bar{x}_0\bar{y}_0$	-0.2109	-0.1944	0.2109	0.1944	$\tilde{A} \prec \tilde{B}$ $-\tilde{B} \succ -\tilde{A}$
Wang and Lee's Revised $\bar{x}_0$	-0.5625	-0.5833	0.5625	0.5833	$\tilde{A} \succ \tilde{B}$ $-\tilde{A} \prec -\tilde{B}$
Cheng's distance method $R = \sqrt{\bar{x}_0^2 + \bar{y}_0^2}$	0.6760	0.6719	0.6760	0.6719	$\tilde{A} \succ \tilde{B}$ $-\tilde{A} \prec -\tilde{B}$

Table 3.3 : Comparative results of Example 3.3.4

Fuzzy number	Improved Centroid Point	Sign Distance p=1	Sign Distance p=2	Chu and Tsao	Cheng's Distance method
$\tilde{A}$	0.4714	0.0500	0.8583	0.1111	0.4714
$\tilde{B}$	0.0000	0.0000	0.8206	0.1111	0.4714
Results	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \sim \tilde{B}$

Table 3.4 : Comparative results of Example 3.4.3

Fuzzy number	New approach Mag_f $f(\alpha)$	Sign Distance p=1	Sign Distance p=2	Distance minimization
$\tilde{A}$	0.6666	3.2000	2.5820	0.0000
$\tilde{B}$	0.1666	1.5313	1.2417	0.0000
$\tilde{C}$	1.3334	2.9444	4.0414	0.0000
$\tilde{D}$	0.3334	3.000	2.3094	0.0000
Results	$\tilde{B} \prec \tilde{D} \prec \tilde{A} \prec \tilde{C}$	$\tilde{B} \prec \tilde{C} \prec \tilde{D} \prec \tilde{A}$	$\tilde{B} \prec \tilde{D} \prec \tilde{A} \prec \tilde{C}$	$\tilde{A} \sim \tilde{B} \sim \tilde{C} \sim \tilde{D}$

Table 3.5 : Comparative results of Example 3.4.5

Authors	Fuzzy number	set 1	set 2	set 3	set 4
New approach	$\tilde{A}$	0.5334	0.5584	0.5000	0.5250
Mag_f	$\tilde{B}$	0.7000	0.6334	0.6416	0.5084
$f(\alpha)$	$\tilde{C}$	0.8666	0.7000	0.5166	0.5750
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A} \prec \tilde{C}$
Sign Distance	$\tilde{A}$	1.2	1.15	1	0.95
method	$\tilde{B}$	1.4	1.3	1.25	1.05
p=1	$\tilde{C}$	1.6	1.4	1.1	1.05
Results		$A \prec B \prec C$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \sim \tilde{C}$
Sign Distance	$\tilde{A}$	0.8869	0.8756	0.7257	0.7853
method	$\tilde{B}$	1.0194	0.9522	0.9416	0.7958
p=2	$\tilde{C}$	1.1605	1.0033	0.8165	0.8386
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Choobineh	$\tilde{A}$	0.333	0.458	0.333	0.50
and Li	$\tilde{B}$	0.50	0.583	0.4167	0.5833
	$\tilde{C}$	0.667	0.667	0.5417	0.6111
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$

Table 3.5 : (Continued)

Authors	Fuzzy number	set 1	set 2	set 3	set 4
Yager	$\tilde{A}$	0.60	0.575	0.5	0.45
	$\tilde{B}$	0.70	0.65	0.55	0.525
	$\tilde{C}$	0.80	0.7	0.625	0.55
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chen	$\tilde{A}$	0.3375	0.4315	0.375	0.52
	$\tilde{B}$	0.50	0.5625	0.425	0.57
	$\tilde{C}$	0.667	0.625	0.55	0.625
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Baldwin and Guild	$\tilde{A}$	0.30	0.27	0.27	0.40
	$\tilde{B}$	0.33	0.27	0.37	0.42
	$\tilde{C}$	0.44	0.37	0.45	0.42
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \sim \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \sim \tilde{C}$
Chu and Tsao	$\tilde{A}$	0.299	0.2847	0.25	0.24402
	$\tilde{B}$	0.350	0.32478	0.31526	0.26243
		0.3993	0.350	0.27475	0.2619
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$

Table 3.5 : (Continued)

Authors	Fuzzy number	set 1	set 2	set 3	set 4
Yao and Wu	$\tilde{A}$	0.6	0.575	0.5	0.475
	$\tilde{B}$	0.7	0.65	0.625	0.525
	$\tilde{C}$	0.8	0.7	0.55	0.525
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \sim \tilde{C}$
Cheng Distance	$\tilde{A}$	0.79	0.7577	0.7071	0.7106
	$\tilde{B}$	0.8602	0.8149	0.8037	0.7256
	$\tilde{C}$	0.9268	0.8602	0.7458	0.7241
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$
Cheng CV uniform distribution	$\tilde{A}$	0.0272	0.0328	0.0133	0.0693
	$\tilde{B}$	0.0214	0.0246	0.0304	0.0385
	$\tilde{C}$	0.0225	0.0095	0.0275	0.0433
Results		$B \prec C \prec A$	$\tilde{C} \prec \tilde{B} \prec \tilde{A}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{B} \prec \tilde{C} \prec \tilde{A}$
Cheng CV proportional distribution	$\tilde{A}$	0.0183	0.026	0.008	0.0471
	$\tilde{B}$	0.0128	0.0146	0.0234	0.0236
	$\tilde{C}$	0.0137	0.0057	0.0173	0.0255
Results		$\tilde{B} \prec \tilde{C} \prec \tilde{A}$	$\tilde{C} \prec \tilde{B} \prec \tilde{A}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{B} \prec \tilde{C} \prec \tilde{A}$

Chapter 4

Approximation of Fuzzy Numbers

4.1 Introduction

Thousands of scientific papers and many applications have proved that fuzzy set theory let us effectively model and transform imprecise as we mentioned in previous chapters. It is not surprising that fuzzy numbers play a significant role among all fuzzy sets since the predominant are numbers. The results of our calculation strongly depend on the shape of the membership function of these numbers. Less regular membership functions lead to calculations that are more complicated. Moreover, fuzzy numbers with shape of membership functions often have more intuitive and more natural interpretation. All these reasons cause a natural need of simple approximations of fuzzy numbers that are easy to handle and have a natural interpretation. For the sake of simplicity, the trapezoidal or triangular fuzzy numbers are most common in current applications. The importance of the approximation of fuzzy numbers by trapezoidal fuzzy numbers is pointed out in many papers [1, 2, 3, 11, 43, 44, 81, 82, 85].

The symmetric triangular approximation was presented by Ma et al. [55], Chanas [19] derived a formula for determining the interval approximations under the hamming distance. The trapezoidal approximation was proposed by Abbasbandy et al. [1, 2, 3]. Grzegorzewski [43] proposed the trapezoidal approximation of a fuzzy number, which is tread as a reasonable compromise between two opposite tendencies: to loose too much information and to introduce too sophisticated form of approximation from the point of view of computation. The method of the Lagrangian multipliers is used in paper [43] to find the nearest (with respect to a well-known metric between fuzzy numbers) trapezoidal approximation operator preserving the expected interval. Due to some conditions are avoided in the solving, the result of approximation is not always a trapezoidal fuzzy number as it was proved in [7]. Then in the paper [44], the Karush-Kuhn-Tucker theorem is used by the authors to improve the previous nearest trapezoidal approximation operator.

In 2008, Ban [11] pointed out the result is wrong again. Accordingly, he solved in a correct and complete way, the problem of the nearest trapezoidal approximation of a fuzzy number, which preserves the expected interval, in case of the same metric between fuzzy numbers. Other different approximations have also been investigated, such as weighted triangular approximation of fuzzy numbers by Zeng and Li [85].

In 2008, Yeh [81] has showed the approximation of a fuzzy numbers presented by Zeng and Li [85] may fail to be fuzzy numbers. Then he improved trapezoidal and triangular approximation of fuzzy number. These works show that the approximation and ordring of fuzzy numbers are a meaningful topic.

4.2 Preliminaries

Consider two arbitrary fuzzy numbers $\tilde{A}$ and $\tilde{B}$ with $\alpha-$cut sets $[\tilde{A}_L(\alpha), \tilde{A}_U(\alpha)]$ and $[\tilde{B}_L(\alpha), \tilde{B}_U(\alpha)]$, respectively, the equality

$$d(\tilde{A}, \tilde{B}) = \sqrt{\int_0^1 \left(\tilde{A}_L(\alpha) - \tilde{B}_L(\alpha)\right)^2 d\alpha + \int_0^1 \left(\tilde{A}_U(\alpha) - \tilde{B}_U(\alpha)\right)^2 \mathrm{d}\alpha}, \tag{4.2.1}$$

is the distance between $\tilde{A}$ and $\tilde{B}$. For more details we refer the reader to [41].

Zeng and Li [85] introduced the weighted distance for arbitrary fuzzy numbers $\tilde{A}$ and $\tilde{B}$ as follows:

$$d_f(\tilde{A}, \tilde{B}) = \sqrt{\int_0^1 f(\alpha)\Big(\left(\tilde{A}_L(\alpha) - \tilde{B}_L(\alpha)\right)^2 + \left(\tilde{A}_U(\alpha) - \tilde{B}_U(\alpha)\right)^2\Big)\mathrm{d}\alpha}, \tag{4.2.2}$$

where the function $f(\alpha)$ is non-negative and increasing on $[0, 1]$ with $f(0) = 0, f(1) = 1$ and $\int_0^1 f(\alpha)d\alpha = \frac{1}{2}$. The function $f(\alpha)$ is also called weighting function. The property of monotone increasing of function $f(\alpha)$ means that the higher the cut level is, the more important its weight is in determining the distance of fuzzy numbers $\tilde{A}$ and $\tilde{B}$. Both conditions $f(0) = 0$ and $\int_0^1 f(\alpha)\mathrm{d}\alpha = \frac{1}{2}$ ensure that the distance defined by Eq. (4.2.2) is the extension of ordinary distance in $\mathbb{R}$ defined by absolute value. That means, this distance becomes an absolute value in $\mathbb{R}$ when fuzzy number reduces to a real number. In applications, the function $f(\alpha)$ can be chosen according to the actual situation. Clearly, the weighted distance defined by Eq. (4.2.2) synthetically reflects the information on every membership degree, and the importance of the core.

