ISLAMIC AZAD UNIVERSITY
SCIENCE AND RESEARCH BRANCH

METHODS FOR SOLVING N-TH ORDER FUZZY DIFFERENTIAL EQUATIONS

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Methods for solving n-th order fuzzy differential equations**" by **Elham Ahmady** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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To my mother and father

and sisters

Contents

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Abstract

In this thesis we proposed two methods for solving n-order linear differential equations.
The first method is a numerical method based on the collocation method and applied for solving n-order linear differential equations with fuzzy initial values. In this method we consider three cases for coefficients equation, and seek an approximate solution for each case.
The second method is an analytic method used for solving n-order linear fuzzy differential equations with fuzzy initial values. In this method we converts n-order fuzzy linear differential equation to the fuzzy system by changing variable, finally we obtain the solution of fuzzy system by eigenvalue-eigenvector method.

Originality

The following chapters are proposed in this thesis:

In chapter 1, some basic definitions and results that will be used are brought.

In chapter 2, one numerical method for solving n-order linear differential equations with fuzzy initial values is considered. The idea is based on the collocation method. The existence theorem of the fuzzy solution is considered. This method is illustrated by solving several examples.

In chapter 3, an analytic method (eigenvalue-eigenvector method) for solving n-order fuzzy differential equations is considered. In this method three cases are introduced, in each case, it is shown that the solution of differential equation is a fuzzy number. Finally the method is illustrated by solving several numerical examples.

Articles

1- A method for solving N-th order fuzzy differential, *in journal of International journal of computer mathematics*, 2007, In press

2-Nth-order fuzzy linear differential equations, *in journal of "Information Sciences*, Volume 178, Issue 5, 1 March 2008, Pages 1309-1324.

3-Numerical solution of fuzzy differential equations by predictor-corrector method,*in journal of Information Sciences* , Volume 177, Issue 7, 1 April 2007, Pages 1633-1647

4-Block Jacobi two-stage method with Gauss-Sidel inner iterations for fuzzy system of linear equations, *in journal of Applied Mathematics and Computation*, Volume 175, Issue 2, 15 April 2006, Pages 1217-1228.

5-Two step method for fuzzy differential equations, *in journal of "International Mathematical Forum*, 2006, Pages 823-832.

6-Triangular approximation of fuzzy numbers, *Proceedings of the 6th Iranian Conference on Fuzzy Systems and 1st Islamic World Conference on Fuzzy Systems*, 2006, Shiraz, Iran.

7-Numerical solution for N-order fuzzy linear differential equations, *Proceedings of the 6th Iranian Conference on Fuzzy Systems and 1st Islamic World Conference on Fuzzy Systems*, 2006, Shiraz, Iran.

8-Adams-Bashforth two step method for fuzzy differential equations, *Proceedings of the 36th Annual Iranian Mathematics*, 10-13 Sep 2005, Yazd, Iran.

9-Predictor-corrector method for fuzzy initial value problem, *Proceedings of the 7th Iranian Conference on Fuzzy Systems and 8th Conference on Intelligent Systems*, 29-31 Aug 2007, Mashhad, Iran.

Introduction

The topic of Fuzzy Differential Equations (FDEs) has been rapidly growing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [15]; it was followed up by Dubois and Prade [18], who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [40] and Goetschel and Voxman [22]. Kandel and Byatt [29, 30] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by O. Kaleva [27, 28], S. Seikkala [41], O. He and W. Yi [24], Kloeden [31] and W. Menda [33], and by other researchers (see [37, 11, 13, 14, 38, 17, 26]). The numerical methods for solving fuzzy differential equations are introduced in [1, 2, 6].

In this thesis, we proposed two numerical and analytic methods for solving n-order fuzzy differential equation, the numerical method seeks an approximate solution based on the collocation method. The other method, is an analytic method, which converts n-order fuzzy differential equation to fuzzy system by changing variables. Then the solution of fuzzy system will be obtained by eigenvalue-eigenvector method. Finally we compare this method with Buckley-Feuring method by an example.[12]

Chapter 1

Fuzzy Mathematics

1.1 Introduction

Fuzziness is not a priori an obvious concept and demands some explanation. "Fuzziness" is what Black calls "vagueness" when he distinguishes it from "generality" and from "ambiguity". Generalizing refers to the application of a symbol to a multiplicity of objects in the field of reference, ambiguity to the association of a finite number of alternative meanings having the same phonetic form. But, the fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol applies.

More specifically, let X be a field of reference, also called a universe of discourse or universe for short, covering a definite range of objects. Consider a subset $\tilde{A}$ where transition between membership and nonmembership is gradual rather than abrupt. This "fuzzy subset" obviously has no well-defined boundaries. Fuzzy classes of objects are often encountered in real life. For instance, $\tilde{A}$ may be the set of tall men in a community X. Usually, there are members of X who are definitely tall, others who are definitely not tall, but there exist also borderline cases. Traditionally, the grade of membership 1 is assigned to the objects that completely belong to $\tilde{A}$-here the men who are definitely tall, conversely the objects that do not belong to $\tilde{A}$ at all are assigned a membership value 0. Quite naturally, the grades of membership of the borderline cases lie between 0 and 1. The more an element or object x belongs to $\tilde{A}$, the closer to 1 is its grade of membership $\mu_{\tilde{A}}(x)$. The use of a numerical scale such as the interval [0, 1] allows a convenient representation of the gradation in membership. Precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or a group. Moreover, they are context-dependent. The grades of membership reflect an "ordering" of the objects in the universe, induced by the predicate associated with $\tilde{A}$; this "ordering", when it exists, is more important than the membership values themselves. The membership assessment of objects can sometimes be made easier by the use of a similarity measure with respect to an ideal element. Note that a membership value $\mu_{\tilde{A}}(x)$ can be interpreted as the degree of compatibility of the predicate associated with $\tilde{A}$ and the object x. For concepts such as "tallness", related to a physical measurement scale, the assignment of membership values will often be less controversial than for more complex and subjective concepts such as "beauty".

The above approach, developed by Zadeh (1964), provides a tool for modeling human-centered systems. As a matter of fact, fuzziness seems to pervade most human perception and thinking processes. Parikh (1977) has pointed out that no nontrivial first-order-logic-like observational predicate (i.e., one pertaining to perception) can be defined on an observationally connected space; the only possible observational predicates on such a space are not classical predicates but "vague" ones. Moreover, according to Zadeh (1973), one of the most important facets of human thinking is the ability to summarize information "into labels of fuzzy sets which bear an approximate relation to the primary data". Linguistic descriptions, which are usually summary descriptions of complex situations, are fuzzy in essence.

It must be noticed that fuzziness differs from imprecision. In tolerance analysis imprecision refers to lack of knowledge about the value of a parameter and is thus expressed as a crisp tolerance interval. This interval is the set of possible values of the parameters. Fuzziness occurs when the interval has no sharp boundaries, i.e., is a fuzzy set $\tilde{A}$. Then, $\mu_{\tilde{A}}(x)$ is interpreted as the degree of possibility (Zadeh, 1978) that x is the value of the parameter fuzzily restricted by $\tilde{A}$.

The word fuzziness has also been used by Sugeno (1977) in a radically different context. Consider an arbitrary object x of the universe X; to each nonfuzzy subset A of X is assigned a value $g_x(A) \in [0, 1]$ expressing the "grade of fuzziness" of the statement "x belongs to A". In fact this grade of fuzziness must be understood as a grade of certainty: according to the mathematical definition of g, $g_x(A)$ can be interpreted as the probability, the degree of subjective belief, the possibility, that x belongs to A.

Generally, g is assumed increasing in the sense of set inclusion, but not necessarily additive as in the probabilistic case. The situation modeled by Sugeno is more a matter of guessing whether $x \in A$ rather than a problem of vagueness in the sense of Zadeh. The existence of two different points of view on "fuzziness" has been pointed out by MacVicar-Whelan (1977) and Skala. The monotonicity assumption for g seems to be more consistent with human guessing than does the additivity assumption. For instance, seeing a piece of Indian pottery in a shop, we may try to guess whether it is genuine or counterfeit; obviously, genuineness is a fuzzy concept. Hence x is the Indian pottery; A is the crisp set of genuine Indian artifacts; and $g_x(A)$ expresses, for instance, a subjective belief that the pottery is indeed genuine. The situation is slightly more complicated when we try to guess whether the pottery is old: actually, the set $\tilde{A}$ of old Indian pottery is fuzzy because "old" is a vague predicate.

1.2 Fuzzy Sets

Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X is often viewed as a characteristic function, μ_A from X to $\{0, 1\}$ such that

$$\mu_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in A. \end{cases} \tag{1.1}$$

If the valuation set, i.e., 0 and 1, is allowed to be the real interval $[0, 1]$, A is called a fuzzy set (Zadeh, 1965), and $\mu_A(x)$ is the grade of membership of x in A. Sometimes we denoted $\mu_A(x)$ by $A(x)$. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A. Clearly, A is a subset of X that has no sharp boundary. Hence A is completely characterized by the set of pairs

$$A = \{(x, \mu_A(x)) | x \in X\}.$$

A more convenient notation was proposed by Zadeh [42]. When X is a finite set $\{x_1, ..., x_n\}$, a fuzzy set on X is expressed as

$$A = \frac{\mu_A(x_1)}{x_1} + \cdots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_n)}{x_n}.$$

When X is not finite, we write

$$A = \int_x \mu_A(x).$$

Definition 1.2.1. (Equality of fuzzy sets) Two fuzzy sets A and B are said to be equal (denoted $A = B$) iff

$$\forall\ x \in X,\ \ \mu_A(x) = \mu_B(x).$$

Definition 1.2.2. (Support) The support of a fuzzy set A is the ordinary subset of X:

$$supp\ A = \{x \in X,\ \ \mu_A(x) > 0\}.$$

Definition 1.2.3. (Core) The core of a fuzzy set A is the set of all points with the membership degree one in A:

$$coreA = \{x \in X |\ \mu_A(x) = 1\}.$$

Definition 1.2.4. (Height of a fuzzy set) The height of A is $hgt(A) = sup_{x \in X}\ \mu_A(x)$, i.e., the least upper bound of $\mu_A(x)$.

Definition 1.2.5. (Normal fuzzy set) A is said to be normal if and only if there exists an $x \in X$, such that $\mu_A(x) = 1$, otherwise A is subnormal.

Definition 1.2.6. (Empty fuzzy set) The empty set ϕ is defined as $\forall\ x \in X,\ \mu_\phi(x) = 0$; of course, $\forall\ X,\ \mu_X(x) = 1$.

Definition 1.2.7. $(r - Cuts)$ An r-level set of a fuzzy set A of X is a non-fuzzy set denoted by $[A]_r$ and is defined by

$$[A]_r = \begin{cases} \{\ x \in X|\ \mu_A(x) \geq r\ \} & if\ \ r > 0 \\ cl(suppA) & if\ \ r = 0 \end{cases} \tag{1.2}$$

where $cl(suppA)$ denotes the closure of the support of A. One also defines the strong $h - cut$ $[A]_{\overline{r}} = \{\ x \in X,\ \mu_A(x) > r\ \}$.

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its $r - cuts$ according to the formula [**?**]

$$\mu_A(x) = \sup_{r \in]0,1]} \min(r, \mu_{A_r}(x)),$$

where

$$\mu_{A_r}(x) = \begin{cases} 1, & x \in A_r \\ 0, & otherwise. \end{cases} \tag{1.3}$$

It is easily checked that the following properties hold:

$$[A \cup B]_r = [A]_r \cup [B]_r, \qquad [A \cap B]_r = [A]_r \cap [B]_r. \tag{1.4}$$

Definition 1.2.8. (Convexity) A fuzzy set A of X is convex if and only if $r-$cuts are convex. An equivalent definition of convexity is: A is convex if and only if $\forall x_1 \in X, \forall x_2 \in X, \forall \lambda \in [0, 1]$,

$$A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{A(x_1), A(x_2)\} \tag{1.5}$$

Definition 1.2.9. (Fuzzy number) fuzzy number

In this thesis, the family of fuzzy numbers will be denoted E.

Definition 1.2.10. (Quasi fuzzy number) A quasi fuzzy number A is a fuzzy set of the real line with a normal, fuzzy convex and continuous membership function satisfying the limit conditions

$$\lim_{t \to +\infty} A(t) = 0, \qquad \lim_{t \to -\infty} A(t) = 0.$$

Remark 1.2.1. Let A be a fuzzy number. Then $[A]_r$ is a closed convex (compact) subset of $\mathbb{R}$ for all $r \in [0,1]$.

Definition 1.2.11. (Trapezoidal fuzzy number) A fuzzy number A is called trapezoidal fuzzy number with tolerance interval $[a,b]$, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{\alpha} & if \quad a - \alpha \le t \le a \\ 1 & if \quad a \le t \le b \\ 1 - \frac{(t-b)}{\beta} & if \quad b \le t \le b + \beta \\ 0 & otherwise \end{cases}$$

and we use for it the notation $A = (a, b, \alpha, \beta)$. It can easily be shown that

$$[A]_r = [a - (1-r)\alpha, b + (1-r)\beta], \forall r \in [0,1].$$

The support of A is $[a - \alpha, b + \beta]$.

Definition 1.2.12. We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \le r \le 1$, which satisfy the following requirements [20]:

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,

2. $\overline{u}(r)$ is a bounded left continuous non-increasing function over $[0,1]$,

3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

A crisp number λ is simply represented by $\underline{u}(r) = \overline{u}(r) = \lambda, 0 \leq r \leq 1$. By appropriate definitions the fuzzy number space $\{\underline{u}(r), \overline{u}(r)\}$ becomes a convex cone, E^1.

Lemma 1.2.1. *Let v and w be fuzzy number and s be real number. Then for $0 \leq r \leq 1$*

$$u = v \text{ if and only if } \underline{u}(r) = \underline{v}(r) \text{ and } \overline{u}(r) = \overline{v}(r),$$

$$v + w = (\underline{v}(r) + \underline{w}(r), \overline{v}(r) + \overline{w}(r)),$$

$$v - w = (\underline{v}(r) - \overline{w}(r), \overline{v}(r) - \underline{w}(r)),$$

$$v.w = (\min\{\underline{v}(r).\underline{w}(r), \underline{v}(r).\overline{w}(r), \overline{v}(r).\underline{w}(r), \overline{v}(r).\overline{w}(r)\},$$

$$\max\{\underline{v}(r).\underline{w}(r), \underline{v}(r).\overline{w}(r), \overline{v}(r).\underline{w}(r), \overline{v}(r).\overline{w}(r)\}),$$

$$sv = s(\underline{v}(r), \overline{v}(r)).$$

Definition 1.2.13. Any fuzzy number $A \in E$ can be described as

$$A(t) = \begin{cases} L(\frac{a-t}{\alpha}) & if \quad t \in [a-\alpha, a] \\ 1 & if \quad t \in [a, b] \\ R(\frac{t-b}{\beta}) & if \quad t \in [b, b+\beta] \\ 0 & otherwise \end{cases}$$

where $[a, b]$ is the core of A,

$$L : [0,1] \to [0,1], \quad R : [0,1] \to [0,1]$$

are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $R(1) = L(1) = 0$. We call this fuzzy interval of LR-type and refer to it by

$$A = (a, b, \alpha, \beta)_{LR}.$$

The support of A is $(a - \alpha, b + \beta)$.