4.2.1 Criteria for approximation

Suppose we want to approximate a fuzzy number by a trapezoidal fuzzy number. Thus, we have to use an operator $T : F(\mathbb{R}) \to F^T(\mathbb{R})$, which transforms a family of all fuzzy numbers into a family of trapezoidal fuzzy numbers, i.e., $T : \tilde{A} \to T(\tilde{A})$. Since trapezoidal approximation could be also performed in many ways, we propose a number of criteria which the approximation operator should or just can possess. Some of the criteria are similar to those specified for defuzzification operators or for interval approximation operators [55]. However, there are some points that have no counterpart in the defuzzification strategies, are formulated strictly for fuzzy numbers and relate exclusively to problems typical for this frame work.

(*i*) $\alpha-$cut invariance

We say that an approximation operator T is α_0-invariant if

$$(T(A))_{\alpha_0} = A_{\alpha_0}. \tag{4.2.3}$$

Zero invariant operator preserves the support of a fuzzy number $\tilde{A}$, i.e.,

$$\text{supp}(T(\tilde{A})) = \text{supp}(\tilde{A}). \tag{4.2.4}$$

Similarly, unity invariant operator preserves the core of a fuzzy number $\tilde{A}$, i.e.,

$$core(T(\tilde{A})) = core(\tilde{A}), \tag{4.2.5}$$

while 0.5-invariant operator preserves a set of values that belong to $\tilde{A}$ to the same extent as they belong to its complement $\neg\tilde{A}$. It is easily seen that specifying two different levels α_1 and α_2 ($\alpha_1 \neq \alpha_2$) we obtain one and only one approximation operator which is both α_1 and α_2-invariant.

(*ii*) Translation invariance

We say that an approximation operator T is invariant to translation if

$$T(\tilde{A} + z) = T(\tilde{A}) + z \quad \forall z \in \mathbb{R}. \tag{4.2.6}$$

Thus, translation invariance means that the relative position of approximation remains constant when the membership function is moved to the left or the right.

(*iii*) Scale invariance

We say that an approximation operator T is scale invariant if

$$T(\lambda\tilde{A}) = \lambda T(\tilde{A}) \quad \forall \lambda \in \mathbb{R}/[0]. \tag{4.2.7}$$

It is worth nothing that for $\lambda = -1$ we get, so called, symmetry constraint, which means that the relative position of the approximation does not vary if the orientation of the support interval changes.

(iv) Identity

The criterion of identity states that the approximation of trapezoidal fuzzy numbers is equivalent to that number, i.e.,

$$if \quad \tilde{A} \in F^T(\mathbb{R}) \quad then \quad T(\tilde{A}) = \tilde{A}. \tag{4.2.8}$$

(v) Monotony

The criterion of monotony states that for any two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ holds,

$$if \quad \tilde{A} \subseteq \tilde{B} \quad then \quad T(\tilde{A}) \subseteq T(\tilde{B}). \tag{4.2.9}$$

4.3 Weighted trapezoidal approximation of a fuzzy number

The idea of the trapezoidal approximation considered by Delgado et al. [26]. In their paper, the authors suggested that the approximation should preserve at least some parameters of the original fuzzy number. There are a lot of methods for approximating fuzzy numbers by trapezoidal form. For example, it suggests itself to substitute a fuzzy number $\tilde{A}$ by $T(\tilde{A})$ described by following four values: $t_1 = \inf \mathrm{supp}\tilde{A}$, $t_2 = \inf core\tilde{A}$, $t_3 = \sup core\tilde{A}$ and $t_4 = \sup \mathrm{supp}\tilde{A}$.

Abbasbandy et al. [1, 2, 3] also presented some approximations of fuzzy numbers. Since one can easily propose many other approximation methods, a natural question arises: how to construct a good approximation operator? Moreover, we face the same problem in defuzzification and interval approximation of fuzzy sets. In addition, in these areas different criteria for "good" operators have been considered. For more details, we refer the reader to [40, 42, 43, 62, 85].

In this section, we use the weighted trapezoidal approximation operator $T : F(\mathbb{R}) \rightarrow F^T(\mathbb{R})$, which produces a trapezoidal fuzzy number, that is the closest to given original fuzzy number and preserves its core with respect to distance d_f defined by (4.2.2). Therefore, this operator will be called the weighted trapezoidal approximation-preserving core of a fuzzy number.

Suppose $\tilde{A}$ is an arbitrary fuzzy number, we will try to find a trapezoidal fuzzy number $T(\tilde{A})$, which is the nearest to $\tilde{A}$, with respect to metric d_f defined by (4.2.2). Let $[T_L(\alpha), T_U(\alpha)]$ denotes the $\alpha-$cut of $T(\tilde{A})$, we would like to minimize

$$d_f(\tilde{A}, T(\tilde{A})) = \sqrt{\int_0^1 f(\alpha)\big(\tilde{A}_L(\alpha) - T_L(\alpha)\big)^2 \mathrm{d}\alpha + \int_0^1 f(\alpha)\big(\tilde{A}_U(\alpha) - T_U(\alpha)\big)^2 \mathrm{d}\alpha}, \tag{4.3.1}$$

with respect to $T_L(\alpha)$ and $T_U(\alpha)$, where function $f(\alpha)$ is weighting function. However, since trapezoidal fuzzy number is completely by four real numbers that are borders of its support and core, our goal reduces to finding such real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$ that characterize $T(\tilde{A}) = (t_1, t_2, t_3, t_4)$. It is easily seen that the $\alpha-$cut of $T(\tilde{A})$ is equal to $\big[t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha\big]$. Therefore, (4.3.1) reduces to

$$d_f(\tilde{A}, T(\tilde{A})) = \Big[\int_0^1 f(\alpha)\Big(\tilde{A}_L(\alpha) - \big(t_1 + (t_2 - t_1)\alpha\big)\Big)^2 \mathrm{d}\alpha + \int_0^1 f(\alpha)\Big(\tilde{A}_U(\alpha) - \big(t_4 - (t_4 - t_3)\alpha\big)\Big)^2 \mathrm{d}\alpha\Big]^{\frac{1}{2}}, \quad (4.3.2)$$

and we will try to minimize (4.3.2) with respect to t_1, t_2, t_3, t_4. To emphasize, we want to find a trapezoidal fuzzy number, which is not only closest to given fuzzy number but which preserves the core of that fuzzy number.

It is easily seen that in order to minimize $d_f(\tilde{A}, T(\tilde{A}))$ it suffices function $d_f^2(\tilde{A}, T(\tilde{A}))$ with respect to following conditions;

$core(\tilde{A}) = core(T(\tilde{A}))$ or $t_2 = \tilde{A}_L(1)$, and $t_3 = \tilde{A}_U(1)$. Using well-known method of the Lagrangian multipliers, our problem reduces to find such real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$ that minimize

$$\begin{aligned} D(t_1, t_2, t_3, t_4) = &\int_0^1 f(\alpha)\Big(\tilde{A}_L(\alpha) - \big(t_1 + (t_2 - t_1)\big)\alpha\Big)^2 \mathrm{d}\alpha \\ &+ \int_0^1 f(\alpha)\Big(\tilde{A}_U(\alpha) - \big(t_4 - (t_4 - t_3)\alpha\big)\Big)^2 \mathrm{d}\alpha \\ &+ \lambda_1 f(\alpha)\big(t_2 - \tilde{A}_L(1)\big) + \lambda_2 f(\alpha)\big(t_3 - \tilde{A}_U(1)\big), \end{aligned} \quad (4.3.3)$$

where λ_1 and λ_2 are real numbers, called the Lagrangian multipliers.

Consequently, we can get their partial derivatives

$$\begin{cases} \frac{\partial D(t_1,t_2,t_3,t_4)}{\partial t_1} = 2\int_0^1 f(\alpha)\Big(\tilde{A}_L(\alpha) - \big(t_1 + (t_2 - t_1)\alpha\big)\Big)(\alpha - 1)\mathrm{d}\alpha, \\ \\ \frac{\partial D(t_2,t_2,t_3,t_4)}{\partial t_2} = 2\int_0^1 f(\alpha)\Big(\tilde{A}_L(\alpha) - \big(t_1 + (t_2 - t_1)\alpha\big)\Big)(-\alpha)\mathrm{d}\alpha - \lambda_1 f(\alpha), \\ \\ \frac{\partial D(t_1,t_2,t_3,t_4)}{\partial t_3} = 2\int_0^1 f(\alpha)\Big(\tilde{A}_U(\alpha) - \big(t_4 - (t_4 - t_3)\alpha\big)\Big)(-\alpha)\mathrm{d}\alpha - \lambda_2 f(\alpha), \\ \\ \frac{\partial D(t_1,t_2,t_3,t_4)}{\partial t_4} = 2\int_0^1 f(\alpha)\Big(\tilde{A}_U(\alpha) - \big(t_4 - (t_4 - t_3)\alpha\big)\Big)(\alpha - 1)\mathrm{d}\alpha. \end{cases} \tag{4.3.4}$$

Let $\frac{\partial D(t_1,t_2,t_3,t_4)}{\partial t_1} = \frac{\partial D(t_2,t_2,t_3,t_4)}{\partial t_2} = \frac{\partial D(t_1,t_2,t_3,t_4)}{\partial t_3} = \frac{\partial D(t_1,t_2,t_3,t_4)}{\partial t_4} = 0,$

then we solve

$$\begin{cases} t_1\int_0^1 (\alpha-1)^2 f(\alpha)\mathrm{d}\alpha - t_2\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha = -\int_0^1 (\alpha-1)\tilde{A}_L(\alpha)f(\alpha)\mathrm{d}\alpha, \\ \\ -t_1\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha + t_2\int_0^1 \alpha^2 f(\alpha)\mathrm{d}\alpha - \frac{\lambda_1 f(\alpha)}{2} = \int_0^1 \alpha\tilde{A}_L(\alpha)f(\alpha)\mathrm{d}\alpha, \\ \\ t_3\int_0^1 \alpha^2 f(\alpha)d\alpha - t_4\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha - \frac{\lambda_2 f(\alpha)}{2} = \int_0^1 \alpha\tilde{A}_U(\alpha)f(\alpha)\mathrm{d}\alpha, \\ \\ -t_3\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha + t_4\int_0^1 (\alpha-1)^2 f(\alpha)\mathrm{d}\alpha = -\int_0^1 (\alpha-1)\tilde{A}_U(\alpha)f(\alpha)\mathrm{d}\alpha. \end{cases} \tag{4.3.5}$$

The solution is

$$\begin{cases} t_1 = \frac{-\int_0^1(\alpha-1)\tilde{A}_L(\alpha)f(\alpha)\mathrm{d}\alpha+\tilde{A}_L(1)\int_0^1\alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha}{\int_0^1(\alpha-1)^2f(\alpha)\mathrm{d}\alpha}, \\ t_2 = A_L(1), \\ t_3 = A_U(1), \\ t_4 = \frac{-\int_0^1(\alpha-1)\tilde{A}_U(\alpha)f(\alpha)\mathrm{d}\alpha+\tilde{A}_U(1)\int_0^1\alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha}{\int_0^1(\alpha-1)^2f(\alpha)\mathrm{d}\alpha}. \end{cases} \tag{4.3.6}$$

Considering weighting function $f(\alpha) = \alpha$, then we have $\int_0^1 f(\alpha)d\alpha = \frac{1}{2}, \int_0^1 \alpha(\alpha - 1)f(\alpha)\mathrm{d}\alpha = \frac{-1}{12}$ and $\int_0^1(\alpha-1)^2f(\alpha)\mathrm{d}\alpha = \frac{1}{12}$. Therefore, the solution in Eqs. (4.3.6) can also express as follows:

$$\begin{cases} t_1 = -12\int_0^1 \alpha(\alpha-1)\tilde{A}_L(\alpha)\mathrm{d}\alpha - \tilde{A}_L(1), \\ t_2 = \tilde{A}_L(1), \\ t_3 = \tilde{A}_U(1), \\ t_4 = -12\int_0^1 \alpha(\alpha-1)\tilde{A}_U(\alpha)\mathrm{d}\alpha - \tilde{A}_U(1). \end{cases} \tag{4.3.7}$$

Moreover, in the case that $f(\alpha) = 1$, then we have $\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha = \frac{-1}{6}$ and

$\int_0^1 (\alpha-1)^2 f(\alpha)\mathrm{d}\alpha = \frac{1}{3}$. Then, the solution in Eqs. (4.3.6) will be summarized as,

$$\begin{cases} t_1 = -3\int_0^1 (\alpha-1)\tilde{A}_L(\alpha)\mathrm{d}\alpha - \frac{1}{2}\tilde{A}_L(1), \\ \\ t_2 = \tilde{A}_L(1), \\ \\ t_3 = \tilde{A}_U(1), \\ \\ t_4 = -3\int_0^1 (\alpha-1)\tilde{A}_U(\alpha)\mathrm{d}\alpha - \frac{1}{2}\tilde{A}_U(1). \end{cases} \tag{4.3.8}$$

To obtain good results, we must prove that the system (4.3.6) is a fuzzy number. For that reason, we follow next theorem.

Theorem 4.3.1. *The system (4.3.6) determines a fuzzy number. In other words, we have the following conditions*

$$t_1 \le t_2, \quad t_2 \le t_3, \quad t_3 \le t_4.$$

Proof. Since $A_L(\alpha)$ is non-decreasing for all $\alpha \in [0,1]$, hence

$$\tilde{A}_L(\alpha) \le \tilde{A}_L(1),$$

from $f(\alpha) \ge 0$ and $(1-\alpha) \succeq 0, \quad \forall \alpha \in [0,1]$, we can get that

$$(1-\alpha)\tilde{A}_L(\alpha)f(\alpha) \le (1-\alpha)\tilde{A}_L(1)f(\alpha),$$

$$(1-\alpha)\tilde{A}_L(\alpha)f(\alpha) \leq (1-\alpha+\alpha^2-\alpha^2+\alpha-\alpha)\tilde{A}_L(1)f(\alpha),$$

then

$$-(\alpha-1)\tilde{A}_L(\alpha)f(\alpha) \leq \big[(\alpha-1)^2-\alpha(\alpha-1)\big]\tilde{A}_L(1)f(\alpha).$$

By the theorem of integration we have

$$\int_0^1 -(\alpha-1)\tilde{A}_L(\alpha)f(\alpha)\mathrm{d}\alpha \leq \int_0^1 [(\alpha-1)^2-\alpha(\alpha-1)]\tilde{A}_L(1)f(\alpha)\mathrm{d}\alpha,$$

$$-\int_0^1 (\alpha-1)\tilde{A}_L(\alpha)f(\alpha)\mathrm{d}\alpha \leq \int_0^1 (\alpha-1)^2\tilde{A}_L(1)f(\alpha)\mathrm{d}\alpha - \int_0^1 \alpha(\alpha-1)\tilde{A}_L(1)f(\alpha)\mathrm{d}\alpha.$$

Consequently,

$$-\int_0^1 (\alpha-1)\tilde{A}_L(\alpha)f(\alpha)\mathrm{d}\alpha + \tilde{A}_L(1)\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha \leq \tilde{A}_L(1)\int_0^1 (\alpha-1)^2f(\alpha)\mathrm{d}\alpha,$$

since $\int_0^1(\alpha-1)^2f(\alpha)\mathrm{d}\alpha > 0$, the following will be obtained

$$\frac{-\int_0^1(\alpha-1)\tilde{A}_L(\alpha)f(\alpha)\mathrm{d}\alpha + \tilde{A}_L(1)\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha}{\int_0^1(\alpha-1)^2f(\alpha)\mathrm{d}\alpha} \leq \tilde{A}_L(1),$$

it means

$$t_1 \leq t_2.$$

It is evident that

$$t_2 \leq t_3.$$

Similarly, we can show that $t_3 \leq t_4$.□

Theorem 4.3.2. $\left(\int_0^1 \alpha(1-\alpha)f(\alpha)\mathrm{d}\alpha \right)^2 \leq \int_0^1(1-\alpha)^2f(\alpha)\mathrm{d}\alpha \times \int_0^1 \alpha^2f(\alpha)\mathrm{d}\alpha.$

Proof. We can write

$$\left(\int_0^1 \alpha(1-\alpha)f(\alpha)\mathrm{d}\alpha \right)^2 = \left(\int_0^1 \alpha\sqrt{f(\alpha)}(1-\alpha)\sqrt{f(\alpha)}\mathrm{d}\alpha \right)^2,$$

from Schwartz's inequality, we get that

$$\left(\int_0^1 \alpha(1-\alpha)f(\alpha)\mathrm{d}\alpha \right)^2 \leq \int_0^1 \alpha^2 f(\alpha)\mathrm{d}\alpha \times \int_0^1 (1-\alpha)^2 f(\alpha)\mathrm{d}\alpha. \square$$

On the other hand, we can get the Hessian matrix,

$$H = \left[H_{ij}\right], \quad i,j = 1,2,3,4,$$

which

$$H_{ij} = \frac{\partial^2 D(t_1,t_2,t_3,t_4)}{\partial t_i \partial t_j} \quad i,j = 1,2,3,4.$$

Therefore,

$$H_{11} = H_{44} = 2\int_0^1 (\alpha-1)^2 f(\alpha)\mathrm{d}\alpha$$

$$H_{12} = H_{21} = H_{34} = H_{43} = -2\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha$$

$$H_{22} = H_{33} = 2\int_0^1 \alpha^2 f(\alpha)\mathrm{d}\alpha$$

$$H_{13} = H_{14} = H_{23} = H_{24} = H_{31} = H_{32} = H_{41} = H_{42} = 0$$

hence,

$$\det H = H_{11}\det M - H_{12}\det N$$

such that

$$M = \begin{pmatrix} H_{22} & 0 & 0 \\ 0 & H_{33} & H_{34} \\ 0 & H_{43} & H_{44} \end{pmatrix},$$

$$N = \begin{pmatrix} H_{21} & 0 & 0 \\ 0 & H_{33} & H_{34} \\ 0 & H_{43} & H_{44} \end{pmatrix}.$$

It is obvious to see that

$$H_{11} = 2\int_0^1 (\alpha - 1)^2 f(\alpha)\mathrm{d}\alpha > 0$$

and

$$H_{22} = 2\int_0^1 \alpha^2 f(\alpha)\mathrm{d}\alpha > 0$$

moreover, by Theorem 4.3.2 $H_{33}.H_{44} - H_{34}.H_{43} > 0$, in other words,

$$4\int_0^1 (\alpha - 1)^2 f(\alpha)\mathrm{d}\alpha \times \int_0^1 \alpha^2 f(\alpha)\mathrm{d}\alpha - \left(\; 2\int_0^1 \alpha(1-\alpha) f(\alpha)\mathrm{d}\alpha \; \right)^2 > 0.$$

Consequently, $\det H > 0$.