Definition 1.2.14. (Fuzzy point) Let A be a fuzzy number. If supp(A)= $\{x_0\}$ then A is called a fuzzy point and we use the notation $A = \overline{x_0}$.

Definition 1.2.15. The space E^n is all of fuzzy subsets U of $\mathbb{R}^n$ which satisfy the following conditions:

1. U is normal,
2. U is fuzzy convex,
3. U is upper semi-continuous,
4. $[U]_0$ is bounded subset of $\mathbb{R}^n$,

when $n = 1$, elements of E^1 are Fuzzy numbers.

1.3 The extension principle

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to *add*, *subtract*, *multiply* and *divide* with fuzzy quantities. The process of doing these operations is called *fuzzy arithmetic.*

We shall first introduce an important concept from fuzzy set theory called the *extension principle*. We then use it to provide for these arithmetic operations on fuzzy numbers.

In general the extension principle plays a fundamental role in enabling us to extend any point operations to operations involving fuzzy sets. In the sequel, we define this principle.

Definition 1.3.1. (*extension principle*) Assume X and Y are crisp sets and let f be a mapping from X to Y,

$$f : X \to Y,$$

such that for each $x \in X$, $f(x) = y \in Y$. Assume A is a fuzzy subset of X, using extension principle, we can define $f(A)$ as a fuzzy subset of Y such that

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Definition 1.3.2. (*sup-min extension n-place functions*) Let X_1, X_2,...,X_n and Y be a family of sets. Assume f is a mapping from the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ into Y. Let A_1, A_2,...,A_n be fuzzy subsets of X_1, X_2,...,X_n, respectively, then we use the extension principle for the evaluation of $f(A_1, A_2, ..., A_n)$. $f(A_1, A_2, ..., A_n)$ is a fuzzy set such that

$$f(A_1, A_2, ..., A_n)(y) = \begin{cases} \sup\{\min\{A_1(x_1), A_2(x_2), \dots, A_n(x_n)\} \mid x \in f^{-1}(y)\} \\ \qquad \text{if } \quad f^{-1}(y) \neq \emptyset, \\ 0 \qquad \text{otherwise,} \end{cases}$$

where $x = (x_1, x_2, ..., x_n)$.

Let $f : X \times X \to X$ be defined as

$$f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2,\ \lambda_1,\ \lambda_2 \in \mathbb{R}.$$

Suppose A_1 and A_2 are fuzzy subsets of X. Then using the extension principle we get

$$f(A_1, A_2)(y) = \sup_{\lambda_1 x_1 + \lambda_2 x_2 = y} \min\{A_1(x_1), A_2(x_2)\}$$

and we use the notation $f(A_1, A_2) = \lambda_1 A_1 + \lambda_2 A_2$.

Definition 1.3.3. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be a function from $E^1(X)$ to $E^1(Y)$. Then f is called a fuzzy function (or mapping) and we use the notation

$$f : E^1(X) \to E^1(Y).$$

Theorem 1.3.1. *Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets. Then every fuzzy mapping $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined by the extension principle is monotonic increasing.*

Proof. Let $A, A' \in \mathcal{F}(X)$ such that $A \subset A'$. Then using the definition of sup-min extension principle we get

$$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \leq \sup_{x \in f^{-1}(y)} A'(x) = f(A')(y)$$

for all $y \in Y$. $\square$

Let $A = (a_1, a_2, \alpha_1, \alpha_2)_{LR}$ and $B = (b_1, b_2, \beta_1, \beta_2)_{LR}$ be fuzzy numbers of $LR-type$. Using the (sup-min) extension principle, we can verify the following rules for addition and subtraction of fuzzy numbers of $LR-type$:

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_{LR}$$

$$A - B = (a_1 - b_1, a_2 - b_2, \alpha_1 + \beta_2, \alpha_2 + \beta_1)_{LR}$$

furthermore, if $\lambda \in \mathbb{R}$ is a real number then λA can be represented as

$$\lambda A = \begin{cases} (\lambda a_1, \lambda a_2, \lambda \alpha_1, \lambda \alpha_2)_{LR} & \text{if } \lambda \geq 0, \\ (\lambda a_2, \lambda a_1, \mid \lambda \mid \alpha_2, \mid \lambda \mid \alpha_1)_{LR} & \text{if } \lambda < 0. \end{cases}$$

In particular, if $A = (a_1, a_2, \alpha_1, \alpha_2)$ and $B = (b_1, b_2, \beta_1, \beta_2)$ are fuzzy numbers of trapezoidal form, then

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$A - B = (a_1 - b_2, a_2 - b_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1).$$

If $A = (a, \alpha_1, \alpha_2)$ and $B = (b, \beta_1, \beta_2)$ are fuzzy numbers of triangular form, then

$$A + B = (a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$A - B = (a - b, \alpha_1 + \beta_2, \alpha_2 + \beta_1),$$

and if $A = (a, \alpha)$ and $B = (b, \beta)$ are fuzzy numbers of symmetrical triangular form, then

$$A + B = (a + b, \alpha + \beta)$$

$$A - B = (a - b, \alpha + \beta),$$

$$\lambda A = (\lambda a, \mid \lambda \mid \alpha).$$

The above results can be generalized to linear combination of fuzzy numbers.

Let A and B be fuzzy numbers with $[A]_r = [a_1(r), a_2(r)]$ and $[B]_r = [b_1(r), b_2(r)]$, $0 \leq r \leq 1$. Then it can easily be shown that

$$[A + B]_r = [a_1(r) + b_1(r), a_2(r) + b_2(r)],$$

$$[-A]_r = [-a_2(r), -a_1(r)],$$

$$[A - B]_r = [a_1(r) - b_2(r), a_2(r) - b_1(r)],$$

$$[\lambda A]_r = [\lambda a_1(r), \lambda a_2(r)] \quad if \quad \lambda \geq 0,$$

$$[\lambda A]_r = [\lambda a_2(r), \lambda a_1(r)] \quad if \quad \lambda < 0,$$

for all $r \in [0, 1]$, i.e. any r-level set of the extended sum of two fuzzy numbers is equal to the sum of their r-level sets. The following two theorems show that this property is valid for any continuous function.

Theorem 1.3.2. *[37] Let $f : X \to X$ be a continuous function and let A be a fuzzy number. Then,*

$$[f(A)]_r = f([A]_r),$$

where $f(A)$ is defined by the extension principle and

$$f([A]_r) = \{f(x) \mid x \in [A]_r\}.$$

If $[A]_r = [a_1(r), a_2(r)]$ and f is continuous and monotone increasing then from the above theorem we get

$$[f(A)]_r = f([A]_r) = f([a_1(r), a_2(r)]) = [f(a_1(r)), f(a_2(r))].$$

Theorem 1.3.3. *[37] Let $f : X \times X \to X$ be a continuous function and let A and B be fuzzy numbers. Then*

$$[f(A, B)]_r = f([A]_r, [B]_r)$$

where,

$$f([A]_r, [B]_r) = \{f(x_1, x_2) \mid x_1 \in [A]_r, x_2 \in [B]_r\}.$$

Let $f(x, y) = xy$ and let $[A]_r = [a_1(r), a_2(r)]$, $[B]_r = [b_1(r), b_2(r)]$ be the $r-$level sets of two fuzzy numbers A and B. Applying above theorem we get

$$[f(A, B)]_r = f([A]_r, [B]_r) = [A]_r[B]_r$$

However the equation

$$[AB]_r = [A]_r[B]_r = [a_1(r)b_1(r), a_2(r)b_2(r)]$$

holds if and only if A and B are both nonnegative, i.e. $A(x) = B(x) = 0$ for $x \leq 0$. If B is nonnegative then we have

$$[A]_r[B]_r = [\min\{a_1(r)b_1(r), a_1(r)b_2(r)\}, \max\{a_2(r)b_1(r), a_2(r)b_2(r)\}].$$

In general case, we obtain a very complicated expression for the r-level sets of the product AB

$$[A]_r[B]_r = [\min\{a_1(r)b_1(r), a_1(r)b_2(r), a_2(r)b_1(r), a_2(r)b_2(r)\},$$

$$\max\{a_1(r)b_1(r), a_1(r)b_2(r), a_2(r)b_1(r), a_2(r)b_2(r)\}].$$

1.4 Hausdorff distance for fuzzy numbers

Let A and B be fuzzy numbers with $[A]_r = [a_1(r), a_2(r)]$ and $[B]_r = [b_1(r), b_2(r)]$.

Definition 1.4.1. (Hausdorff distance) The Hausdorff distance between two (nonempty) sets $X,\ Y \subseteq \mathbb{R}$ is given as

$$d_H(X,Y) = \max\{\beta(X,Y), \beta(Y,X)\},$$

where $\beta(X,Y) = \sup_{x\in X} \rho(x,Y)$ and $\rho(x,Y) = \inf_{y\in Y} \mid x-y \mid$. The generalization

$$d_H(A,B) \quad = \quad \sup_{r\in(0,1]} d_H([A]_r,[B]_r) \ \ \forall\ A,B \in E^1,$$

defines a distance measure [23]. It is clear that

$$d_H(A,B) = \sup_{h\in[0,1]} \max\{\mid a_1(r) - b_1(r) \mid, \mid a_2(r) - b_2(r) \mid\},$$

i.e. $d_H(A,B)$ is the maximal distance between r-level sets of A and B [38].

Definition 1.4.2. For arbitrary fuzzy quantities $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$, the quantity

$$D(u,v) = [\int_0^1 (\underline{u}(r) - \underline{v}(r))^2 dr + \int_0^1 (\overline{u}(r) - \overline{v}(r))^2 dr]^{\frac{1}{2}} \tag{1.6}$$

is the distance between u and v. []

1.5 Operation on fuzzy numbers

Some previous works related to operations on fuzzy numbers are those of Jain [25], Nahmias [36], Mizumoto and Tanaka [35],[34] Baas and Kwakernaak [7].

1.5.1 Addition and Multiplication

Addition: Addition is an increasing operation. Hence, the extended addition ($\oplus$) of fuzzy numbers gives a fuzzy number. Note that $-(M \oplus N) = (-M) \oplus (-N)$. ($\oplus$)

is commutative and associative but has no group structure. The identity of $(\oplus)$ is the nonfuzzy number 0. But M has no symmetrical element in the sense of a group structure. In particular, $M \oplus (-M) \neq 0, \quad \forall M \in E - \mathbb{R}$.

Multiplication: Multiplication is an increasing operation on $\mathbb{R}^+$ and a decreasing operation on $\mathbb{R}^-$. Hence, the product of fuzzy numbers $(\odot)$ that are all either positive or negative gives a positive fuzzy number. Note that $-(M) \odot N = -(M \odot N)$, so that the factors can have different signs. $(\odot)$ is commutative and associative. The set of positive fuzzy numbers is not a group for $(\odot)$: although $\forall M, \quad M \odot 1 = M$, the product $M \odot M^{-1} \neq 1$ as soon as M is not a real number. M has no inverse in the sense of group structure.

1.5.2 Subtraction

Subtraction is neither increasing nor decreasing. However, it is easy to check that $M \ominus N = M \oplus (-N), \ \forall (M, N) \in E^2$ so that $M \ominus N$ is a fuzzy number whenever M and N are.

1.5.3 Division

Division is neither increasing nor decreasing. But, since $M \oslash N = M \odot (N^{-1}), \forall (M, N) \in E^2$, $M \oslash N$ is a fuzzy number when M and N are positive or negative fuzzy numbers. The division of ordinary fuzzy numbers can be performed similarly to multiplication, by decomposition.

1.6 Fuzzy System

Definition 1.6.1. The $N \times N$ linear system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N &= y_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N &= y_2, \\ &\vdots \\ a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N &= y_N, \end{aligned} \tag{1.7}$$

where the coefficient matrix $A = (a_{ij})$, $1 \leq i, j \leq N$ is a crisp $N \times N$ matrix and $y_i \in E$, $1 \leq i \leq N$ is called a fuzzy system of linear equations (FSLEs).

Definition 1.6.2. [5], A fuzzy number vector $(x_1, x_2, \ldots, x_N)^t$ given by $x_i = (\underline{x}_i(r), \overline{x}_i(r))$, $1 \leq i \leq N$, $0 \leq r \leq 1$, is called a solution of the FSLE if

$$\underline{\sum_{j=1}^{N} a_{ij}x_j} = \sum_{j=1}^{N} \underline{a_{ij}x_j} = \underline{y}_i,$$

$$\overline{\sum_{j=1}^{N} a_{ij}x_j} = \sum_{j=1}^{N} \overline{a_{ij}x_j} = \overline{y}_i.$$

Consider the i-th equation of the system (1.7)

$$a_{i1}(\underline{x}_1, \overline{x}_1) + \ldots + a_{ii}(\underline{x}_i, \overline{x}_i) + \ldots + a_{in}(\underline{x}_n, \overline{x}_n) = (\underline{y}_i, \overline{y}_i),$$

we have

$$\begin{aligned} \underline{a_{i1}x}_1 + \ldots + \underline{a_{ii}x}_i + \ldots + \underline{a_{in}x}_n &= \underline{y}_i(r), \\ \overline{a_{i1}x}_1 + \ldots + \overline{a_{ii}x}_i + \ldots + \overline{a_{in}x}_n &= \overline{y}_i(r), \end{aligned} \tag{1.8}$$

for $1 \leq i \leq N, 0 \leq r \leq 1$.

From (1.8) we have two crisp $N \times N$ linear systems for all i that there can be extended to a $2N \times 2N$ crisp linear system as follows:

$$SX = Y \Rightarrow \begin{bmatrix} S_1 \geq 0 & S_2 \leq 0 \\ S_2 \leq 0 & S_1 \geq 0 \end{bmatrix} \begin{bmatrix} \underline{X} \\ \overline{X} \end{bmatrix} = \begin{bmatrix} \underline{Y} \\ \overline{Y} \end{bmatrix}, \tag{1.9}$$

where s_{ij} are determined as follows:

$$if \quad a_{ij} \geq 0 \Longrightarrow s_{ij} = a_{ij}, \quad s_{i+n,j+n} = a_{ij}, \tag{1.10}$$

$$if \quad a_{ij} < 0 \Longrightarrow s_{i,j+n} = a_{ij}, \quad s_{i+n,j} = a_{ij},$$

and any s_{ij} which is not determined by (1.10) is zero.