Furthermore, in the case that $f = 1$,

$$H \;=\; \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix},$$

so, $\det H = \frac{1}{9} > 0$.

They show that t_1, t_2, t_3 and t_4 given by Eqs. (4.3.6) minimize $D = (t_1, t_2, t_3, t_4)$ and actually minimize $d = (t_1, t_2, t_3, t_4)$ simultaneously.

Example 4.3.1. *Considering the Gaussian membership function* $\tilde{A}(x) = e^{-(x-\mu_0)^2/\sigma_0^2}$, *then we have its* $\alpha-$*cut set* $\tilde{A}_\alpha = \left[\mu_0 - \sigma_0\sqrt{-\ln\alpha},\ \mu_0 + \sigma_0\sqrt{-\ln\alpha}\right], \quad \alpha \in [0,1]$, *[85].*

Applying Eqs. (4.3.7), then the weighted approximation-preserving core $T(\tilde{A})$ of the fuzzy number $\tilde{A}$ is characterized by

$$\begin{cases} t_1 = -12\int_0^1 \alpha(\alpha-1)(\mu_0 - \sigma_0\sqrt{-\ln\alpha})\mathrm{d}\alpha - \mu_0 = \mu_0 - \frac{(9\sqrt{2}-4\sqrt{3})\pi}{6}\sigma_0, \\ t_2 = \mu_0, \\ t_3 = \mu_0, \\ t_4 = -12\int_0^1 \alpha(\alpha-1)(\mu_0 - \sigma_0\sqrt{-\ln\alpha})\mathrm{d}\alpha - \mu_0 = \mu_0 + \frac{(9\sqrt{2}-4\sqrt{3})\pi}{6}\sigma_0. \end{cases}$$

The results are the same as Zeng and Li's [85]. Fig. 4.1 indicates the weighted trapezoidal approximation-preserving core for a Gaussian membership function with mean 3 and deviation 0.9.

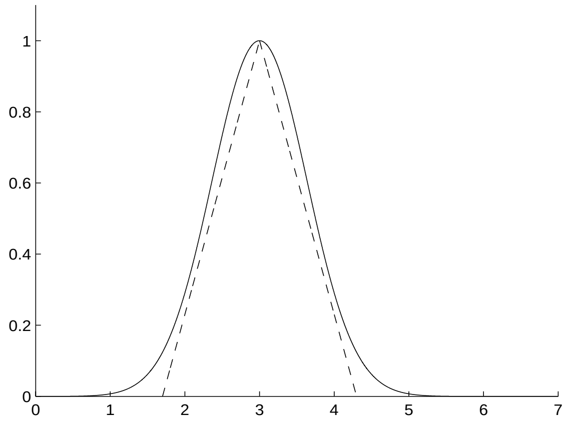

Figure. 4.1

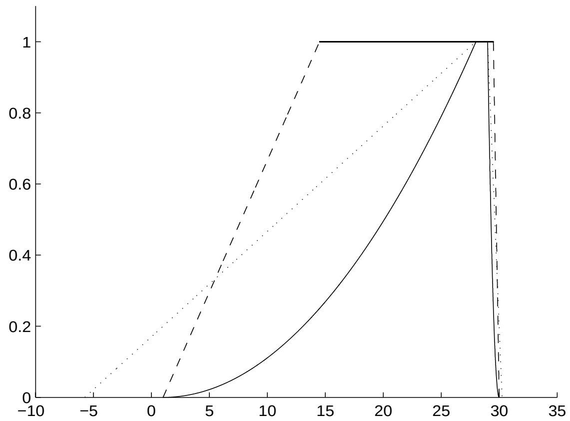

Figure. 4.2

Let us considering the fuzzy number $\tilde{A}(x) = (1, 28, 29, 30)_2$ or in the parametric form, $\tilde{A}_\alpha = \left[1 + 27\sqrt{\alpha}, 30 - \sqrt{\alpha}\right], \quad \alpha \in [0, 1]$, [11].

By using Eqs. (4.3.7) $T(\tilde{A}) = (-\frac{23}{4}, 28, 29, \frac{121}{4})$. However, by Zeng and Li's is not a triangular fuzzy number. In addition, as Ban [11] pointed out by Grzegorzewski's proposed $T(\tilde{A})$ is also not a trapezoidal fuzzy number and from conditions in his work $T(\tilde{A}) = (1, \frac{29}{2}, \frac{59}{2}, 30)$. Fig. 4.2 shows the fuzzy number $\tilde{A}$ and the trapezoidal approximations by proposed method and Ban's method [11].

4.3.1 Properties of weighted approximation-preserving core

In this part, some properties of the approximation including translation invariance, scale invariance, identity and continuity will be proved.

Theorem 4.3.3. *The weighted trapezoidal approximation operator-preserving core is invariant to translation.*

Proof. Let z be a real number and $\tilde{A}$ denotes a fuzzy number with $\alpha-$cut $\tilde{A}_\alpha = \big[\tilde{A}_L(\alpha), \tilde{A}_U(\alpha)\big]$. Then the $\alpha-$cut of a fuzzy number $\tilde{A}$ translated by a number z is

$$(\tilde{A} + z)_\alpha = \big[\tilde{A}_L(\alpha) + z, \tilde{A}_U(\alpha) + z\big].$$

accordingly, from Eqs. (4.3.6) we can get that

$$t_1(\tilde{A} + z) = \frac{-\int_0^1 (\alpha - 1)\big(\tilde{A}_L(\alpha) + z\big) f(\alpha) \mathrm{d}\alpha + \big(\tilde{A}_L(1) + z\big) \int_0^1 \alpha(\alpha - 1) f(\alpha) \mathrm{d}\alpha}{\int_0^1 (\alpha - 1)^2 f(\alpha) \mathrm{d}\alpha}$$

$$= \frac{-\int_0^1 (\alpha - 1)\tilde{A}_L(\alpha) f(\alpha) \mathrm{d}\alpha + \tilde{A}_L(1) \int_0^1 \alpha(\alpha - 1) f(\alpha) d\alpha + z \int_0^1 (\alpha - 1)^2 f(\alpha) \mathrm{d}\alpha}{\int_0^1 (\alpha - 1)^2 f(\alpha) \mathrm{d}\alpha}.$$

Consequently,

$$t_1(\tilde{A} + z) = t_1(\tilde{A}) + z.$$

Similar computations can be done for $t_i(\tilde{A}+z), i = 2, 3, 4$.

Since $t_i(\tilde{A}+z) = t_i(\tilde{A})+z$ for $i = 1, 2, 3, 4$, then we can get that $T(\tilde{A}+z) = T(\tilde{A})+z$, which proves the translation invariance. □

Theorem 4.3.4. *The weighted trapezoidal approximation operator-preserving core is invariant to scale.*

Proof. Consider a real number λ such that $\lambda \neq 0$. Using Eqs. (4.3.6), $T(\lambda\tilde{A})$ will be described as follows:

$$t_1(\lambda\tilde{A}) = \frac{-\int_0^1 (\alpha-1)\lambda\tilde{A}_L(\alpha)f(\alpha)\mathrm{d}\alpha + \lambda\tilde{A}_L(1)\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha}{\int_0^1 (\alpha-1)^2 f(\alpha)\mathrm{d}\alpha} = \lambda t_1(\tilde{A}).$$

Similarly it will be obtained $t_2(\lambda A) = \lambda t_2(A),\ t_3(\lambda\tilde{A}) = \lambda t_3(\tilde{A})$ and $t_4(\lambda\tilde{A}) = \lambda t_4(\tilde{A})$. Then we have $T(\lambda\tilde{A}) = \lambda T(\tilde{A})$, which proves the scale invariance. □

Theorem 4.3.5. *The weighted trapezoidal approximation operator-preserving core satisfies the identity property.*

Proof. Suppose a trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ where $a \leq b \leq c \leq d$ and its $\alpha-$cut set $\tilde{A}(\alpha) = \left[a+(b-a)\alpha, d-(d-c)\alpha\right]$. Since $t_2(\tilde{A}) = \tilde{A}_L(1) = b,\ t_3(\tilde{A}) = \tilde{A}_U(1) = c$, it is sufficient to prove $t_1(\tilde{A}) = a,\ t_4(\tilde{A}) = d$. From Eqs. (4.3.6), the weighted trapezoidal approximation $T(\tilde{A})$ is determined in the following

$$t_1(\tilde{A}) = \frac{-\int_0^1(\alpha-1)\big(a+(b-a)\alpha\big)f(\alpha)\mathrm{d}\alpha + b\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha}{\int_0^1(\alpha-1)^2 f(\alpha)\mathrm{d}\alpha},$$

$$= \frac{-a\int_0^1(\alpha-1)f(\alpha)\mathrm{d}\alpha - (b-a)\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha + b\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha}{\int_0^1(\alpha-1)^2 f(\alpha)\mathrm{d}\alpha},$$

$$= \frac{-a\int_0^1(\alpha-1)f(\alpha)\mathrm{d}\alpha + a\int_0^1 \alpha(\alpha-1)f(\alpha)\mathrm{d}\alpha}{\int_0^1(\alpha-1)^2 f(\alpha)\mathrm{d}\alpha}.$$

Hence, $t_1(\tilde{A}) = a$, and $t_4(\tilde{A}) = d$ can be proved as the same before. Therefore, $T(\tilde{A}) = \tilde{A} = (a, b, c, d)$, and the weighted trapezoidal approximation satisfies the identity property. □

Grzegorzewski [40] utilized the continuity of mappings of fuzzy sets to describe properties of fuzzy sets transformations and introduced continuity to describe approximation operation. For $\tilde{A}, \tilde{B} \in \mathbb{R}$, $T(\tilde{A})$ and $T(\tilde{B})$ denote trapezoidal approximation of two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively. A mapping T between fuzzy sets is continuous, if

$$\forall\varepsilon > 0,\ \exists\delta > 0,\ d(\tilde{A}, \tilde{B}) < \delta \Rightarrow d(T(\tilde{A}), T(\tilde{B})) < \varepsilon,$$

where d is a metric defined on fuzzy number space.