1.7 Fuzzy Differential equations

1.7.1 Fuzzy Derivatives

Let $\widetilde{X}(t)$ is a fuzzy number for each $tinI$. Also, let $[\widetilde{X}(t)]^{\alpha} = [x_1(t,\alpha), x_2(t,\alpha)]$ and write $x_i'(t,\alpha)$ with respect ro t, $i = 1, 2$. We assume these partial always exist in this section. Now we will discuss the Goetchel-Voxman derivative, the Seikkala derivative, the Dubois-Prade derivative, the puri-Ralescu derivative, and the Kandel-Friedman-Ming derivative of $\widetilde{X}(t)$.[11]

Goetschel-Voxman derivative

we first must give the metric used for this derivative. Let $\widetilde{X}(t)$ and $\widetilde{Z}(t)$ be two fuzzy fuctions for $t \in I$. Both $\widetilde{X}(t)$ and $\widetilde{Z}(t)$ are fuzzy numbers for each $t \in I$. Set

$[\widetilde{X}(t)]_r = [x_1(t,r), x_2(t,r)]$ and $[\widetilde{X}(t)]_r = [z_1(t,r), z_2(t,r)]$ for all t and r.

Then metric D is

$$D(\widetilde{X}(t), \widetilde{Z}(t)) = \sup_r \{\max[|x_1(t,r) - z_1(t,r)|, |x_2(t,r) - z_2(t,r)|]\}, \tag{1.11}$$

for all t in I.

The derivative of $\widetilde{X}(t)$ at t_0 defined as

$$GVD\widetilde{X}(t_0) = (\frac{\widetilde{X}(t_0+h) - \widetilde{X}(t_0)}{h}), \tag{1.12}$$

provided the limit exists with respect to the metric D. However, the subtraction in Eq.(1.12) is not standard fuzzy subtraction because

$$[\widetilde{X}(t_0+h) - \widetilde{X}(t_0)]_r = [x_1(t_0+h,r) - x_1(t_0,r), x_2(t_0+h,r) - x_2(t_0,r)], \tag{1.13}$$

for all t, r. Standard fuzzy arithmetic would produce

$$[x_1(t_0+h,r) - x_2(t_0,r), x_2(t_0+h,r) - x_1(t_0,r)].$$

However, $GVD\widetilde{X}(t)$ may not be a fuzzy number for some t in I and non-standard fuzzy subtraction is used in the definition of the derivative.

Seikkala derivative

The Seikkala derivative of $\widetilde{X}(t)$, written $SD\widetilde{X}(t)$ define as follows: if $[x_1'(t,r), x_2'(t,r)]$ are the r-cuts of a fuzzy number for each $t \in I$, then $SD\widetilde{X}(t)$ exists and $SD\widetilde{X}(t) = [x_1'(t,r), x_2'(t,r)]$.

$SD\widetilde{X}(t)$ is a fuzzy number for all $t \in I$.

Dubois-Prade derivative

The Dubois-Prade derivative of $\widetilde{X}(t)$, written $DPD\widetilde{X}(t)$, always exists and its membership function is given by

$$DPD\widetilde{X}(t)(x) = \sup\{r|x = x_1'(t,r), x = x_2'(t,r)\}. \tag{1.14}$$

However, $DPD\widetilde{X}(t)$ may not be a fuzzy number.

Let us consider the situation where $DPD\widetilde{X}(t)$ can be a fuzzy number for t in I. Assume that $x_1'(t,r)$ and $x_2'(t,r)$ satisfy the sufficient conditions for $[x_1'(t,r), x_2'(t,r)]$ to define $r-$cut of a fuzzy number. We may have to add something to the definition of $DPD\widetilde{X}(t)$ to obtain a fuzzy number. If $x_1'(t,1) < x_2'(t,1)$ for some value of t, then we separately define $DPD\widetilde{X}(t) = 1$ for all x satisfying $x_1'(t,1) < x < x_2'(t,1)$. The $DPD\widetilde{X}(t)$ will be fuzzy number.

Puri-Ralescu derivative

The Puri-Ralescu derivative of $\widetilde{X}(t)$, and written by $PRD\widetilde{X}(t)$. We first specify the metric used for this derivative. Let $\widetilde{X}(t)$ and $\widetilde{Z}(t)$ be two fuzzy fuctions for $t \in I$. Both $\widetilde{X}(t)$ and $\widetilde{Z}(t)$ are fuzzy numbers for each $t \in I$. Set $[\widetilde{X}(t)]_r = [x_1(t,r), x_2(t,r)]$ and $[\widetilde{X}(t)]_r = [z_1(t,r), z_2(t,r)]$ for all t and r. Then metric D is

$$D(\widetilde{X}(t), \widetilde{Z}(t)) = \sup_r H([\widetilde{X}(t)]_r, [\widetilde{Z}(t)]_r), \tag{1.15}$$

for all t, where H is the Hausdorff metric on non-empty compact subsets of R.

Next, we need to define the Hukuhara difference between two fuzzy numbers $\widetilde{A}$ and

$\widetilde{B}$. If there exists a fuzzy number $\widetilde{C}$ so that $\widetilde{C}+\widetilde{A}=\widetilde{B}$, then $\widetilde{C}$ is called the Hukuhara difference between $\widetilde{B}$ and $\widetilde{A}$ and we write this as

$$\widetilde{B} \sim \widetilde{A} = \widetilde{C}.$$

$\widetilde{X}(t)$ is differentiable at t_0 in I if there exists fuzzy number $PRD\widetilde{X}(t_0)$ so that

$$\lim_{h\to 0^+} (\frac{\widetilde{X}(t_0+h) \sim \widetilde{X}(t_0)}{h}) = PRD\widetilde{X}(t_0), \tag{1.16}$$

and

$$\lim_{h\to 0^+} (\frac{\widetilde{X}(t_0) \sim \widetilde{X}(t_0-h)}{h}) = PRD\widetilde{X}(t_0). \tag{1.17}$$

Both limits are taken with respect to the metric D in Eq. (1.15).

If $PRD\widetilde{X}(t)$ exists, then

$$PRD\widetilde{X}(t)[r] = [x_1'(t,r), x_2'(t,r)],$$

for all $t \in I$, all $r \in [0,1]$. $PRD\widetilde{X}(t)$ is always a fuzzy number for each $t \in I$. however, non-standard fuzzy subtraction is used in that they employ the Hukuhara difference of fuzzy sets.

Kandel-Friedman-Ming derivative

The Puri-Ralescu derivative of $\widetilde{X}(t)$, and written by $PRD\widetilde{X}(t)$.

First, fuzzy numbers now do not need to have compact support. The metric D used is

$$D_p(\widetilde{X}(t), \widetilde{Z}(t)) = \max\{[\int_0^1 |x_1(t,r) - z_1(t,r)|^p dr]^{1/p}, [\int_0^1 |x_1(t,r) - z_1(t,r)|^p dr]^{1/p}\},$$

for $x_1(t,r), x_2(t,r), z_1(t,r)$ and $z_1(t,r)$ all in $L_p[0,1]$ for all t in I.

$\widetilde{X}(t)$ is differentiable at $t_0 \in I$ if there is a fuzzy number $KFMD\widetilde{X}(t_0)$ so that

$$\lim_{h \to 0} D_p[\frac{\widetilde{X}(t_0+h) - \widetilde{X}(t_0)}{h}, KFMD\widetilde{X}(t_0)] = 0.$$

However, the subtraction $\widetilde{X}(t_0+h) - \widetilde{X}(t_0)$ in the above equation is not standard fuzzy subtraction since it is defined as in Eq. (1.13).

When this derivative exists

$$KFMD\widetilde{X}(t)(r) = [x_1'(t,r), x_2'(t,r)],$$

for all $t \in I$, all $r \in [0,1]$. This derivative also equals a fuzzy number for all $t \in I$. Also, non-standard fuzzy subtraction is used.

Relationships

Theorem 1.7.1. *1. If $GVD\widetilde{X}(t)$ exists and is a fuzzy number for each $t \in I$, then $SD\widetilde{X}(t)$ exists and $GVD\widetilde{X}(t) = SD\widetilde{X}(t)$.*
2. If $PRD\widetilde{X}(t)$ exists, then $SD\widetilde{X}(t)$ exists and $PRD\widetilde{X}(t) = SD\widetilde{X}(t)$.
3. If $KFMD\widetilde{X}(t)$ exists, then so does $SD\widetilde{X}(t)$ and they are equal.
4. If $SD\widetilde{X}(t)$ exists and if $x_1'(t,r)$ and $x_2'(t,r)$ are both continuous in α for each t in I, then $SD\widetilde{X}(t) = DPD\widetilde{X}(t)$.

Proof. See [11].

Theorem 1.7.2. *Assume the continuity condition holds. If $SD\widetilde{X}(t)$ exists, then $SD\widetilde{X}(t) = DPD\widetilde{X}(t)$ then $SD\widetilde{X}(t) = DPD\widetilde{X}(t) = GVD\widetilde{X}(t) = PRD\widetilde{X}(t) = KFMD\widetilde{X}(t)$.*

Proof. See [11].

Theorem 1.7.3. *Assume the continuity condition holds. If one of the derivatives SD and GVD and it is a fuzzy number, PRD, or $KMFD$ exist, then so do the others and they are all equal.*

Proof. See [11].

1.7.2 Buckley-Feuring method

Buckley-Feuring [12] introduced two analytical methods for solving n th-order linear differential equations with fuzzy initial conditions. Their first method of solution was to fuzzify the crisp solution and then check to see if it satisfies the differential equation with fuzzy initial conditions; and the second method was the reverse of the first method, in that they first solved the fuzzy initial value problem and the checked to see if it defined a fuzzy function. They study solutions to

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y^{(1)} + a_0(x)y = g(x) \tag{1.18}$$

where $a_i(x), 0 \leq i \leq n-1$, and $g(x)$ are continuous on some interval I, subject to initial conditions $y(0) = \overline{\gamma}_0, y^{(1)}(0) = \overline{\gamma}^1, ..., y^{(n-1)}(0) = \overline{\gamma}_{n-1}$, for fuzzy numbers $\overline{\gamma}_i, 0 \leq i \leq n-1$. the interval I can be $[0, T]$ for some $T > 0$ or $I = [0, \infty)$. This problem is called the fuzzy initial value problem for linear differential equations.

Method 1

their first method of solution is to simply fuzzify the crisp solution to obtain a fuzzy function $Y(x)$, and then check to see if it satisfies the differential equation with fuzzy initial conditions. Let $y_i(x)$ be n linearly independent solutions to

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y^{(1)} + a_0(x)y = 0 \tag{1.19}$$

for x in I.

Let G(x) be any solution to Eq. (1). The general solution to equation (1) is

$$y(x) = \sum_{j=1}^{n} c_j y_j(x) + G(x) \tag{1.20}$$

for arbitrary constants $c_j, 1 \leq j \leq n$. Let the crisp initial conditions be $y(0) = \gamma_0, y^{(1)}(0) = \gamma_1, ..., y^{(n-1)}(0) = \gamma_{n-1}$. Using these initial conditions they solve Eq.(3) for the unique values of the $c_j, 1 \leq j \leq n$. However, we must be careful to show the exact dependence of the c_j on the γ_j because we will fuzzify the γ_j.

Define $\Delta_0 = G(0), ..., \Delta_{n-1} = G^{(n-1)}(0)$ and $\Delta = (\Delta_0, ..., \Delta_{n-1}), c = (c_1, ..., c_n), \gamma = (\gamma_0, ..., \gamma_{n-1})$. Let W be the $n \times n$ matrix

$$W = \begin{bmatrix} y_1(0) & ... & y_n(0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(0) & ... & y_n^{(n-1)} \\ & & \end{bmatrix} \tag{1.21}$$

Using the crisp initial conditions we must solve the following system for c

$$Wc^T + \Delta^T = \gamma^T \tag{1.22}$$

or

$$Wc^T = \gamma^T - \Delta^T. \tag{1.23}$$

If W_j is W with its jth column replaced by $\gamma^T - \Delta^T$, then (Cramer's rule)

$$c_j = \frac{|W_j|}{|W|}, \tag{1.24}$$

where $|.|$ represents the determinant. $|W| \neq 0$ because it is the Wronskian evaluated at $x = 0$. Now set $W_{j1}(W_{j2})$ as W with its jth column replaced by $\gamma^T(\Delta^T)$. Then we see that

$$c_j = \frac{|W_{j1}|}{|W|} - \frac{|W_{j2}|}{|W|}. \tag{1.25}$$

Define $c_{j1} = \frac{|W_{j1}|}{|W|}, c_{j2} = \frac{|W_{j2}|}{|W|}$. Now c_{j2} is just a constant and c_{j1} contains all $\gamma_i, 0 \leq i \leq n-1$. To show the dependence of c_{j1} on the γ_i let

$$c_{j1} = f_j(\gamma_0, ..., \gamma_{n-1}), \tag{1.26}$$

f_j is a continuous function of the $\gamma_i \in R, 0 \leq i \leq n-1$.

Next, they rewrite the unique solution, Eq.(1.20) with $c_j = c_{j1} - c_{j2}, 1 \leq j \leq n$, as follows

$$y(x) = \sum_{j=1}^{n} f_j(\gamma) y_j(x) + \Psi(x) \tag{1.27}$$

where

$$\Psi(x) = -\sum_{j=1}^{n} c_{j2} y_j(x) + G(x). \tag{1.28}$$

They have split the unique solution into two parts:

(1) the first part contains all the γ_i-values, $0 \leq i \leq n$, and (2) the second part $\Psi(x)$ is independent of all the $\gamma_i, 0 \leq i \leq n-1$.

Now they fuzzify Eq.(1.27) using the extension principle, to get

$$\overline{Y}(x) = \sum_{j=1}^{n} f_j(\overline{\gamma}) y_j(x) + \Psi(x), \tag{1.29}$$

where $\overline{\gamma} = (\overline{\gamma}_0, ..., \overline{\gamma}_{n-1})$, the $\overline{\gamma}_i$ all triangular fuzzy numbers. $\overline{Y}(X)$ will be a triangular-shaped fuzzy number for all $x \in I$. Let $\overline{Y}(x)[r] = [y_1(x, r), y_2(x, r)], x \in$

$I, r \in [0,1]$. We will show how to obtain $y_i(x,r), i = 1,2$, later on in this section. Right now they assume that the $y_i(x,r)$ have continuous derivatives of order n, on x, for fixed r in $[0,1]$.