Definition 4.3.1. The weighted trapezoidal approximation operator-preserving core is called continuous, if it satisfies the aforementioned condition for any $\tilde{A}, \tilde{B} \in \mathbb{R}$.

Theorem 4.3.6. *The weighted trapezoidal approximation operator-preserving core is continuous.*

Proof. Since $T_L^{\tilde{A}}(\alpha) = t_1(\tilde{A}) + \big(t_2(\tilde{A}) - t_1(\tilde{A})\big)\alpha$ and $T_U^{\tilde{A}}(\alpha) = t_4(\tilde{A}) - \big(t_4(\tilde{A}) - t_3(\tilde{A})\big)\alpha$, by Eq. (4.3.8) we can get that

$$d^2\big(T(\tilde{A}), T(\tilde{B})\big) = \int_0^1 \Big[T_L^{\tilde{A}}(\alpha) - T_L^{\tilde{B}}(\alpha)\Big]^2 \mathrm{d}\alpha + \int_0^1 \Big[T_U^{\tilde{A}}(\alpha) - T_U^{\tilde{B}}(\alpha)\Big]^2 \mathrm{d}\alpha$$

$$= \int_0^1 \Big[3(\alpha - 1)\int_0^1 (\gamma - 1)\big(\tilde{A}_L(\gamma) - \tilde{B}_L(\gamma)\big)\mathrm{d}\gamma + \frac{3\alpha - 1}{2}\big(\tilde{A}_L(1) - \tilde{B}_L(1)\big)\Big]^2 \mathrm{d}\alpha$$

$$+ \int_0^1 \Big[3(\alpha - 1)\int_0^1 (\gamma - 1)\big(\tilde{A}_U(\gamma) - \tilde{B}_U(\gamma)\big)\mathrm{d}\gamma + \frac{3\alpha - 1}{2}\big(\tilde{A}_U(1) - \tilde{B}_U(1)\big)\Big]^2 \mathrm{d}\alpha.$$

Now, for abbreviation, let us adopt for a moment a following notation;

$$\Upsilon_L = \int_0^1 (\gamma - 1)\big(\tilde{A}_L(\gamma) - \tilde{B}_L(\gamma)\big)\mathrm{d}\gamma$$

$$\Upsilon_U = \int_0^1 (\gamma - 1)\big(\tilde{A}_U(\gamma) - \tilde{B}_U(\gamma)\big)\mathrm{d}\gamma.$$

Thus, we have

$$d^2\big(T(\tilde{A}),T(\tilde{B})\big) = \int_0^1 \Big[3(\alpha-1)\Upsilon_L + \frac{3\alpha-1}{2}\big(\tilde{A}_L(1)-\tilde{B}_L(1)\big)\Big]^2 \mathrm{d}\alpha$$

$$+\int_0^1 \Big[3(\alpha-1)\Upsilon_U + \frac{3\alpha-1}{2}\big(\tilde{A}_U(1)-\tilde{B}_U(1)\big)\Big]^2 \mathrm{d}\alpha$$

$$= 3\Upsilon_L^2 + \frac{1}{4}\big(\tilde{A}_L(1)-\tilde{B}_L(1)\big)^2 + 3\Upsilon_U^2 + \frac{1}{4}\big(\tilde{A}_U(1)-\tilde{B}_U(1)\big)^2.$$

Going back to our original notation it will be obtained

$$d^2\big(T(\tilde{A}),T(\tilde{B})\big) = 3\Big[\int_0^1 (\gamma-1)\big(\tilde{A}_L(\gamma)-\tilde{B}_L(\gamma)\big)\mathrm{d}\gamma\Big]^2 + \frac{1}{4}\big(\tilde{A}_L(1)-\tilde{B}_L(1)\big)^2$$

$$+3\Big[\int_0^1 (\gamma-1)\big(\tilde{A}_U(\gamma)-\tilde{B}_U(\gamma)\big)\mathrm{d}\gamma\Big]^2 + \frac{1}{4}\big(\tilde{A}_U(1)\tilde{B}_U(1)\big)^2$$

$$\leq 3\int_0^1 (\gamma-1)^2\big(\tilde{A}_L(\gamma)-\tilde{B}_L(\gamma)\big)^2\mathrm{d}\gamma + \frac{1}{4}\big(\tilde{A}_L(1)-\tilde{B}_L(1)\big)^2$$

$$+3\int_0^1 (\gamma-1)^2\big(\tilde{A}_U(\gamma)-\tilde{B}_U(\gamma)\big)^2\mathrm{d}\gamma + \frac{1}{4}\big(\tilde{A}_U(1)-\tilde{B}_U(1)\big)^2$$

$$\leq 3\int_0^1 \Big[\big(\tilde{A}_L(\gamma)-\tilde{B}_L(\gamma)\big)^2 + \big(\tilde{A}_U(\gamma)-\tilde{B}_U(\gamma)\big)^2\Big]\mathrm{d}\gamma = 3d^2(\tilde{A},\tilde{B}).$$

It means $\forall \varepsilon > 0,\ \exists \delta = \frac{\sqrt{3}}{3}\varepsilon > 0$, when $d(\tilde{A}, \tilde{B}) < \delta$, then we have $d\big(T(\tilde{A}), T(\tilde{B})\big) < \varepsilon$. It shows that our weighted trapezoidal approximation-preserving core is continuous. □

4.4 Concluding Remarks

In this chapter, we used ordinary distance between two fuzzy numbers to investigate a trapezoidal approximation of an arbitrary fuzzy number. The proposed operator, called the weighted trapezoidal approximation operator-preserving core. A satisfactory approximation operator should be easy to implement, computational inexpensive and should have convenience interpretation. We have formulated a list of criteria, which would be desirable for approximation operators to possess. We also discuss some properties of the approximation including translation invariance, scale invariance, identity and continuity. The advantage is that the proposed method is simpler than other methods computationally and natural.

Chapter 5

Fuzzy Distance

5.1 Introduction

As we discussed in previous chapters, ranking or ordering of fuzzy numbers is a fundamental problem of fuzzy optimization or fuzzy decision-making. Since fuzzy numbers do not form a natural linear order, like real numbers, different comparison approaches are used. As we mentioned, many methods of ranking fuzzy numbers have been proposed in the literature . Each method has its own advantages and disadvantages and it would be a hard task to decide which one is the best. Therefore, each time the ordering method should be chosen to the particular problem.

Although, so many defuzzification methods have been proposed so far, no one method gives a right effective defuzzification output. The computational results of these methods are often conflict. We often face difficultly in selecting appropriate defuzzification methods for some specific application problems. Most of the existing defuzzification methods have tried to make the estimation of a fuzzy set in an objective way. However, an important aspect of the fuzzy set application is that it can represent the subjective knowledge of the decision maker; different decision makers may have different perception for the defuzzification results.

All of the aforementioned methods were used real numbers to rank fuzzy numbers. Other methods introduced a crisp number as distance between two fuzzy numbers, and then it was utilized to rank them [41]. A natural question at this stage is whether it is reasonable to define a crisp distance between fuzzy objects. If we are not certain about the numbers themselves, how can we be 'certain' about the distances among them. Several authors have considered this question. In 1997, Voxman [68] considered some possible approaches to assign a distance between fuzzy numbers. A pseudo-metric on the set of fuzzy numbers arising from the idea of the value of a fuzzy number described. In addition, it was shown that the proposed fuzzy distance has some, but not all of the characteristics of a usual metric. In 2008, Chen and Wang [21] introduced a fuzzy distance of trapezoidal fuzzy numbers by using graded mean integration representation of generalized fuzzy numbers and the span of fuzzy number. Furthermore, they discussed the distance of the linguistic data of 'greater or less than x' and 'about x'.

The present study has been motivated by the authors' interest in making decision in fuzzy environment. In present chapter, we attempt to combine some commonly used defuzzification methods to define some new definition into a certain uniform.

The reminder of this chapter includes the following three parts: In Part 1 we first describe positive fuzzy numbers, negative fuzzy numbers and fuzzy zero to represent some new definitions. Part 2 discusses equality and inequality of fuzzy numbers based on part 1 and some useful properties. The concept of fuzzy absolute value is proposed in the third part. In Section 4, the point of view is taken that if numbers are fuzzy, then it is reasonable to assume that the 'distance' between these numbers is also fuzzy. The aforementioned concepts use to produce a new distance between two fuzzy numbers as a trapezoidal fuzzy number. A fuzzy 'metric' exhibiting all of the properties of normal metrics is defined. The metric properties are also studied. Several examples demonstrate these ideas and finally, concluding remarks are given in the end of this chapter.

5.2 Some new algebraic properties of fuzzy numbers

Throughout this chapter, we concentrate our attention on the trapezoidal fuzzy numbers. It recalls most of the definitions and properties hold for all fuzzy numbers. In the first part of this section, we are going to describe the concept of the positive, negative fuzzy number and fuzzy zero, which is different from what has been already defined. In other words, we will divide the fuzzy numbers to positive fuzzy numbers, negative fuzzy numbers and fuzzy zero number. So far, it is said if the support of a fuzzy number is positive (negative) then it is a positive fuzzy number. It will be proposed an easy way to distinguish positive and negative fuzzy number from each other. In the second subsection, it will be presented a partial ordering and the last part of this section, in order to define a metric for $\mathbb{F}(\mathbb{R})$ we introduce the concept of the fuzzy absolute value of a fuzzy number.

5.2.1 Positive (negative) trapezoidal fuzzy numbers and fuzzy zero number

In the point of our view, a fuzzy number is considered as a positive (negative) fuzzy number if its centroid point ($\bar{x}_0$'s value) be positive (negative). Therefore, in the case that the $\bar{x}_0 = 0$, this fuzzy number equals to fuzzy zero number. This is consistent with our intuitive judgment. To further, justify this idea, we present in the following figure from the point of view of analytical geometry.

Now, it is convenient to give the definitions of positive (negative) fuzzy number and fuzzy zero number as follow:

Definition 5.2.1. Let $\bar{x}_0(\tilde{A})$ be the representative location of fuzzy number $\tilde{A}$ on the real line which is indicated by formulae (2.3.10) in Chapter 2. Then $\tilde{A}$ is said to be zero ($\tilde{A} \approx \tilde{0}$), positive ($\tilde{A} \succ \tilde{0}$) or negative ($\tilde{A} \prec \tilde{0}$) fuzzy number if $\bar{x}_0(\tilde{A}) = 0$, > 0 or < 0, respectively.

Remark 5.2.1. Let $\tilde{A}$ be a trapezoidal fuzzy number then $\tilde{A} \succ \tilde{0}$ if and only if $-\tilde{A} \prec \tilde{0}$.

Remark 5.2.2. The addition of two positive (negative) trapezoidal fuzzy numbers is positive (negative). In other words, if $\tilde{A} \succ \tilde{0}$ ($\tilde{A} \prec \tilde{0}$) and $\tilde{B} \succ \tilde{0}$ ($\tilde{B} \prec \tilde{0}$), then $\tilde{A} + \tilde{B} \succ \tilde{0}$ ($\tilde{A} + \tilde{B} \prec \tilde{0}$).

Remark 5.2.3. A symmetric trapezoidal fuzzy number $\tilde{A}$ is a fuzzy zero if and only if $MOM(\tilde{A}) = 0$ (defuzzification MOM, defined by relation (2.3.6)).

In particular, for normal trapezoidal fuzzy number, it can be find that

$$(-a_1, -a_2, a_2, a_1) \approx \tilde{0}.$$

5.2.2 Almost equality and inequality of fuzzy numbers

Definition 5.2.2. Consider two trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, then it is said to be almost equal ($\tilde{A} \approx \tilde{B}$) if and only if $\tilde{A} - \tilde{B} \approx \tilde{0}$.

Theorem 5.2.1. *Let two fuzzy numbers $\tilde{A}$, $\tilde{B}$ satisfied the following conditions*

- $core(\tilde{A}) = core(\tilde{B})$,
- $supp(\tilde{A}) \subseteq supp(\tilde{B})$ or $supp(\tilde{B}) \subseteq supp(\tilde{A})$,
- $\min supp(\tilde{A}) - \min supp(\tilde{B}) = \max supp(\tilde{B}) - \max supp(\tilde{A})$.

Then $\tilde{A}$ almost equals to $\tilde{B}$ ($\tilde{A} \approx \tilde{B}$). In other words, $\tilde{A}$ and $\tilde{B}$ in aforementioned conditions can be analytically expressed as $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (a_1-k, a_2, a_3, a_4+k)$ such that $k \geq 0$ or $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (a_1 + k, a_2, a_3, a_4 - k)$ such that $0 \leq k \leq \min\{a_2 - a_1, a_4 - a_3\}$, (see Fig. 5.1).

Proof. It is sufficient to show $\tilde{A} - \tilde{B} \approx \tilde{0}$. Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (a_1 - k, a_2, a_3, a_4 + k)$ such that $k \geq 0$, therefore

$$\tilde{A} - \tilde{B} \approx (a_1 - a_4 - k, 0, 0, a_4 - a_1 + k),$$

or

$$\tilde{A} - \tilde{B} \approx \big(- (a_4 - a_1 + k), 0, 0, (a_4 - a_1 + k)\big),$$

and hence by Remark 5.2.3 we can conclude $\tilde{A} - \tilde{B} \approx \tilde{0}$. $\square$

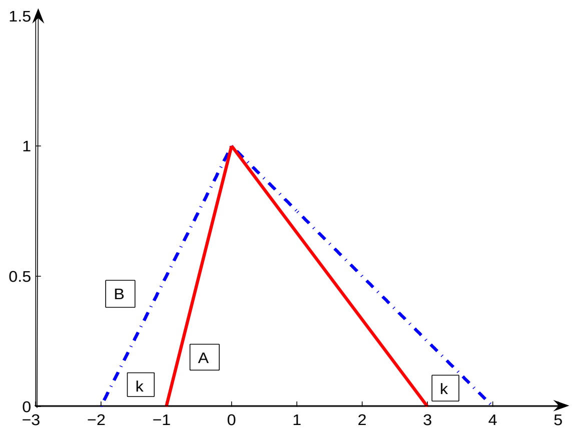

Figure. 5.1

Definition 5.2.3. Consider two trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ then it is said $\tilde{A} \prec \tilde{B}$ if and only if $\tilde{A} - \tilde{B} \prec \tilde{0}$.

Remark 5.2.4. Consider two trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ then $\tilde{A} - \tilde{B} \prec \tilde{0}$ if and only if $\tilde{B} - \tilde{A} \succ \tilde{0}$. In other words, $\tilde{A} \prec \tilde{B}$ if and only if $-\tilde{A} \succ -\tilde{B}$.

It is easy to show that almost equal fuzzy numbers (by Definition 5.2.2) have the same Val, FOM, MOM, LOM and $BADD$ (for $\gamma = 0, \gamma = 1$ and $\gamma \to \infty$). If these almost equal fuzzy numbers be triangular then their COG and centroid points are equal. In addition, they have the same order by some ranking methods such as Cheng's centroid-based [22], Chu and Tsao's area [24], sing distance method (p=1) [4], distance minimization method [8] and Wang and Lee's method [72].

There are some properties in the following by the above-mentioned definitions.

Theorem 5.2.2. *Consider three trapezoidal fuzzy numbers $\tilde{A}, \tilde{B}$ and $\tilde{C}$, then*

(1) $\tilde{A} + \tilde{0} = \tilde{0} + \tilde{A} \approx \tilde{A}$,

(2) $\tilde{A} + (-\tilde{A}) = (-\tilde{A}) + \tilde{A} \approx \tilde{0}$,

(3) $\tilde{A} \approx \tilde{B} \Longleftrightarrow \tilde{A} + \tilde{C} \approx \tilde{B} + \tilde{C}$,

(4) $\tilde{A} \approx \tilde{B} \Longleftrightarrow \tilde{A} - \tilde{C} \approx \tilde{B} - \tilde{C}$.

Proof. All parts (1) to (4) follow immediately from Definition 5.2.2 and Theorem 5.2.1. Then we formulate the order $\succeq$ and $\preceq$ as $\tilde{A} \succeq \tilde{B}$ if and only if $\tilde{A} \succ \tilde{B}$ or $\tilde{A} \approx \tilde{B}$, $\tilde{A} \preceq \tilde{B}$ if and only if $\tilde{A} \prec \tilde{B}$ or $\tilde{A} \approx \tilde{B}$. □

Theorem 5.2.3. *Consider four fuzzy numbers $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and positive nonzero constant k then*

(1) If $\tilde{A} \preceq \tilde{B}$ and $\tilde{C} \preceq \tilde{D}$ then $\tilde{A} + \tilde{C} \preceq \tilde{B} + \tilde{D}$,

(2) $\tilde{A} \preceq \tilde{B} \Leftrightarrow \tilde{A} \pm \tilde{C} \preceq \tilde{B} \pm \tilde{C}$,

(3) $\tilde{A} \preceq \tilde{B} \Leftrightarrow |k|\ \tilde{A} \preceq |k|\ \tilde{B}$.

Proof. (1) Since $\tilde{A} \preceq \tilde{B}$ and $\tilde{C} \preceq \tilde{D}$, by Definition 5.2.3 we have $\tilde{A} - \tilde{B} \preceq \tilde{0}$ and $\tilde{C} - \tilde{D} \preceq \tilde{0}$. From Remark 5.2.2 the addition of two negative fuzzy numbers is negative. In other words, $(\tilde{A} - \tilde{B}) + (\tilde{C} - \tilde{D}) \preceq \tilde{0}$ and by Definition 5.2.3 inequality is hold.

(2) By considering before part (1) and so that $\tilde{C} \approx \tilde{0}$, it is trivial.

(3) It is easy to verify that the inequality is satisfied by Definition 5.2.3. □

Lemma 5.2.4. "$\preceq$" *is partial ordering relation on fuzzy numbers. It means for any three fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$*

(i) "$\preceq$" *is reflexive, i.e. $\tilde{A} \preceq \tilde{A}$.*

(ii) "$\preceq$" *is anti-symmetric, i.e. if $\tilde{A} \preceq \tilde{B}$ and $\tilde{B} \preceq \tilde{A}$ then $\tilde{A} \approx \tilde{B}$.*

(iii) "$\preceq$" *is transitive, i.e. if $\tilde{A} \preceq \tilde{B}$ and $\tilde{B} \preceq \tilde{C}$ then $\tilde{A} \preceq \tilde{C}$.*

5.2.3 Fuzzy absolute value

Definition 5.2.4. The fuzzy absolute value of a trapezoidal fuzzy number $\tilde{A}$ denoted by $\widetilde{Abs}(\tilde{A})$ is defined by

$$\widetilde{Abs}(\tilde{A}) = \begin{cases} \tilde{A}, & if \quad \tilde{A} \succ \tilde{0}, \\ \tilde{0}, & if \quad \tilde{A} \approx \tilde{0}, \\ -\tilde{A}, & if \quad \tilde{A} \prec \tilde{0}. \end{cases} \tag{5.2.1}$$

Theorem 5.2.5. *Suppose $\tilde{A}$ and $\tilde{B}$ are two trapezoidal fuzzy numbers and k be a constant, then*

(1) $\widetilde{Abs}(\tilde{A}) \approx \widetilde{Abs}(-\tilde{A})$,

(2) $\widetilde{Abs}(k\tilde{A}) \approx |k|\widetilde{Abs}(\tilde{A})$,

(3) If $\tilde{A} \succeq \tilde{0}$, $\tilde{B} \succeq \tilde{0}$ and $\tilde{A} \preceq \tilde{B}$ then $\widetilde{Abs}(\tilde{A}) \preceq \widetilde{Abs}(\tilde{B})$,

(4) If $\tilde{A} \preceq \tilde{0}$, $\tilde{B} \preceq \tilde{0}$ and $\tilde{A} \preceq \tilde{B}$ then $\widetilde{Abs}(\tilde{A}) \succeq \widetilde{Abs}(\tilde{B})$,

(5) $\widetilde{Abs}(\tilde{A} + \tilde{B}) \preceq \widetilde{Abs}(\tilde{A}) + \widetilde{Abs}(\tilde{B})$.