Let $\overline{\gamma}_i[r] = [\gamma_{i1}(r), \gamma_{i2}(r)], 0 \le i \le n-1$. They say $\overline{Y}(x)$ is a solution if $y_i(x,r)$ satisfies Eq.(1.18) for each $r \in [0,1], i = 1,2$, and

$$y_1^{(k)}(0,r) = \gamma_{k1}(r), \tag{1.30}$$

$$y_2^{(k)}(0,r) = \gamma_{k2}(r), \tag{1.31}$$

$0 \le k \le n-1, r \in [0,1]$.

Now they turn to finding the r-cuts of $\overline{Y}(x)$. Since the f_j are continuous they know

$$y_1(x,r) = \min\{\sum_{j=1}^{n} f_j(\gamma)y_j(x) + \Psi(x) | \gamma_i \in \overline{\gamma}_i[r], 0 \le i \le n-1\}, \tag{1.32}$$

$$y_2(x,r) = \max\{\sum_{j=1}^{n} f_j(\gamma)y_j(x) + \Psi(x) | \gamma_i \in \overline{\gamma}_i[r], 0 \le i \le n-1\}. \tag{1.33}$$

The max and min are evaluated for each $x \in I, r \in [0,1]$. We must look more closely on how one obtains the max and min in Eqs.(1.32) and (1.33). For each $x \in I$ and $\alpha \in [0,1]$ there is a $\gamma_i^*(x,r) \in \overline{\gamma}_i[r], 0 \le i \le n-1$, so that if we set $c_j^*(x,r) = f_j(\gamma^*), \gamma^* = (\gamma_0^*(x,r), ..., \gamma_{n-1}^*(x,r))$, then

$$y_1(x,r) = \sum_{j=1}^{n} c_j^*(x,r)y_j(x) + \Psi(x). \tag{1.34}$$

similarly, for each $x \in I$ and $r \in [0,1]$ there is a $\gamma_i^{**}(x,r) \in \overline{\gamma}_i[r], 0 \le i \le n-1$, so that if they define $c_j^{**}(x,r) = f_j(\gamma^{**}), \gamma^{**} = (\gamma_0^{**}(x,r), ..., \gamma_{n-1}^{**}(x,r))$, then

$$y_2(x,r) = \sum_{j=1}^{n} c_j^{**}(x,r)y_j(x) + \Psi(x). \tag{1.35}$$

For $\overline{Y}(x)$ to be a solution c_j^* and c_j^{**} must be independent of x on the same interval. This brings use to the "interval condition" in order that $\overline{Y}(x)$ is a solution. We first define the intervals. If $I = [0, T]$, then $0 = \delta_0 < \delta_1 < ... < \delta_K = T$ and $I_k = [\delta_{k-1}, \delta_k], 1 \leq k \leq K$. If $I = [0, \infty)$, then $0 = \delta_0 < \delta_1 < ...$ with $\delta_k \rightarrow \infty$ and $I_k = [\delta_{k-1}, \delta_k], 1 \leq k < \infty$. We say that the interval condition holds if there are intervals I_k so that

$$c_j^*(x, r) = c_j^*(r), \tag{1.36}$$

$$c_j^{**}(x, r) = c_j^{**}(r), \tag{1.37}$$

$1 \geq j \leq n, r \in [0, 1]$, all $x \in I_k$, all K. What this means is that both c_j^* and $c_j^{**}(r)$ are independent of x on each I_k.

Method 2

This procedure is the reverse of Method 1 in that they first solve the fuzzy initial value problem and then check to see if it defines a fuzzy function for x in I.

Let $\overline{Y}(x)$ denote the fuzzy subset of R for each $x \in I$ so that its r-cuts are closed, bounded, intervals for all x. Set $\overline{Y}(x)[x] = [y_1(x, r), y_2(x, r)], x \in I, r \in [0, 1]$. They substitute the α-cuts of $\overline{Y}(x)$ into the differential equation and then solve for $y_1(x, r)$ and $y_2(x, r)$. Then $y_i(x, r)$ are assumed to have continuous derivatives on x of order n for all α. From Eq.(1.18) we obtain

$$[y_1^{(n)}(x, r), y_2^{(n)}(x, r)] + a_{n-1}(x)[y_1^{(n-1)}(x, r), y_2^{(n-1)}(x, r)] + ... + a_0(x) \tag{1.38}$$

$$[y_1(x, r), y_2(x, r)] = [g(x), g(x)],$$

subject to the initial conditions:(1) $y_1(0, r) = \gamma_{01}(r), ..., y_1^{n-1}(0, r) = \gamma_{n-1,1}(r)$, and

(2) $y_2(0,r) = \gamma_{02}(r), ..., y_2^{n-1}(0,r) = \gamma_{n-1,2}(r)$, to be solved for the $y_i(x,r), i = 1,2$. The symbol $y_i^k(x,r)$ is the kth derivative on x for fixed $r \in [0,1], i = 1,2$. One dose interval arithmetic in Eq.(1.38) to obtain two equations to solve simultaneously for $y_1(x,r)$ and $y_2(x,r)$.

Chapter 2

Numerical method

2.1 N-order differential equation with fuzzy initial conditions

In this section one numerical method for solving nth-order linear differential equations with fuzzy initial conditions is considered. The idea is based on the collocation method. The existence theorem of the fuzzy solution is considered. This method is illustrated by solving several examples.

We are going to solve the following problem (taken from [12], Eq.1)

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y = g(t), \tag{2.1}$$

where $a_i(t),\ 0 \le i \le n-1$, are continuous on some interval I, subject to initial conditions

$$\widetilde{y}(0) = \widetilde{b}_0,\ \widetilde{y}'(0) = \widetilde{b}_1, \ldots, \widetilde{y}^{(n-1)}(0) = \widetilde{b}_{n-1} \tag{2.2}$$

for fuzzy numbers $\widetilde{b}_i,\ 0 \le i \le n-1$. The interval I can be $[0, T]$ for some $T > 0$.

In this thesis we proposed one method for solving n-order fuzzy differential equation.

This method is to seek an approximate solution as

$$\widetilde{y}_N(t) = \sum_{k=0}^{N} \widetilde{\alpha}_k \phi_k(t), \tag{2.3}$$

where $\phi_k(t)$ are positive basic functions whose all differentiations are positive. Now, the aim is to compute the fuzzy coefficients in (2.3) by setting the error to zero as follows,

$$Error = D(\widetilde{y}^{(n)} + a_{n-1}(t)\widetilde{y}^{(n-1)} + \ldots + a_1(t)\widetilde{y}' + a_0(t)\widetilde{y}, \widetilde{g}(t)) + D(\widetilde{y}(0), \widetilde{b}_0) + \tag{2.4}$$

$$D(\widetilde{y}'(0), \widetilde{b}_1) + \ldots + D(\widetilde{y}^{(n-1)}(0), \widetilde{b}_{n-1}),$$

to illustrate this approach, let $\widetilde{y}_N(t)$ be the fuzzy solution of (2.1) such that

$$\underline{y}_N^{(i)}(t,r) = \underline{\sum_{k=0}^{N} \alpha_k(r)\phi_k^{(i)}(t)} = \sum_{k=0}^{N} \underline{\alpha_k(r)\phi_k^{(i)}(t)} = \sum_{k=0}^{N} \underline{\alpha_k}(r)\phi_k^{(i)}(t), \tag{2.5}$$

$$\overline{y}_N^{(i)}(t,r) = \overline{\sum_{k=0}^{N} \alpha_k(r)\phi_k^{(i)}(t)} = \sum_{k=0}^{N} \overline{\alpha_k(r)\phi_k^{(i)}(t)} = \sum_{k=0}^{N} \overline{\alpha_k}(r)\phi_k^{(i)}(t), \tag{2.6}$$

for $i = 0, 1, \ldots, n$.

We substitute (2.5) and (2.6) in (2.4), then

$$\begin{cases} \underline{y^{(n)}(t,r) + a_{n-1}(t)y^{(n-1)}(t,r) + \ldots + a_1(t)y'(t,r) + a_0(t)y(t,r)} = \underline{g(t,r)}, \\ \underline{y}(0,r) = \underline{b_0}(0,r), \\ \vdots \\ \underline{y}^{(n-1)}(0,r) = \underline{b_{n-1}}(0,r), \\ \\ \overline{y^{(n)}(t,r) + a_{n-1}(t)y^{(n-1)}(t,r) + \ldots + a_1(t)y'(t,r) + a_0(t)y(t,r)} = \overline{g(t,r)}, \\ \overline{y}(0,r) = \overline{b_0}(0,r), \\ \vdots \\ \overline{y}^{(n-1)}(0,r) = \overline{b_{n-1}}(0,r). \end{cases} \tag{2.7}$$

Lemma 2.1.1. *Let basic functions $\phi_k(t)$ and all of their differentiations be positive, without loss of generality. Then $(\underline{y}_N)^{(i)}(t) = \underline{y_N^{(i)}}(t)$ and $(\overline{y}_N)^{(i)}(t) = \overline{y_N^{(i)}}(t)$.*

proof. *Since $\widetilde{y}_N(t) = \sum_{k=0}^{N} \widetilde{\alpha}_k \phi_k(t)$, and because $\phi_k(t)$, $k = 0, 1, ..., N$, are positive, we have*

$$(\underline{y}_N)(t) = \sum_{k=0}^{N} \underline{\alpha_k} \phi_k(t), \tag{2.8}$$

thus

$$(\underline{y}_N)^{(i)}(t) = \sum_{k=0}^{N} \underline{\alpha_k} \phi_k^{(i)}(t). \tag{2.9}$$

Moreover $\widetilde{y}_N^{(i)}(t) = \sum_{k=0}^{N} \widetilde{\alpha}_k \phi_k^{(i)}(t)$ and also $\phi_k^{(i)}(t)$, $i = 0, 1, ..., n$ are positive, therefore

$$\underline{y_N^{(i)}}(t) = \sum_{k=0}^{N} \underline{\alpha_k} \phi_k^{(i)}(t), \tag{2.10}$$

and thus from (2.2.9) and (2.2.10) we have $(\underline{y}_N)^{(i)}(t) = \underline{y_N^{(i)}}(t)$. In a similar way, we can prove that $(\overline{y}_N)^{(i)}(t) = \overline{y_N^{(i)}}(t)$.

Now, to simplify and solve (2.7), we consider the following three cases. The existence conditions of solutions provided, following the three cases below. Also, in the examples we obtain exact solution by Buckly-Feuring method.[12].

2.1.1 Case 1

Suppose that coefficients $a_{n-1}(t), a_{n-2}(t), \ldots, a_0(t)$ are nonnegative. From (2.7),

$$\underline{y^{(n)}}(t,r)+a_{n-1}(t)\underline{y^{(n-1)}}(t,r)+a_{n-2}(t)\underline{y^{(n-2)}}(t,r)\ldots+a_1(t)\underline{y'}(t,r)+a_0(t)\underline{y}(t,r) = \underline{g}(t,r), \tag{2.11}$$

$$\overline{y^{(n)}}(t,r)+a_{n-1}(t)\overline{y^{(n-1)}}(t,r)+a_{n-2}(t)\overline{y^{(n-2)}}(t,r)\ldots+a_1(t)\overline{y'}(t,r)+a_0(t)\overline{y}(t,r) = \overline{g}(t,r), \tag{2.12}$$

$\underline{y}(0,r)=\underline{b_0}(0,r),\dots,\underline{y^{(n-1)}}(0,r)=\underline{b_{n-1}}(0,r),\overline{y}(0,r)=\overline{b_0}(0,r),\dots,$

$\overline{y^{(n-1)}}(0,r)=\overline{b_{n-1}}(0,r).$

If (2.5) and (2.6) are substituted in (2.11) and (2.12), respectively, then

$$\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k^{(n)}(t)+a_{n-1}(t)\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k^{(n-1)}(t)+\ldots+a_0(t)\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k(t)=\underline{g}(t,r),$$

and

$$\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k^{(n)}(t)+a_{n-1}(t)\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k^{(n-1)}(t)+\ldots+a_0(t)\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k(t)=\overline{g}(t,r),$$

also

$$\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k(0)=\underline{b}_0(r),\dots,\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k^{(n-1)}(0)=\underline{b}_{n-1}(r),$$
$$\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k(0)=\overline{b}_0(r),\dots,\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k^{(n-1)}(0)=\overline{b}_{n-1}(r).$$

By setting

$$\begin{cases}\phi_k^{(n)}(t)+a_{n-1}(t)\phi_k^{(n-1)}(t)+a_{n-2}(t)\phi_k^{(n-2)}(t)+\ldots+a_0(t)\phi_k(t)=\beta_k,\\ \phi_k^{(j)}(0)=\sigma_{jk},\quad j=0,1,\dots,n-1,k=0,1,\dots,N,\end{cases}$$

the following system is obtained:

$$\begin{cases}\sum_{k=0}^{N}\underline{\alpha}_k(r)\beta_k=\underline{g}(t,r),\\ \sum_{k=0}^{N}\underline{\alpha}_k(r)\sigma_{0k}=\underline{b}_0(r),\\ \vdots\\ \sum_{k=0}^{N}\underline{\alpha}_k(r)\sigma_{n-1\ k}=\underline{b}_{n-1}(r),\\ \\ \sum_{k=0}^{N}\overline{\alpha}_k(r)\beta_k=\overline{g}(t,r),\\ \sum_{k=0}^{N}\overline{\alpha}_k(r)\sigma_{0k}=\overline{b}_0(r),\\ \vdots\\ \sum_{k=0}^{N}\overline{\alpha}_k(r)\sigma_{n-1\ k}=\overline{b}_{n-1}(r).\end{cases}\tag{2.13}$$

Eqs. (2.13) are a system of linear equation $SX = Y$ such that

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}_{2(N+1)\times 2(N+1)}, S_1 = \begin{bmatrix} \beta_0 & \beta_1 & \dots & \beta_N \\ \sigma_{00} & \sigma_{01} & \dots & \sigma_{0N} \\ \vdots & \vdots & \dots & \vdots \\ \sigma_{n-1\ 0} & \sigma_{n-1\ 1} & \dots & \sigma_{n-1\ N} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$S_2 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$X = [\underline{\alpha}_0, \underline{\alpha}_1, \underline{\alpha}_2, \dots, \underline{\alpha}_N, \overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_N]$,

$Y = [\underline{g}(t,r), \underline{b}_0(r), \dots, \underline{b}_{n-1}(r), \overline{g}(t,r), \overline{b}_0(r), \dots, \overline{b}_{n-1}(r)]$.

The parameters $\underline{\alpha}_0, \dots, \underline{\alpha}_N, \overline{\alpha}_0, \dots, \overline{\alpha}_N$, are obtained by solving (2.13) by setting $t = a$, $a \in [0, T]$. These parameters yield the fuzzy approximate solution $(\underline{y}(t,r), \overline{y}(t,r))$.

Example 2.1.1. Consider the following second-order fuzzy linear differential equation

$$\begin{cases} y'' + y = -t, \quad t \in [0,1] \\ \widetilde{y}(0) = (0.1r - 0.1, 0.1 - 0.1r), \\ \widetilde{y}'(0) = (0.088 + 0.1r, 0.288 - 0.1r). \end{cases} \tag{2.14}$$

The exact solution is as follows:

$\underline{Y}(t,r) = (0.1r - 0.1)cos(t) + (1.088 + 0.1r)sin(t) - t$,

$\overline{Y}(t,r) = (0.1 - 0.1r)cos(t) + (1.288 - 0.1r)sin(t) - t$

If $\phi_k(t) = t^k,\ k = 0, 1, 2$, from (2.5)and (2.6),

$$\underline{y}_2(t) = \underline{\alpha}_0(r) + \underline{\alpha}_1(r)t + \underline{\alpha}_2(r)t^2, \qquad \overline{y}_2(t) = \overline{\alpha}_0(r) + \overline{\alpha}_1(r)t + \overline{\alpha}_2(r)t^2. \tag{2.15}$$

From the initial conditions in (2.14) and (2.15), we have

$$\begin{cases} 2\underline{\alpha}_2(r) + \underline{\alpha}_0(r) + \underline{\alpha}_1(r)t + \underline{\alpha}_2(r)t^2 = -t, \\ \underline{\alpha}_0(r) = 0.1r - 0.1, \\ \underline{\alpha}_1(r) = 0.088 + 0.1r, \\ 2\overline{\alpha}_2(r) + \overline{\alpha}_0(r) + \overline{\alpha}_1(r)t + \overline{\alpha}_2(r)t^2 = -t, \\ \overline{\alpha}_0(r) = 0.1 - 0.1r, \\ \overline{\alpha}_1(r) = 0.288 - 0.1r. \end{cases}$$

Then the following system is obtained,

$$\begin{bmatrix} 1 & t & 2+t^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 2+t^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \underline{\alpha}_0(r) \\ \underline{\alpha}_1(r) \\ \underline{\alpha}_2(r) \\ \overline{\alpha}_0(r) \\ \overline{\alpha}_1(r) \\ \overline{\alpha}_2(r) \end{bmatrix} = \begin{bmatrix} -t \\ 0.1r - 0.1 \\ 0.088 + 0.1r \\ -t \\ 0.1 - 0.1r \\ 0.288 - 0.1r \end{bmatrix}.$$

By setting $t = \frac{1}{2}$, the parameters $\underline{\alpha}_0(r), \underline{\alpha}_1(r), \underline{\alpha}_2(r), \overline{\alpha}_0(r), \overline{\alpha}_1(r), \overline{\alpha}_2(r)$ are obtained, and by putting them into (2.15) we have:

$\underline{y}(t,r) = 0.1r - .1 + (0.88e - 1 + 0.1r)t + (-0.1973333334 - 0.6666666666e - 1r)t^2,$

$\overline{y}(t,r) = 0.1 - .1r + (0.288 - 0.1r)t + (-.3306666666 + 0.6666666666e - 1r)t^2.$

Tables 3.1, 3.2 and 3.3 show the comparison of the exact and approximate solutions at $t = 0, 0.001, 0.01$ for any $r \in (0,1]$.

r	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	-0.1	-0.1	0	0.1	0.1	0
0.1	-0.09	-0.09	0	0.09	0.09	0
0.2	-0.08	-0.08	0	0.08	0.08	0
0.3	-0.07	-0.07	0	0.07	0.07	0
0.4	-0.06	-0.06	0	0.06	0.06	0
0.5	-0.05	-0.05	0	0.05	0.05	0
0.6	-0.04	-0.04	0	0.04	0.04	0
0.7	-0.03	-0.03	0	0.03	0.03	0
0.8	-0.02	-0.02	0	0.02	0.02	0
0.9	-0.01	-0.01	0	0.01	0.01	0
1	0	0	0	0	0	0

Table. 3.1, t=0.

r	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	-0.9991219733e-1	-0.9991195018e-1	0.24715e-6	.1002876693	.1002879498	0.2805e-6
0.1	-0.8990220400e-1	-0.8990195518e-1	0.24882e-6	0.9027767600e-1	0.9027795479e-1	0.27879e-6
0.2	-0.7989221067e-1	-0.7989196018e-1	0.25049e-6	0.8026768267e-1	0.8026795979e-1	0.27712e-6
0.3	-0.6988221733e-1	-0.6988196519e-1	0.25214e-6	0.7025768933e-1	0.7025796479e-1	0.27546e-6
0.4	-0.5987222400e-1	-0.5987197019e-1	0.25381e-6	0.6024769600e-1	0.6024796979e-1	0.27379e-6
0.5	-0.4986223067e-1	-0.4986197519e-1	0.25548e-6	0.5023770267e-1	0.5023797479e-1	0.27212e-6
0.6	-0.3985223733e-1	-0.3985198019e-1	0.25714e-6	0.4022770933e-1	0.4022797980e-1	0.27047e-6
0.7	-0.2984224400e-1	-0.2984198519e-1	0.25881e-6	0.3021771600e-1	0.3021798480e-1	0.26880e-6
0.8	-0.1983225067e-1	-0.1983199020e-1	0.26047e-6	0.2020772267e-1	0.2020798980e-1	0.26713e-6
0.9	-0.9822257333e-2	-0.9821995196e-2	0.262137e-6	0.1019772933e-1	0.1019799480e-1	0.26547e-6
1	0.1877360000e-3	0.187999802e-3	0.2638020e-6	0.1877360000e-3	0.187999802e-3	0.2638020e-6

Table. 3.2, t=0.001.

r	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	-0.9913973333e-1	-0.9911518137e-1	0.2455196e-4	.1028469333	.1028747854	0.278521e-4
0.1	-0.8904040000e-1	-0.8901568304e-1	0.2471696e-4	0.9274760000e-1	0.927752870e-1	0.2768700e-4
0.2	-0.7894106667e-1	-0.7891618470e-1	0.2488197e-4	0.8264826667e-1	0.8267578870e-1	0.2752203e-4
0.3	-0.6884173333e-1	-0.6881668636e-1	0.2504697e-4	0.7254893333e-1	0.7257629036e-1	0.2735703e-4
0.4	-0.5874240000e-1	-0.5871718802e-1	0.2521198e-4	0.6244960000e-1	0.6247679202e-1	0.2719202e-4
0.5	-0.4864306667e-1	-0.4861768969e-1	0.2537698e-4	0.5235026667e-1	0.5237729369e-1	0.2702702e-4
0.6	-0.3854373333e-1	-0.3851819135e-1	0.2554198e-4	0.4225093333e-1	0.4227779535e-1	0.2686202e-4
0.7	-0.2844440000e-1	-0.2841869301e-1	0.2570699e-4	0.3215160000e-1	0.3217829701e-1	0.2669701e-4
0.8	-0.1834506667e-1	-0.1831919468e-1	0.2587199e-4	0.2205226667e-1	0.2207879868e-1	0.2653201e-4
0.9	-0.8245733333e-2	-0.8219696334e-2	0.26036999e-4	0.1195293333e-1	0.1197930033e-1	0.2636700e-4
1	0.1853600000e-2	0.187980200e-2	0.26202000e-4	0.1853600000e-2	0.187980200e-2	0.26202000e-4

Table. 3.3, t=0.01.

2.1.2 Case 2

Suppose that coefficients $a_{n-1}(t), a_{n-2}(t), \ldots, a_0(t)$ are negative. From (2.7),

$$\underline{y}^{(n)}(t,r)+a_{n-1}(t)\overline{y}^{(n-1)}(t,r)+a_{n-2}(t)\overline{y}^{(n-2)}(t,r)\ldots+a_1(t)\overline{y}'(t,r)+a_0(t)\overline{y}(t,r)=\underline{g}(t,r), \tag{2.16}$$

$$\overline{y}^{(n)}(t,r)+a_{n-1}(t)\underline{y}^{(n-1)}(t,r)+a_{n-2}(t)\underline{y}^{(n-2)}(t,r)\ldots+a_1(t)\underline{y}'(t,r)+a_0(t)\underline{y}(t,r)=\overline{g}(t,r). \tag{2.17}$$

If (2.5) and (2.6) are substituted in (2.16) and (2.17), respectively, then

$$\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k^{(n)}(t)+a_{n-1}(t)\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k^{(n-1)}(t)+\ldots+a_0(t)\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k(t)=\underline{g}(t,r),$$

$$\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k^{(n)}(t)+a_{n-1}(t)\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k^{(n-1)}(t)+\ldots+a_0(t)\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k(t)=\overline{g}(t,r),$$

also

$$\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k(0)=\underline{b}_0(r),\ldots,\sum_{k=0}^{N}\underline{\alpha}_k(r)\phi_k^{(n-1)}(0)=\underline{b}_{n-1}(r),$$
$$\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k(0)=\overline{b}_0(r),\ldots,\sum_{k=0}^{N}\overline{\alpha}_k(r)\phi_k^{(n-1)}(0)=\overline{b}_{n-1}(r),$$

let

$$\begin{cases}\phi_k^{(n)}(t)=\gamma_{nk},\phi_k^{(j)}(0)=\sigma_{jk}, \quad j=0,1,\ldots,n-1, \quad k=0,1,\ldots,N,\\ \\ a_{n-1}(t)\phi_k^{(n-1)}(t)+a_{n-2}(t)\phi_k^{(n-2)}(t)+\ldots+a_0(t)\phi_k(t)=\delta_k,\end{cases}$$

then the following system is obtained:

$$\begin{cases}\sum_{k=0}^{N}(\underline{\alpha}_k(r)\gamma_{nk}+\overline{\alpha}_k(r)\delta_k)=\underline{g}(t,r),\\ \sum_{k=0}^{N}\underline{\alpha}_k(r)\sigma_{0k}=\underline{b}_0(r),\\ \vdots\\ \sum_{k=0}^{N}\underline{\alpha}_k(r)\sigma_{n-1\ k}=\underline{b}_{n-1}(r),\\ \\ \sum_{k=0}^{N}(\overline{\alpha}_k(r)\gamma_{nk}+\underline{\alpha}_k(r)\delta_k)=\overline{g}(t,r),\\ \sum_{k=0}^{N}\overline{\alpha}_k(r)\sigma_{0k}=\overline{b}_0(r),\\ \vdots\\ \sum_{k=0}^{N}\overline{\alpha}_k(r)\sigma_{n-1\ k}=\overline{b}_{n-1}(r),\end{cases} \tag{2.18}$$

Eqs. (2.18) are a system of linear equation $SX=Y$ such that

$$S=\begin{bmatrix}S_1 & S_2\\ S_2 & S_1\end{bmatrix}_{2(N+1)\times 2(N+1)},S_1=\begin{bmatrix}\gamma_{n0} & \gamma_{n1} & \ldots & \gamma_{nN}\\ \sigma_{00} & \sigma_{01} & \ldots & \sigma_{0N}\\ \vdots & \vdots & \ldots & \vdots\\ \sigma_{n-1\ 0} & \sigma_{n-1\ 1} & \ldots & \sigma_{n-1\ N}\end{bmatrix}_{(N+1)\times(N+1)},$$

$$S_2=\begin{bmatrix}\delta_0 & \delta_1 & \ldots & \delta_N\\ 0 & 0 & \ldots & 0\\ \vdots & \vdots & \ldots & \vdots\\ 0 & 0 & \ldots & 0\end{bmatrix}_{(N+1)\times(N+1)},$$

$X = [\underline{\alpha}_0, \underline{\alpha}_1, \underline{\alpha}_2, \ldots, \underline{\alpha}_N, \overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_N]$,

$Y = [\underline{g}(t,r), \underline{b}_0(r), \ldots, \underline{b}_{n-1}(r), \overline{g}(t,r), \overline{b}_0(r), \ldots, \overline{b}_{n-1}(r)]$.

The parameters $\underline{\alpha}_0, \ldots, \underline{\alpha}_N, \overline{\alpha}_0, \ldots, \overline{\alpha}_N$, are obtained by solving (2.18) by setting $t = a$, $a \in [0,T]$. These parameters yield the fuzzy approximate solution $(\underline{y}(t,r), \overline{y}(t,r))$.

Example 2.1.2. Consider the following second-order fuzzy linear differential equation

$$\begin{cases} y'' - \frac{2}{t^2}y = (\frac{2}{t}), \ t \geq 1 \\ \widetilde{y}(1) = (0.1r - 0.1, 0.1 - 0.1r), \\ \widetilde{y}'(1) = (-0.25 + 0.25r, 0.25 - 0.25r), \end{cases} \tag{2.19}$$

The exact solution is as follows:

$\underline{Y}(t,r) = (\frac{1}{t})(-0.500000001e - 1r + .3833333334) + (.15r + .5166666666)t^2 - t$,

$\underline{Y}(t,r) = (\frac{1}{t})(0.5000000003e - 1r + .2833333337) + (-.15 + .8166666666)t^2 - t$.