Proof. It is easy to verify that part (1) and part (2) are satisfied by considering two basic cases respect to position $\tilde{A}$. Parts (3) and (4) follow immediately from Definition 5.2.4.

(5) We consider four basic cases, which reflect the potential "positions" of $\tilde{A}$ and $\tilde{B}$.

(a) It is obvious that the equality stands when $\tilde{A} \approx \tilde{0}$ or $\tilde{B} \approx \tilde{0}$.

(b) In the case that $\tilde{A} \succ \tilde{0}$ ($\tilde{A} \prec \tilde{0}$) and $\tilde{B} \succ \tilde{0}$ ($\tilde{B} \prec \tilde{0}$) by remark 5.2.2 and definition 5.2.4 the equality is hold too.

(c) Let $\tilde{A} \succ \tilde{0}$, $\tilde{B} \prec \tilde{0}$. We also consider two subcases: $\tilde{A} + \tilde{B} \succeq \tilde{0}$ and $\tilde{A} + \tilde{B} \preceq \tilde{0}$. Suppose $\tilde{A} + \tilde{B} \succeq \tilde{0}$, hence $\widetilde{Abs}(\tilde{A} + \tilde{B}) = \tilde{A} + \tilde{B}$. On the other hand $\tilde{A} + \tilde{B} \preceq \tilde{A}$, and from part(3), $\widetilde{Abs}(\tilde{A} + \tilde{B}) \preceq \widetilde{Abs}(\tilde{A})$, and furthermore $\tilde{0} \preceq \widetilde{Abs}(\tilde{B})$. Now by considering Theorem 5.2.3, part (3), the result will be obtained.

Other sub-case ($\tilde{A} + \tilde{B} \preceq \tilde{0}$) can be proved as the same before.

(d) Let $\tilde{A} \preceq \tilde{0}$, $\tilde{B} \succeq \tilde{0}$. The present case will be proved similar to part (c). □

5.3 Fuzzy distance and fuzzy metric

A natural question at this stage is whether it is reasonable to define a crisp distance between fuzzy objects. If we are not certain about the numbers themselves how can we be "certain" about the distances among them?

In order to answer the above question, with the idea of positive (negative) trapezoidal fuzzy number, fuzzy zero and equality of trapezoidal fuzzy numbers, we will propose a fuzzy distance between two fuzzy numbers as a trapezoidal fuzzy numbers in addition, discuss its properties. Then, it will be shown that the fuzzy distance is a fuzzy metric.

Definition 5.3.1. Let $\tilde{A}$ and $\tilde{B}$ be two trapezoidal fuzzy numbers. The fuzzy distance between $\tilde{A}, \tilde{B}$ denoted by $\widetilde{dist}(\tilde{A}, \tilde{B})$, is defined as

$$\widetilde{dist}(\tilde{A}, \tilde{B}) = \widetilde{Abs}(\tilde{A} - \tilde{B}). \tag{5.3.1}$$

5.3.1 Properties of fuzzy distance

Theorem 5.3.1. *If $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are arbitrary trapezoidal fuzzy numbers and k is a real number, then*

(1) $\widetilde{dist}(\tilde{A}, \tilde{B}) \approx \widetilde{dist}(\tilde{A} \pm \tilde{C}, \tilde{B} \pm \tilde{C})$,

(2) $|k|\widetilde{dist}(\tilde{A}, \tilde{B}) \approx \widetilde{dist}(k\tilde{A}, k\tilde{B})$.

Proof. (1) From Definition 5.3.1 and Theorem 5.2.2

$$\widetilde{dist}(\tilde{A}, \tilde{B}) \approx \widetilde{dist}(\tilde{A} - \tilde{B} + \tilde{C} - \tilde{C}),$$

therefore, we can get out that

$$\widetilde{dist}(\tilde{A}, \tilde{B}) \approx \widetilde{dist}\big((\tilde{A} \pm \tilde{C}) - (\tilde{B} \pm \tilde{C})\big).$$

(2) It follows immediately from Definition 5.3.1 and Theorem 5.2.5. □

Theorem 5.3.2. *If $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are arbitrary trapezoidal fuzzy numbers, then*

(1) $\widetilde{dist}(\tilde{A}, \tilde{A}) \approx \tilde{0}$,

(2) $\widetilde{dist}(\tilde{A}, \tilde{B}) \approx \widetilde{dist}(\tilde{B}, \tilde{A})$,

(3) $\widetilde{dist}(\tilde{A},\tilde{B}) \approx \tilde{0}$ if and only if $\tilde{A} \approx \tilde{B}$,

(4) $\widetilde{dist}(\tilde{A},\tilde{B}) \preceq \widetilde{dist}(\tilde{A},\tilde{C}) + \widetilde{dist}(\tilde{C},\tilde{B})$.

Proof. Part (1) and (2) can be easily verified by using Definition 5.3.1 and Theorem 5.2.5.

(3) It is obvious that if $\tilde{A} \approx \tilde{B}$ then $\widetilde{dist}(\tilde{A},\tilde{B}) \approx \tilde{0}$. Now suppose that we have $\widetilde{Abs}(\tilde{A},\tilde{B}) \approx \tilde{0}$. We consider two subcases $\tilde{A} - \tilde{B} \succeq \tilde{0}$ and $\tilde{A} - \tilde{B} \preceq \tilde{0}$.

Let us consider $\tilde{A} - \tilde{B} \succeq \tilde{0}$. From Definition 5.3.1

$$\widetilde{Abs}(\tilde{A} - \tilde{B}) \approx \tilde{A} - \tilde{B},$$

then $\tilde{A} - \tilde{B} \approx \tilde{0}$ by using Theorem 5.2.2, we can get the result. Other sub-case is as the same.

(4) In general there are eight cases, which reflect the positions $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ respect to each other.

Case 1: $\tilde{A} - \tilde{B} \succeq \tilde{0}$, $\tilde{A} - \tilde{C} \succeq \tilde{0}$ and $\tilde{C} - \tilde{B} \succeq \tilde{0}$,

Case 2: $\tilde{A} - \tilde{B} \succeq \tilde{0}$, $\tilde{A} - \tilde{C} \succeq \tilde{0}$ and $\tilde{C} - \tilde{B} \preceq \tilde{0}$,

Case 3: $\tilde{A} - \tilde{B} \succeq \tilde{0}$, $\tilde{A} - \tilde{C} \preceq \tilde{0}$ and $\tilde{C} - \tilde{B} \succeq \tilde{0}$,

Case 4: $\tilde{A} - \tilde{B} \succeq \tilde{0}$, $\tilde{A} - \tilde{C} \preceq \tilde{0}$ and $\tilde{C} - \tilde{B} \preceq \tilde{0}$,

Case 5: $\tilde{A} - \tilde{B} \preceq \tilde{0}$, $\tilde{A} - \tilde{C} \succeq \tilde{0}$ and $\tilde{C} - \tilde{B} \succeq \tilde{0}$,

Case 6: $\tilde{A} - \tilde{B} \preceq \tilde{0}$, $\tilde{A} - \tilde{C} \succeq \tilde{0}$ and $\tilde{C} - \tilde{B} \preceq \tilde{0}$,

Case 7: $\tilde{A} - \tilde{B} \preceq \tilde{0}$, $\tilde{A} - \tilde{C} \preceq \tilde{0}$ and $\tilde{C} - \tilde{B} \succeq \tilde{0}$, Case 8: $\tilde{A} - \tilde{B} \preceq \tilde{0}$, $\tilde{A} - \tilde{C} \preceq \tilde{0}$ and $\tilde{C} - \tilde{B} \preceq \tilde{0}$.

From Lemma 5.2.4 cases 4 and 5 are impossible. In addition, cases 1 and 8 easily can be shown. The other cases will be proved in similar way. As an example case 2 will be proved in the following. Since $\tilde{C} - \tilde{B} \preceq \tilde{0}$, then $\tilde{C} \preceq \tilde{B}$. Observe that $-\tilde{B} \preceq -\tilde{C}$ and $-\tilde{B} - \tilde{B} \preceq -\tilde{C} - \tilde{C}$. Now, by adding $\tilde{A} + \tilde{B}$ we have $\tilde{A} + \tilde{B} - \tilde{B} - \tilde{B} \preceq \tilde{A} + \tilde{B} - \tilde{C} - \tilde{C}$, therefore,

$$\tilde{A} - \tilde{B} \preceq \tilde{A} + \tilde{B} - \tilde{C} - \tilde{C}.$$

Due to $\tilde{A} \succeq \tilde{B}$, $\tilde{A} \succeq \tilde{C}$ and $\tilde{C} \preceq \tilde{B}$, from Definitions 5.2.4 and 5.3.1, case 2 holds. □

Consequently, the distance between two fuzzy numbers has all of the characteristics of a usual metric thus, it is a fuzzy metric.