If $\phi_k(t) = t^k, \ k = 0, 1, 2$, from (2.5), (2.6),

$$\underline{y}_2(t) = \underline{\alpha}_0(r) + \underline{\alpha}_1(r)t + \underline{\alpha}_2(r)t^2, \qquad \overline{y}_2(t) = \overline{\alpha}_0(r) + \overline{\alpha}_1(r)t + \overline{\alpha}_2(r)t^2. \tag{2.20}$$

From the initial conditions in (2.20) and (2.19) the following system is obtained

$$\begin{bmatrix} 0 & 0 & 2 & -\frac{2}{t^2} & -\frac{2}{t} & -2 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ -\frac{2}{t^2} & -\frac{2}{t} & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \underline{\alpha}_0(r) \\ \underline{\alpha}_1(r) \\ \underline{\alpha}_2(r) \\ \overline{\alpha}_0(r) \\ \overline{\alpha}_1(r) \\ \overline{\alpha}_2(r) \end{bmatrix} = \begin{bmatrix} \frac{2}{t} \\ 0.1r - 0.1 \\ -0.25 + 0.25r \\ \frac{2}{t} \\ 0.1 - 0.1r \\ 0.25 - 0.25r \end{bmatrix}.$$

By setting $t = \frac{3}{2}$ the parameters $\underline{\alpha}_0(r), \underline{\alpha}_1(r), \underline{\alpha}_2(r), \overline{\alpha}_0(r), \overline{\alpha}_1(r), \overline{\alpha}_2(r)$ are obtained, and by putting them into (2.20) we have:

$$\underline{y}(t,r) = (0.99 - 0.24r) + (-1.93 + 0.43r)t + (0.84 - 0.8999999996e - 1r)t^2,$$

$$\overline{y}(t,r) = (0.51 + 0.24r) + (-1.07 - 0.43r)t + (0.66 + 0.8999999996e - 1r)t^2.$$

Tables 3.4, 3.5 and 3.6 show the comparison of the exact and the approximate solutions at the $t = 1, 1.001, 1.01$ for any $r \in (0, 1]$.

r	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	-0.1	-0.1	0	0.1	0.1	0
0.1	-0.09	-0.09	0	0.09	0.09	0
0.2	-0.08	-0.08	0	0.08	0.08	0
0.3	-0.07	-0.07	0	0.07	0.07	0
0.4	-0.06	-0.06	0	0.06	0.06	0
0.5	-0.05	-0.05	0	0.05	0.05	0
0.6	-0.04	-0.400000001e-1	0.1e-9	0.04	0.04	0
0.7	-0.03	-0.300000001e-1	0.1e-9	0.03	0.03	0
0.8	-0.02	-0.02	0	0.02	0.02	0
0.9	-0.01	-0.100000001e-1	0.1e-9	0.01	0.01	0
1	-0.1e-9	0.1e-9	0	0	0	

Table. 3.4, t=1.

r	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	-.1002491600	-.1003491004	0.999404e-4	.1002506600	.100351100	-0.1004400e-3
0.1	-0.902241690e-1	-0.903140904e-1	0.899214e-4	0.902256690e-1	0.90316090e-1	-0.904210e-4
0.2	-0.801991780e-1	-0.802790804e-1	0.799024e-4	0.802006780e-1	0.80281080e-1	-0.804020e-4
0.3	-0.701741870e-1	-0.702440704e-1	0.698834e-4	0.701756870e-1	0.70246070e-1	-0.703830e-4
0.4	-0.601491960e-1	-0.602090604e-1	0.598644e-4	0.601506960e-1	0.60211060e-1	-0.603640e-4
0.5	-0.501242050e-1	-0.501740504e-1	0.498454e-4	0.501257050e-1	0.50176050e-1	-0.503450e-4
0.6	-0.400992140e-1	-0.401390405e-1	0.398265e-4	0.401007140e-1	0.40141040e-1	-0.403260e-4
0.7	-0.300742230e-1	-0.301040304e-1	0.298074e-4	0.300757230e-1	0.30106030e-1	-0.303070e-4
0.8	-0.200492320e-1	-0.200690204e-1	0.197884e-4	0.200507320e-1	0.20071020e-1	-0.202880e-4
0.9	-0.100242410e-1	-0.100340104e-1	0.97694e-5	0.100257410e-1	0.10036010e-1	-0.102690e-4
1	0.7500e-6	0.1000e-5	-0.2500e-6	0.7500e-6	0.1000e-5	-0.2500e-6

Table. 3.5, t=1.001.

r	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	-.1024160000	-.1034103795	0.9943795e-3	.1025660000	.103609720	-0.10437200e-2
0.1	-0.921669000e-1	-0.930593746e-1	0.8924746e-3	0.923169000e-1	0.93258715e-1	-0.9418150e-3
0.2	-0.819178000e-1	-0.827083696e-1	0.7905696e-3	0.820678000e-1	0.82907710e-1	-0.8399100e-3
0.3	-0.716687000e-1	-0.723573647e-1	0.6886647e-3	0.718187000e-1	0.72556705e-1	-0.7380050e-3
0.4	-0.614196000e-1	-0.620063597e-1	0.5867597e-3	0.615696000e-1	0.62205700e-1	-0.6361000e-3
0.5	-0.511705000e-1	-0.516553548e-1	0.4848548e-3	0.513205000e-1	0.51854695e-1	-0.5341950e-3
0.6	-0.409214000e-1	-0.413043499e-1	0.3829499e-3	0.410714000e-1	0.41503690e-1	-0.4322900e-3
0.7	-0.306723000e-1	-0.309533450e-1	0.2810450e-3	0.308223000e-1	0.31152685e-1	-0.3303850e-3
0.8	-0.204232000e-1	-0.206023400e-1	0.1791400e-3	0.205732000e-1	0.20801680e-1	-0.2284800e-3
0.9	-0.101741000e-1	-0.102513351e-1	0.772351e-4	0.103241000e-1	0.10450675e-1	-0.1265750e-3
1	0.750000e-4	0.99670e-4	-0.246700e-4	0.750000e-4	0.99670e-4	-0.246700e-4

Table. 3.6, t=1.01.

2.1.3 Case 3

Suppose that some coefficients, say, $a_{n-m-1}(t), a_{n-m-2}(t), \ldots, a_1(t), a_0(t)$, are negative. From (2.7),

$$\underline{y^{(n)}}(t,r) + a_{n-1}(t)\underline{y^{(n-1)}}(t,r) + \ldots + a_{n-m}(t)\underline{y^{(n-m)}}(t,r) + a_{n-m-1}(t)\overline{y^{(n-m-1)}} \quad (2.21)$$

$(t,r) + \ldots + a_0(t)\overline{y}(t,r) = \underline{g}(t,r),$

$$\overline{y^{(n)}}(t,r) + a_{n-1}(t)\overline{y^{(n-1)}}(t,r) + \ldots + a_{n-m}(t)\overline{y^{(n-m)}(t,r)} + a_{n-m-1}(t)\underline{y^{(n-m-1)}} \quad (2.22)$$

$(t,r) + \ldots + a_0(t)\underline{y}(t,r) = \overline{g}(t,r).$

If (2.5), (2.6) are substituted in (2.21), (2.22), respectively, then

$\sum_{k=0}^{N} \underline{\alpha}_k(r)\phi_k^{(n)}(t) + a_{n-1}(t)\sum_{k=0}^{N} \underline{\alpha}_k(r)\phi_k^{(n-1)}(t) + \ldots + a_{n-m}(t)\sum_{k=0}^{N} \underline{\alpha}_k(r)\phi_k^{(n-m)}(t) +$

$a_{n-m-1}(t)\sum_{k=0}^{N} \overline{\alpha}_k(r)\phi_k^{(n-m-1)}(t) + \ldots + a_0(t)\sum_{k=0}^{N} \overline{\alpha}_k(r)\phi_k(t) = \underline{g}(t,r),$

and

$\sum_{k=0}^{N} \overline{\alpha}_k(r)\phi_k^{(n)}(t) + a_{n-1}(t)\sum_{k=0}^{N} \overline{\alpha}_k(r)\phi_k^{(n-1)}(t) + \ldots + a_{n-m}(t)\sum_{k=0}^{N} \overline{\alpha}_k(r)\phi_k^{(n-m)}(t) +$

$a_{n-m-1}(t)\sum_{k=0}^{N} \underline{\alpha}_k(r)\phi_k^{(n-m-1)}(t) + \ldots + a_0(t)\sum_{k=0}^{N} \underline{\alpha}_k(r)\phi_k(t) = \overline{g}(t,r),$

also

$$\sum_{k=0}^{N} \underline{\alpha}_k(r)\phi_k(0) = \underline{b}_0(r), \ldots, \sum_{k=0}^{N} \underline{\alpha}_k(r)\phi_k^{(n-1)}(0) = \underline{b}_{n-1}(r),$$
$$\sum_{k=0}^{N} \overline{\alpha}_k(r)\phi_k(0) = \overline{b}_0(r), \ldots, \sum_{k=0}^{N} \overline{\alpha}_k(r)\phi_k^{(n-1)}(0) = \overline{b}_{n-1}(r),$$

let

$$\begin{cases} \phi_k^{(n)}(t) + a_{n-1}(t)\phi_k^{(n-1)}(t) + \ldots + a_{n-m}(t)\phi_k^{(n-m)}(t) = \eta_k, \\ a_{n-m-1}(t)\phi_k^{n-m-1}(t) + a_{n-m-2}(t)\phi_k^{n-m-2}(t) + \ldots + a_1(t)\phi_k'(t) + a_0(t)\phi_k(t) = \xi_k, \\ \phi_k^{(j)}(0) = \sigma_{jk}, \quad j = 0, 1, \ldots, n-1, \end{cases}$$

the following system is obtained:

$$\begin{cases} \sum_{k=0}^{N}(\underline{\alpha}_k(r)\eta_k + \overline{\alpha}_k(r)\xi_k) = \underline{g}(t,r), \\ \sum_{k=0}^{N} \underline{\alpha}_k(r)\sigma_{0k} = \underline{b}_0(r), \\ \vdots \\ \sum_{k=0}^{N} \underline{\alpha}_k(r)\sigma_{n-1\ k} = \underline{b}_{n-1}(r), \\ \\ \sum_{k=0}^{N}(\overline{\alpha}_k(r)\eta_k + \underline{\alpha}_k(r)\xi_k) = \overline{g}(t,r), \\ \sum_{k=0}^{N} \overline{\alpha}_k(r)\sigma_{0k} = \overline{b}_0(r), \\ \vdots \\ \sum_{k=0}^{N} \overline{\alpha}_k(r)\sigma_{n-1\ k} = \overline{b}_{n-1}(r), \end{cases} \tag{2.23}$$

Eqs. (2.23) are a system of linear equation $SX = Y$ such that such that

$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}_{2(N+1)\times 2(N+1)}, S_1 = \begin{bmatrix} \eta_0 & \eta_1 & \dots & \eta_N \\ \sigma_{00} & \sigma_{01} & \dots & \sigma_{0N} \\ \vdots & \vdots & \dots & \vdots \\ \sigma_{n-1\ 0} & \sigma_{n-1\ 1} & \dots & \sigma_{n-1\ N} \end{bmatrix}_{(N+1)\times(N+1)},$$

$$S_2 = \begin{bmatrix} \xi_0 & \xi_1 & \dots & \xi_N \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$X = [\underline{\alpha}_0, \underline{\alpha}_1, \underline{\alpha}_2, \dots, \underline{\alpha}_N, \overline{\alpha}_0, \overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_N]$,

$Y = [\underline{g}(t,r), \underline{b}_0(r), \dots, \underline{b}_{n-1}(r), \overline{g}(t,r), \overline{b}_0(r), \dots, \overline{b}_{n-1}(r)]$.

The parameters $\underline{\alpha}_0, \dots, \underline{\alpha}_N, \overline{\alpha}_0, \dots, \overline{\alpha}_N$, are obtained by solving (2.23) by setting $t = a$, $a \in [0, T]$.These parameters yield the fuzzy approximate solution $(\underline{y}(t,r), \overline{y}(t,r))$.

Example 2.1.3. Consider the following second-order fuzzy linear differential equation

$$\begin{cases} y'' - 4y' + 4y = 0, \ t \geq 0 \\ \widetilde{y}(0) = (2+r, 4-r), \\ \widetilde{y}'(0) = (5+r, 7-r). \end{cases} \tag{2.24}$$

The exact solution is as follows:

$\underline{Y}(t,r) = (2+r)e^{2t} + (1-r)te^{2t}$,

$\overline{Y}(t,r) = (4-r)e^{2t} + (r-1)te^{2t}$.

If $\phi_k(t) = t^k$, $k = 0, 1, 2$, from (2.5) and (2.6),

$$\underline{y}_2(t) = \underline{\alpha}_0(r) + \underline{\alpha}_1(r)t + \underline{\alpha}_2(r)t^2, \qquad \overline{y}_2(t) = \overline{\alpha}_0(r) + \overline{\alpha}_1(r)t + \overline{\alpha}_2(r)t^2. \tag{2.25}$$

From the initial conditions in (2.25) and (2.24), the following system is obtained:

$$\begin{bmatrix} 4 & 4t & 2+4t^2 & 0 & -4 & -8t \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & -8t & 4 & 4t & 2+4t^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \underline{\alpha}_0 \\ \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \overline{\alpha}_0 \\ \overline{\alpha}_1 \\ \overline{\alpha}_2 \end{bmatrix} = \begin{bmatrix} -0.25+0.25r \\ 2+r \\ 5+r \\ 0.25-0.25r \\ 4-r \\ 7-r \end{bmatrix},$$

By setting $t = \frac{1}{2}$, the parameters $\underline{\alpha}_0(r), \underline{\alpha}_1(r), \underline{\alpha}_2(r), \overline{\alpha}_0(r), \overline{\alpha}_1(r), \overline{\alpha}_2(r)$ are obtained, and by putting them into (2.25) we have:

$\underline{y}(t,r) = (2+r) + (5+r)t + (1.428571427 - 1.428571430r)t^2,$

$\overline{y}(t,r) = (4-r) + (7-r)t + (-1.428571429 + 1.428571430r)t^2.$

Table 3.7 show the comparison of the exact and the approximate solutions at $t = 0.01$ for any $r \in (0, 1]$.

r	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	2.050142857	2.050604693	0.461836e-3	4.069857143	4.070603347	0.746204e-3
0.1	2.151128571	2.151604626	0.476055e-3	3.968871429	3.969603414	0.731985e-3
0.2	2.252114286	2.252604559	0.490273e-3	3.867885714	3.868603481	0.717767e-3
0.3	2.353100000	2.353604491	0.504491e-3	3.766900000	3.767603549	0.703549e-3
0.4	2.454085714	2.454604424	0.518710e-3	3.665914286	3.666603616	0.689330e-3
0.5	2.555071429	2.555604357	0.532928e-3	3.564928571	3.565603683	0.675112e-3
0.6	2.656057143	2.656604289	0.547146e-3	3.463942857	3.464603751	0.660894e-3
0.7	2.757042857	2.757604222	0.561365e-3	3.362957143	3.363603818	0.646675e-3
0.8	2.858028571	2.858604155	0.575584e-3	3.261971429	3.262603885	0.632456e-3
0.9	2.959014286	2.959604087	0.589801e-3	3.160985714	3.161603953	0.618239e-3
1	3.060	3.060604020	0.604020e-3	3.060	3.060604020	0.604020e-3

Table. 3.7, t=0.01.

Example 2.1.4. Electrical circuit[10]

Consider the electrical circuit shown in Figure 1, where $L = 1h$, $R = 2\Omega$, $C = 0.25f$ and $E(t) = 20cost$.

If Q is the charge on the capacitor at time $t > 0$, then

$$Q'' + 2Q' + 4Q = 50cost,$$

for $Q(0) = (4 + r, 6 - r)$, $Q'(0) = (r, 2 - r)$.

Fig. 7. Electrical circuit in example 2.4.

The exact solution is

$$\underline{Q}(t,r) = e^{-t}(4+r-\frac{150}{13})\cos(\sqrt{3}t)+(6-(\frac{250}{13})/\sqrt{3}\sin(\sqrt{3}t)+(\frac{150}{13})\cos(t)+(\frac{100}{13})\sin(t),$$

$$\overline{Q}(t,r) = e^{-t}((6-r-\frac{150}{13})\cos(\sqrt{3}t)+(6-(\frac{250}{13})/\sqrt{3}\sin(\sqrt{3}t)+(\frac{150}{13})\cos(t)+(\frac{100}{13})\sin(t).$$

If $\phi_k(t) = e^k$, $k = 0, 1, 2, 3, 4$, from (2.5) and (2.6),

$$\underline{y}_4(t) = \underline{\alpha}_0(r)+\underline{\alpha}_1(r)e^t+\underline{\alpha}_2(r)e^{2t}+\underline{\alpha}_3 e^{3t}+\underline{\alpha}_4 e^{4t},\ \overline{y}_4(t) = \overline{\alpha}_0(r)+\overline{\alpha}_1(r)e^t+\overline{\alpha}_2(r)e^{2t}+\overline{\alpha}_3 e^{3t}+\overline{\alpha}_4 e^{4t}. \tag{2.26}$$

Then the following matrix is obtained:

$$\begin{bmatrix} 4 & 7e^t & 12e^{2t} & 19e^{3t} & 28e^{4t} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 7e^t & 12e^{2t} & 19e^{3t} & 28e^{4t} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By setting $t = 1, 1.1, 1.2$ the following system is obtained:

$$\begin{bmatrix} 4 & 7e^{1} & 12e^{2} & 19e^{3} & 28e^{4} & 0 & 0 & 0 & 0 & 0 \\ 4 & 7e^{1.1} & 12e^{2.2} & 19e^{3.3} & 28e^{4.4} & 0 & 0 & 0 & 0 & 0 \\ 4 & 7e^{1.2} & 12e^{2.4} & 19e^{3.6} & 28e^{4.8} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 7e^{1} & 12e^{2} & 19e^{3} & 28e^{4} \\ 0 & 0 & 0 & 0 & 0 & 4 & 7e^{1.1} & 12e^{2.2} & 19e^{3.3} & 28e^{4.4} \\ 0 & 0 & 0 & 0 & 0 & 4 & 7e^{1.2} & 12e^{2.4} & 19e^{3.6} & 28e^{4.8} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{\alpha}_0 \\ \underline{\alpha}_1 \\ \underline{\alpha}_2 \\ \underline{\alpha}_3 \\ \underline{\alpha}_4 \\ \overline{\alpha}_0 \\ \overline{\alpha}_1 \\ \overline{\alpha}_2 \\ \overline{\alpha}_3 \\ \overline{\alpha}_4 \end{bmatrix} = \begin{bmatrix} 27.01151153 \\ 22.67980607 \\ 18.11788773 \\ 4+r \\ r \\ 27.01151153 \\ 22.67980607 \\ 18.11788773 \\ 6-r \\ 2-r \end{bmatrix},$$

by solving above system, the parameters $\underline{\alpha}_0(r), \underline{\alpha}_1(r), \underline{\alpha}_2(r), \underline{\alpha}_3(r), \underline{\alpha}_4(r), \overline{\alpha}_0(r), \overline{\alpha}_1(r), \overline{\alpha}_2(r),$ $\overline{\alpha}_3(r), \overline{\alpha}_4(r)$, are obtained, and by putting them into (2.26) we have:

$\underline{y}(t,r) = 4.0 + (-2.255809230 + 1.889692363r)e^{t} + (3.042557688 - 1.104464241r)e^{2t} + (-.8603224932 + .2321963367r)e^{3t} + (0.7357403924e-1 - 0.1742445864e-1r)e^{4t}$,

$\overline{y}(t,r) = 4.0 + (1.523575496 - 1.889692363r)e^{t} + (.833629204 + 1.104464241r)e^{2t} + (-.3959298198 - .2321963367r)e^{3t} + (0.3872512196e-1 + 0.1742445864e-1r)e^{4t}$.

Table 3.8 show the comparison of the exact and the approximate solutions at $t = 0.001$ for any $r \in (0, 1]$.

r	$\underline{y}$	$\underline{Q}$	ERROR	$\overline{y}$	$\overline{Q}$	ERROR
0	4.001544314	4.002014989	0.470675e-3	6.002158904	6.000012991	-0.2145913e-2
0.1	4.101575043	4.101914890	0.339847e-3	5.902128174	5.900113092	-0.2015082e-2
0.2	4.201605773	4.201814790	0.209017e-3	5.802097444	5.800213192	-0.1884252e-2
0.3	4.301636502	4.301714690	0.78188e-4	5.702066714	5.700313292	-0.1753422e-2
0.4	4.401667231	4.401614590	-0.52641e-4	5.602035984	5.600413392	-0.1622592e-2
0.5	4.501697961	4.501514490	-0.183471e-3	5.502005255	5.500513492	-0.1491763e-2
0.6	4.601728690	4.601414390	-0.314300e-3	5.401974526	5.400613591	-0.1360935e-2
0.7	4.701759421	4.701314291	-0.445130e-3	5.301943796	5.300713691	-0.1230105e-2
0.8	4.801790150	4.801214191	-0.575959e-3	5.201913068	5.200813791	-0.1099277e-2
0.9	4.901820879	4.901114091	-0.706788e-3	5.101882337	5.100913891	-0.968446e-3
1	5.001851609	5.001013991	-0.837618e-3	5.001851607	5.001013991	-0.837616e-3

Table. 3.8, t=0.001.

The solution of any of the three above-mentioned cases exists if matrix S is nonsingular.

Remark 2.1.1. Matrix S is nonsingular if and only if matrices $S_1 + S_2$ and $S_1 - S_2$ are both nonsingular. See [20].

The following remark shows when each of the three above-mentioned cases has a fuzzy approximated solution.

Remark 2.1.2. The sufficient conditions for $(\underline{y}(t,r), \overline{y}(t,r))$ to define the parametric form of a fuzzy number are as follows:

- $\sum_{k=0}^{N} \underline{\alpha_k(r)\phi_k(t)}$ is a bounded left continuous nondecreasing function over r for each $t \in T$,

- $\sum_{k=0}^{N} \overline{\alpha_k(r)\phi_k(t)}$ is a bounded left continuous nonincreasing function over r for each $t \in T$,
- $\sum_{k=0}^{N} \underline{\alpha_k(1)\phi_k(t)} \leq \sum_{k=0}^{N} \overline{\alpha_k(1)\phi_k(t)}$,$0 \leq r \leq 1$, for each $t \in T$.

Fig. 1. comparing $\underline{Y}(t,r)$ and $\underline{y}(t,r)$ in Example 2.1.

Fig. 2. comparing $\overline{Y}(t,r)$ and $\overline{y}(t,r)$ in Example 2.1.

Fig. 3. comparing $\underline{Y}(t, r)$ and $\underline{y}(t, r)$ in Example 2.2.

Fig. 4. comparing $\overline{Y}(t, r)$ and $\overline{y}(t, r)$ in Example 2.2.

Fig. 5. comparing $\underline{Y}(t,r)$ and $\underline{y}(t,r)$ in Example 2.3.

Fig. 6. comparing $\overline{Y}(t,r)$ and $\overline{y}(t,r)$ in Example 2.3.

Fig. 8. comparing $\underline{Y}(t, r)$ and $\underline{y}(t, r)$ in Example 2.4.

Fig. 9. comparing $\overline{Y}(t, r)$ and $\overline{y}(t, r)$ in Example 2.4.

Chapter 3

Analytic method

3.1 N-th order fuzzy linear differential equation

In this section an analytic method (eigenvalue-eigenvector method) for solving n-th order fuzzy differential equations with fuzzy initial conditions is considered. In this method three cases are introduced, in each case, it is shown that the solution of differential equation is a fuzzy number. In addition the method is illustrated by solving several numerical examples.

we consider $\widetilde{y}^{(n)}(t) = f(\widetilde{y}, \widetilde{y}', \ldots, \widetilde{y}^{(n-1)})$, where $f(\widetilde{y}, \widetilde{y}', \ldots, \widetilde{y}^{(n-1)})$ is a fuzzy function of crisp variable t, and fuzzy variables $\widetilde{y}, \widetilde{y}', \ldots, \widetilde{y}^{(n-1)}$, such that $\widetilde{y}^{(i)}, i = 1, ..., n$ are fuzzy Seikkala derivative, i.e.

$$\widetilde{y}^{(i-1)} \in E \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Therefore we have

$$\begin{cases} \widetilde{y}^{(n)}(t) = f(\widetilde{y}(t), \widetilde{y}'(t), \ldots, \widetilde{y}^{(n-1)}(t)), & a \leq t \leq b, \\ \widetilde{y}(a) = \widetilde{\alpha}_1, \widetilde{y}'(a) = \widetilde{\alpha}_2, \ldots, \widetilde{y}^{(n-1)}(a) = \widetilde{\alpha}_n, & \end{cases} \tag{3.2}$$

where $\widetilde{y}(a)=\widetilde{\alpha}_1, \widetilde{y}'(a)=\widetilde{\alpha}_2, \ldots, \widetilde{y}^{(n-1)}(a)=\widetilde{\alpha}_n$ are fuzzy initial values.

The mentioned n-th order fuzzy linear differential equation by changing variables

$$\widetilde{u}_1(t)=\widetilde{y}(t), \widetilde{u}_2(t)=\widetilde{y'}(t), \ldots, \widetilde{u}_n(t)=\widetilde{y}^{(n-1)}(t), \tag{3.3}$$

converts to the following fuzzy system:

$$\begin{cases} \frac{d\widetilde{u}_1}{dt}=\frac{d\widetilde{y}}{dt}=\widetilde{u}_2, \\ \frac{d\widetilde{u}_2}{dt}=\frac{d\widetilde{y'}}{dt}=\widetilde{u}_3, \\ \vdots \\ \frac{d\widetilde{u}_n}{dt}=\frac{d\widetilde{y}^{(n-1)}}{dt}=\widetilde{y}^{(n)}=f(\widetilde{y}(t), \widetilde{y}'(t), \ldots, \widetilde{y}^{(n-1)}(t)) \\ \qquad =f(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n), \end{cases} \tag{3.4}$$

with fuzzy initial conditions:

$$\begin{cases} \widetilde{u}_1(a)=\widetilde{y}(a)=\widetilde{\alpha}_1, \\ \widetilde{u}_2(a)=\widetilde{y}'(a)=\widetilde{\alpha}_2, \\ \vdots \\ \widetilde{u}_n(a)=\widetilde{y}^{(n-1)}(a)=\widetilde{\alpha}_n, \end{cases}$$

Then, the following system will be solved:

$$\begin{cases} \frac{d\widetilde{u}_1}{dt}=f_1(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n)=a_{11}\widetilde{u}_1+a_{12}\widetilde{u}_2+\ldots+a_{1n}\widetilde{u}_n, \\ \frac{d\widetilde{u}_2}{dt}=f_2(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n)=a_{21}\widetilde{u}_1+a_{22}\widetilde{u}_2+\ldots+a_{2n}\widetilde{u}_n, \\ \vdots \\ \frac{d\widetilde{u}_n}{dt}=f_n(\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n)=a_{n1}\widetilde{u}_1+a_{n2}\widetilde{u}_2+\ldots+a_{nn}\widetilde{u}_n, \\ \widetilde{u}_1(a)=\widetilde{\alpha}_1, \widetilde{u}_2(a)=\widetilde{\alpha}_2, \ldots, \widetilde{u}_n(a)=\widetilde{\alpha}_n, \end{cases} \tag{3.5}$$

Thus

$$\widetilde{U'}=A\widetilde{U}, \tag{3.6}$$

such that $\widetilde{U'} = [\frac{d\widetilde{u}_1}{dt}, \frac{d\widetilde{u}_2}{dt}, \ldots, \frac{d\widetilde{u}_n}{dt}]$, $A = [a_{ij}]_{n\times n}$, $\widetilde{U} = [\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n]$. To obtain the solution of $\widetilde{U'} = A\widetilde{U}$, eigenvalue-eigenvector method is used. In sections 3.2.1, 3.2.2 and 3.2.3 three cases for eigenvalue of matrix A are discussed.

3.1.1 Real and distinct eigenvalues

In this case, suppose that λ_i, for $i = 1, \ldots, n$ are real eigenvalues of matrix A. Therefore the solution of (3.6) is as follows:

$$\widetilde{U}(t) = \sum_{i=1}^{n} \widetilde{c_i} v_i(t), \tag{3.7}$$

where $\widetilde{c_i}$ are fuzzy numbers, $v_i(t) = e^{\lambda_i t}\xi_i$ for $i = 1, \ldots, n$ and λ_i and ξ_i are real eigenvalue and eigenvector of matrix A, respectively.

By setting initial values $t = a$, in (3.7)

$$\widetilde{U}(a) = \sum_{i=1}^{n} \widetilde{c_i} e^{\lambda_i a}\xi_i = \widetilde{\alpha}, \tag{3.8}$$

is obtained, where $\xi_i = [\xi_{i1}, \xi_{i2}, \ldots, \xi_{in}]^t$ and $\widetilde{\alpha} = [\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n]^t$. Thus $\widetilde{c_i}$, $i = 1, \ldots, n$, from the following fuzzy system are obtained:

$$\begin{cases} \widetilde{c}_1 e^{\lambda_1 a}\xi_{11} + \widetilde{c}_2 e^{\lambda_2 a}\xi_{21} + \ldots + \widetilde{c}_n e^{\lambda_n a}\xi_{n1} = \widetilde{\alpha}_1, \\ \widetilde{c}_1 e^{\lambda_1 a}\xi_{12} + \widetilde{c}_2 e^{\lambda_2 a}\xi_{22} + \ldots + \widetilde{c}_n e^{\lambda_n a}\xi_{n2} = \widetilde{\alpha}_2, \\ \vdots \\ \widetilde{c}_1 e^{\lambda_1 a}\xi_{1n} + \widetilde{c}_2 e^{\lambda_2 a}\xi_{2n} + \ldots + \widetilde{c}_n e^{\lambda_n a}\xi_{nn} = \widetilde{\alpha}_n, \end{cases} \tag{3.9}$$

the parametric form of (3.9) is as follows:

$$\begin{cases} (\underline{c}_1(r),\overline{c}_1(r))e^{\lambda_1 a}\xi_{11} + (\underline{c}_2(r),\overline{c}_2(r))e^{\lambda_2 a}\xi_{21} + \ldots + (\underline{c}_n(r),\overline{c}_n(r))e^{\lambda_n a}\xi_{n1} = (\underline{\alpha}_1(r),\overline{\alpha}_1(r)), \\ (\underline{c}_1(r),\overline{c}_1(r))e^{\lambda_1 a}\xi_{12} + (\underline{c}_2(r),\overline{c}_2(r))e^{\lambda_2 a}\xi_{22} + \ldots + (\underline{c}_n(r),\overline{c}_n(r))e^{\lambda_n a}\xi_{n2} = (\underline{\alpha}_2(r),\overline{\alpha}_2(r)), \\ \vdots \\ (\underline{c}_1(r),\overline{c}_1(r))e^{\lambda_1 a}\xi_{1n} + (\underline{c}_2(r),\overline{c}_2(r))e^{\lambda_2 a}\xi_{2n} + \ldots + (\underline{c}_n(r),\overline{c}_n(r))e^{\lambda_n a}\xi_{nn} = (\underline{\alpha}_n(r),\overline{\alpha}_n(r)), \end{cases} \tag{3.10}$$

Now similar to (1.9) there is a $2n \times 2n$ crisp system. Therefore $\widetilde{c}_i = (\underline{c}_i(r), \overline{c}_i(r)), i = 1, 2, \ldots, n$ are obtained from (3.10) and are set in (3.7), finally the solution of (3.2) will be obtained from $\widetilde{U}(t) = [\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n]^t = [\widetilde{y}, \widetilde{y'}, \ldots, \widetilde{y}^{(n-1)}]^t$.