5.4 Numerical examples

Example 5.4.1. *Consider the fuzzy number $\tilde{A}$(about 1)= $(0.9, 1, 1.1)$ and crisp number $\tilde{B} = 1$.*

Intuitively, $\tilde{A}$ equals $\tilde{B}$. By using our proposed, the result is the same. In addition, as we expect the distance between the fuzzy number "about 1" and the crisp number "exactly 1" is "about zero". Therefore, we can get that $\widetilde{dist}(\tilde{A}, \tilde{B}) \approx (-0.1, 0, 0.1)$.

Example 5.4.2. *We assume two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ where,*

$\tilde{A}$(almost greater or less than 3) $= (1, 2, 4, 5)$

and

$\tilde{B}$*(almost greater or less than* $7) = (5, 6, 8, 9)$ *[21].*

The ranking order is $\tilde{B} \succ \tilde{A}$. The fuzzy distance of $\tilde{A}, \tilde{B}$ by our approach is $\widetilde{dist}(\tilde{A}, \tilde{B}) = (0, 2, 6, 8)$. We can say the fuzzy distance of $\tilde{A}, \tilde{B}$ is "greater or less than 4". Hence, one can say the fuzzy distance of "greater or less than 3" and "greater or less than 7" is "greater or less than 4". The result of Chen and Wang's method is "greater or less than 4". They indicated $\widetilde{dist}(\tilde{A}, \tilde{B}) = (2, 3, 5, 6)$. The advantage of our method is that the calculation of the fuzzy distance is far simple than other methods such as Chen and Wang's method.

Example 5.4.3. *Consider the following sets, see Yao and Wu [79].*

Set 1: $\tilde{A} = (0.4, 0.5, 1)$, $\tilde{B} = (0.4, 0.7, 1)$, $\tilde{C} = (0.4, 0.9, 1)$ *(Fig. 3.9).*

Set 2: $\tilde{A} = (0.3, 0.4, 0.7, 0.9)$, $\tilde{B} = (0.3, 0.7, 0.9)$, $\tilde{C} = (0.5, 0.7, 0.9)$ *(Fig. 3.10).*

Set 3: $\tilde{A} = (0.3, 0.5, 0.7)$, $\tilde{B} = (0.3, 0.5, 0.8, 0.9)$, $\tilde{C} = (0.3, 0.5, 0.9)$ *(Fig. 3.11).*

Set 4: $\tilde{A} = (0, 0.4, 0.7, 0.8)$, $\tilde{B} = (0.2, 0.5, 0.9)$, $\tilde{C} = (1, 0.6, 0.8)$ *(Fig. 3.12).*

According to our computation (by Definition 5.2.1 and Remark 5.2.1) one can easily get the following results:

Set 1: $\tilde{A} \prec \tilde{B} \prec \tilde{C}$,

Set 2: $\tilde{A} \prec \tilde{B} \prec \tilde{C}$,

Set 3: $\tilde{A} \prec \tilde{C} \prec \tilde{B}$, Set 4: $\tilde{A} \prec \tilde{B} \prec \tilde{C}$.

In short, we rank any three fuzzy numbers $\tilde{A}, \tilde{B}$ and $\tilde{C}$ based on the sign of their difference, such as the sign of $\tilde{A} - \tilde{B}$, $\tilde{B} - \tilde{C}$ and $\tilde{A} - \tilde{C}$. To compare with other methods we refer the reader to Table 5.1.

Moreover, the distance of any two of them will be computed as follow:

Set 1: $\widetilde{dist}(\tilde{A}, \tilde{B}) \approx (-0.6, 0.2, 0.6)$, $\widetilde{dist}(\tilde{B}, \tilde{C}) \approx (-0.6, 0.2, 0.6)$,

Set 2: $\widetilde{dist}(\tilde{A}, \tilde{B}) \approx (-0.6, 0, 0.3, 0.6)$, $\widetilde{dist}(\tilde{B}, \tilde{C}) \approx (-0.4, 0, 0.6)$,

Set 3: $\widetilde{dist}(\tilde{A}, \tilde{C}) \approx (-0.4, 0, 0.6)$, $\widetilde{dist}(\tilde{B}, \tilde{C}) \approx (-0.6, 0, 0.3, 0.6)$,

Set 4: $\widetilde{dist}(\tilde{A}, \tilde{B}) \approx (-0.6, -0.2, 0.1, 0.9)$, $\widetilde{dist}(\tilde{B}, \tilde{C}) \approx (-0.8, 0.1, 0.6)$.

5.5 Concluding Remarks

In this chapter, we extended the absolute value of a real number to fuzzy absolute value. Then, we used it to introduce a fuzzy distance between two fuzzy numbers as a trapezoidal fuzzy number. The presented method has two advantages in comparing with existing methods. The first advantage is that the presented method is simpler than other methods computationally. The second advantage is that the ranking order of authors' approach is more consistent with our intuitions than other methods. The mentioned fuzzy distance has all of the properties of normal metrics. Furthermore, some numerical examples demonstrated the properties of the metric.

Table 5.1 : Comparative results of Example 5.4.3

Authors	Fuzzy number	set 1	set 2	set 3	set 4
Authors' approach	$\tilde{A}$	$\tilde{A}-\tilde{B}\prec\tilde{0}$	$\tilde{A}-\tilde{B}\prec\tilde{0}$	$\tilde{A}-\tilde{B}\prec\tilde{0}$	$\tilde{A}-\tilde{B}\prec\tilde{0}$
	$\tilde{B}$	$\tilde{B}-\tilde{C}\prec\tilde{0}$	$\tilde{B}-\tilde{C}\prec\tilde{0}$	$\tilde{B}-\tilde{C}\succ\tilde{0}$	$\tilde{B}-\tilde{C}\prec\tilde{0}$
	$\tilde{C}$	$\tilde{A}-\tilde{C}\prec\tilde{0}$	$\tilde{A}-\tilde{C}\prec\tilde{0}$	$\tilde{A}-\tilde{C}\prec\tilde{0}$	$\tilde{A}-\tilde{C}\prec\tilde{0}$
Results		$\tilde{A}\prec\tilde{B}\prec\tilde{C}$	$\tilde{A}\prec\tilde{B}\prec\tilde{C}$	$\tilde{A}\prec\tilde{C}\prec\tilde{B}$	$\tilde{A}\prec\tilde{B}\prec\tilde{C}$
Sign Distance method p=1	$\tilde{A}$	1.2000	1.1500	1.0000	0.9500
	$\tilde{B}$	1.4000	1.3000	1.2500	1.0500
	$\tilde{C}$	1.6000	1.4000	1.1000	1.0500
Results		$\tilde{A}\prec\tilde{B}\prec\tilde{C}$	$\tilde{A}\prec\tilde{B}\prec\tilde{C}$	$\tilde{A}\prec\tilde{C}\prec\tilde{B}$	$\tilde{A}\prec\tilde{B}\approx\tilde{C}$
Sign Distance method p=2	$\tilde{A}$	0.8869	0.8756	0.7257	0.7853
	$\tilde{B}$	1.0194	0.9522	0.9416	0.7958
	$\tilde{C}$	1.1605	1.0033	0.8165	0.8386
Results		$\tilde{A}\prec\tilde{B}\prec\tilde{C}$	$\tilde{A}\prec\tilde{B}\prec\tilde{C}$	$\tilde{A}\prec\tilde{C}\prec\tilde{B}$	$\tilde{A}\prec\tilde{B}\prec\tilde{C}$
Wang and Lee's revised	$\tilde{A}$	0.6333	0.57784	0.5000	0.4636
	$\tilde{B}$	0.7000	0.6333	0.6222	0.5333
	$\tilde{C}$	0.7967	0.7000	0.5667	0.8000
Results		$\tilde{A}\prec\tilde{B}\prec\tilde{C}$	$\tilde{A}\prec\tilde{B}\prec\tilde{C}$	$\tilde{A}\prec\tilde{C}\prec\tilde{B}$	$\tilde{A}\prec\tilde{B}\prec\tilde{C}$

Table 5.1 : Continued

Authors	Fuzzy number	set 1	set 2	set 3	set 4
Cheng's distance method	$\tilde{A}$	0.7157	0.7289	0.6009	0.6284
	$\tilde{B}$	0.7753	0.7157	0.7647	0.6289
	$\tilde{C}$	0.8360	0.7753	0.6574	0.6009
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{B} \prec \tilde{A} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{C} \prec \tilde{A} \prec \tilde{B}$
Chu and Tsao's method	$\tilde{A}$	0.2111	0.2568	0.1667	0.1967
	$\tilde{B}$	0.2333	0.2111	0.2765	0.1778
	$\tilde{C}$	0.2556	0.2333	0.1889	0.1667
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{B} \prec \tilde{C} \prec \tilde{A}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{C} \prec \tilde{B} \prec \tilde{A}$
Sign distance minimization	$\tilde{A}$	0.5334	0.5584	0.5000	0.5250
	$\tilde{B}$	0.7000	0.6334	0.6416	0.5084
	$\tilde{C}$	0.8666	0.7000	0.5166	0.5750
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{B} \prec \tilde{A} \prec \tilde{C}$
Yager	$\tilde{A}$	0.60	0.575	0.5	0.45
	$\tilde{B}$	0.70	0.65	0.55	0.525
	$\tilde{C}$	0.80	0.7	0.625	0.55
Results		$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$

Future Researches

There are three main issues for future researches. Concerning our new approach for fuzzy ranking, two important questions have not been addressed in this thesis.

1. What choice of the weighted function may obtain good results in decision making?

2. The new approach may apply for fuzzy numbers in general.

3. The issue of fuzzy ranking based on membership function deserves considerable attention in future researches.

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