Theorem 3.1.1. *The solution of fuzzy system (3.6) with real eigenvalues is a fuzzy number (3.7).*

proof. It is shown that $\widetilde{U}(t) = \sum_{i=1}^{n} \widetilde{c}_i e^{\lambda_i t}\xi_i$ for $i = 1, \ldots, n$ is the solution of $\widetilde{U}' = A\widetilde{U}$. Let $\widetilde{C} = (\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_n)$ with $\widetilde{c}_i$ which are fuzzy numbers and $[\widetilde{C}]_r = \prod_{i=1}^{n} [\widetilde{c}_i]_r$, then

$$\underline{U}(t,r) = \min\{\sum_{i=1}^{n} c_i e^{\lambda_i t}\xi_i | c_i \in [\widetilde{C}]_r\} = \underline{\sum_{i=1}^{n} c_i e^{\lambda_i t}\xi_i} = \sum_{i=1}^{n} \underline{c_i e^{\lambda_i t}\xi_i}, \tag{3.11}$$

$$\overline{U}(t,r) = \max\{\sum_{i=1}^{n} c_i e^{\lambda_i t}\xi_i | c_i \in [\widetilde{C}]_r\} = \overline{\sum_{i=1}^{n} c_i e^{\lambda_i t}} = \sum_{i=1}^{n} \overline{c_i e^{\lambda_i t}\xi_i}, \tag{3.12}$$

with differentiation of Eqs. (3.11) and (3.12) are obtained:

$$\underline{U'}(t,r) = \sum_{i=1}^{n} \underline{\lambda_i c_i e^{\lambda_i t}\xi_i}, \tag{3.13}$$

$$\overline{U'}(t,r) = \sum_{i=1}^{n} \overline{\lambda_i c_i e^{\lambda_i t}\xi_i}, \tag{3.14}$$

Since λ_i is an eigenvalue and ξ_i is its corresponding eigenvector of matrix A, then $A\xi_i = \lambda_i\xi_i$. Therefore

$$\underline{U'}(t,r) = \sum_{i=1}^{n} \underline{\lambda_i c_i e^{\lambda_i t}\xi_i} = \sum_{i=1}^{n} \underline{\lambda_i \xi_i c_i e^{\lambda_i t}} = A\sum_{i=1}^{n} \underline{c_i e^{\lambda_i t}\xi_i} = A\underline{U}(t,r), \tag{3.15}$$

$$\overline{U'}(t,r) = \sum_{i=1}^{n} \overline{\lambda_i c_i e^{\lambda_i t}\xi_i} = \sum_{i=1}^{n} \overline{\lambda_i \xi_i c_i e^{\lambda_i t}} = A\sum_{i=1}^{n} \overline{c_i e^{\lambda_i t}\xi_i} = A\overline{U}(t,r), \tag{3.16}$$

Such that

$$(\underline{U'}(t,r), \overline{U'}(t,r)) = A(\underline{U}(t,r), \overline{U}(t,r)), \tag{3.17}$$

it means that $\widetilde{U}' = A\widetilde{U}$. From Eqs. (3.1), (3.3) and $\widetilde{U}'(t) = [\widetilde{u}_1', \widetilde{u}_2', \ldots, \widetilde{u'}_n]^t$ it is clear that $\widetilde{U}(t)$ is a fuzzy number vector.

3.1.2 Complex eigenvalues

In this case suppose that some eigenvalues of λ_i for $i = 1, 2, \ldots, k$ are complex numbers. Since entries of matrix A are real, therefore characteristic polynomial have real coefficients therefore complex roots are in conjugate pairs.

Lemma 3.1.2. *[39], Let entries of matrix A are real, and λ is a eigenvalue of matrix A, where $\lambda = \alpha + i\beta$, $\beta \neq 0$, and $\xi = \theta + i\delta$ is corresponding eigenvector of λ, then $u_1(t) = Re[\xi e^{(\alpha+i\beta)t}]$, $u_2(t) = Im[\xi e^{(\alpha+i\beta)t}]$ are solutions.*

Therefore from mentioned lemma the solution of each pair of conjugate complex eigenvalues, $\lambda = \alpha +_{-} i\beta$, is as follows:

$$\widetilde{w}(t) = \widetilde{c}_1 Re[\xi e^{(\alpha+i\beta)t}] + \widetilde{c}_2 Im[\xi e^{(\alpha+i\beta)t}],$$

where ξ is corresponding eigenvector of eigenvalue λ. Hence solution of (3.6) is as follows:

$$\widetilde{U}(t) = \sum_{i=1}^{i=\frac{k}{2}} \widetilde{w}_i(t) + \sum_{i=\frac{k}{2}+1}^{i=n} \widetilde{v}_i(t), \tag{3.18}$$

where $\widetilde{w}_i(t) = \widetilde{c}_{i1} Re(\xi_i e^{\lambda_i t}) + \widetilde{c}_{i2} Im(\xi_i e^{\lambda_i t})$ from each pair of conjugate complex eigenvalues and $v_i(t) = \widetilde{c}_i \xi_i e^{\lambda_i t}$ from real eigenvalues are obtained. Then by setting initial value $t = a$ in (3.18) and by solving a fuzzy system similar to (3.10) fuzzy coefficients are obtained. By setting fuzzy coefficients in (3.18), $\widetilde{U}(t)$ is obtained, finally the solution of (3.2) will be obtained from $\widetilde{U}(t) = [\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n]^t = [\widetilde{y}, \widetilde{y'}, \ldots, \widetilde{y}^{(n-1)}]^t$.

Theorem 3.1.3. *The solution of fuzzy system (3.6) with k complex eigenvalues is a fuzzy number (3.18).*

proof. It is shown that $\widetilde{U}(t) = \sum_{i=1}^{i=\frac{k}{2}} \widetilde{w}_i(t) + \sum_{i=\frac{k}{2}+1}^{i=n} \widetilde{v}_i(t)$ is the solution of $\widetilde{U'} = A\widetilde{U}$. Let $\widetilde{C}_1 = (\widetilde{c}_{11}, \widetilde{c}_{21}, \ldots, \widetilde{c}_{\frac{k}{2},1})$, $\widetilde{C}_2 = (\widetilde{c}_{12}, \widetilde{c}_{22}, \ldots, \widetilde{c}_{\frac{k}{2},2})$ and $\widetilde{C} = (\widetilde{c}_{\frac{k}{2}+1}, \widetilde{c}_{\frac{k}{2}+2}, \ldots, \widetilde{c}_n)$ with $\widetilde{c}_{i1}, \widetilde{c}_{i2}, \widetilde{c}_i$ which are fuzzy numbers and $[\widetilde{C}_1]_r = \prod_{i=1}^{\frac{k}{2}} [c_{i1}]_r$, $[\widetilde{C}_2]_r = \prod_{i=1}^{\frac{k}{2}} [c_{i2}]_r$, $[\widetilde{C}] = \prod_{i=\frac{k}{2}+1}^{n} [c_i]_r$.

$\underline{U}(t,r) = \min\{\sum_{i=1}^{\frac{k}{2}} c_{i1} Re(\xi_i e^{\lambda_i t}) + c_{i2} Im(\xi_i e^{\lambda_i t}) + \sum_{i=\frac{k}{2}+1}^{n} c_i \xi_i e^{\lambda_i t} | c_{i1} \in [\widetilde{C}_1]_r, c_{i2} \in [\widetilde{C}_2]_r, c_i \in [\widetilde{C}]_r\} = \underline{\sum_{i=1}^{\frac{k}{2}} c_{i1} Re(\xi_i e^{\lambda_i t}) + c_{i2} Im(\xi_i e^{\lambda_i t}) + \sum_{i=\frac{k}{2}+1}^{n} c_i \xi_i e^{\lambda_i t}}$
$= \sum_{i=1}^{\frac{k}{2}} \underline{c_{i1} Re(\xi_i e^{\lambda_i t}) + c_{i2} Im(\xi_i e^{\lambda_i t})} + \sum_{i=\frac{k}{2}+1}^{n} \underline{c_i \xi_i e^{\lambda_i t}}$.

$\overline{U}(t,r) = \max\{\sum_{i=1}^{\frac{k}{2}} c_{i1} Re(\xi_i e^{\lambda_i t}) + c_{i2} Im(\xi_i e^{\lambda_i t}) + \sum_{i=\frac{k}{2}+1}^{n} c_i \xi_i e^{\lambda_i t} | c_{i1} \in [\widetilde{C}_1]_r, c_{i2} \in [\widetilde{C}_2]_r, c_i \in [\widetilde{C}]_r\} = \overline{\sum_{i=1}^{\frac{k}{2}} c_{i1} Re(\xi_i e^{\lambda_i t}) + c_{i2} Im(\xi_i e^{\lambda_i t}) + \sum_{i=\frac{k}{2}+1}^{n} c_i \xi_i e^{\lambda_i t}}$

$$= \sum_{i=1}^{\frac{k}{2}} \overline{c_{i1}Re(\xi_i e^{\lambda_i t}) + c_{i2}Im(\xi_i e^{\lambda_i t})} + \sum_{i=\frac{k}{2}+1}^{n} \overline{c_i \xi_i e^{\lambda_i t}},$$

with differentiation of above equations are obtained:

$$\underline{U}'(t,r) = \sum_{i=1}^{\frac{k}{2}} \underline{c_{i1}Re(\xi_i \lambda_i e^{\lambda_i t}) + c_{i2}Im(\xi_i \lambda_i e^{\lambda_i t})} + \sum_{i=\frac{k}{2}+1}^{n} \underline{c_i \lambda_i \xi_i e^{\lambda_i t}}, \tag{3.19}$$

$$\overline{U}'(t,r) = \sum_{i=1}^{\frac{k}{2}} \overline{c_{i1}Re(\xi_i \lambda_i e^{\lambda_i t}) + c_{i2}Im(\xi_i \lambda_i e^{\lambda_i t})} + \sum_{i=\frac{k}{2}+1}^{n} \overline{c_i \lambda_i \xi_i e^{\lambda_i t}}, \tag{3.20}$$

since λ_i is an eigenvalue and ξ_i is its corresponding eigenvector of matrix A, then $A\xi_i = \lambda_i \xi_i$ therefore:

$$\begin{aligned}
\underline{U}'(t,r) &= \sum_{i=1}^{\frac{k}{2}} \underline{c_{i1}Re(A\xi_i e^{\lambda_i t}) + c_{i2}Im(A\xi_i e^{\lambda_i t})} + \sum_{i=\frac{k}{2}+1}^{n} \underline{c_i A\xi_i e^{\lambda_i t}} \\
&= A(\sum_{i=1}^{\frac{k}{2}} \underline{c_{i1}Re(\xi_i e^{\lambda_i t}) + c_{i2}Im(\xi_i e^{\lambda_i t})} + \sum_{i=\frac{k}{2}+1}^{n} \underline{c_i \xi_i e^{\lambda_i t}} \\
&= A\underline{U}(t,r),
\end{aligned}$$

and

$$\begin{aligned}
\overline{U}'(t,r) &= \sum_{i=1}^{\frac{k}{2}} \overline{c_{i1}Re(A\xi_i e^{\lambda_i t}) + c_{i2}Im(A\xi_i e^{\lambda_i t})} + \sum_{i=\frac{k}{2}+1}^{n} \overline{c_i A\xi_i e^{\lambda_i t}} \\
&= A(\sum_{i=1}^{\frac{k}{2}} \overline{c_{i1}Re(\xi_i e^{\lambda_i t}) + c_{i2}Im(\xi_i e^{\lambda_i t})} + \sum_{i=\frac{k}{2}+1}^{n} \overline{c_i \xi_i e^{\lambda_i t}} \\
&= A\overline{U}(t,r),
\end{aligned}$$

then $(\underline{U}'(t,r), \overline{U}'(t,r)) = A(\underline{U}(t,r), \overline{U}(t,r))$, it means $\widetilde{U}' = A\widetilde{U}$. From Eqs. (3.1), (3.3) and $\widetilde{U}'(t) = [\widetilde{u}_1', \widetilde{u}_2', \ldots, \widetilde{u}'_n]^t$ it is clear that $\widetilde{U}(t)$ is a fuzzy number vector.

3.1.3 Multiple eigenvalues

In this case suppose that some eigenvalues of matrix A are multiple. suppose that λ_0 is a eigenvalue of matrix A with multiplicity m_0, and corresponding eigenvectors of eigenvalue λ_0 are $\xi_1, \xi_2, \ldots, \xi_k$, if all ξ_i be linear independent, i.e, $k = m_0$, then the is as follows:

$$\widetilde{v}(t) = e^{\lambda_0 t}(\widetilde{c}_1\xi_1 + \widetilde{c}_2\xi_2 + \ldots + \widetilde{c}_k\xi_k),$$

if ξ_i be linear dependent, i.e, $k < m_0$, then following lemma is brought.

Lemma 3.1.4. *[39], Let λ_0 is a eigenvalue of matrix A with multiple $m_0 > 1$ and the number of ξ_i which are linear independent are less than m_0, therefore at least one non-zero vector exist such that*

$$(A - \lambda I)^2\xi = 0, \qquad (A - \lambda I)\xi \neq 0, \tag{3.21}$$

if ξ is satisfied in (3.21) the solution is as follows:

$$v'(t) = e^{\lambda_0 t}[\xi + t(A - \lambda_0 I)\xi].$$

For more details refer to [39].

Hence with mentioned lemma the solution of (3.2) is as follows:

$$\widetilde{U}(t) = \sum_{i=1}^{i=k} \widetilde{c}_i v'_i(t) + \sum_{i=k+1}^{i=n} \widetilde{c}_i v_i(t), \tag{3.22}$$

where $v'_i(t) = e^{\lambda_i t}[\xi_i + t(A - \lambda_i I)\xi_i]$ for λ_i which are satisfied in lemma 3.2 and $v_i = \xi_i e^{\lambda_i t}$ for real eigenvalues are obtained. Then by setting initial value $t = a$ in (3.21) and by solving a fuzzy system similar to (3.10) fuzzy coefficients are obtained and by setting fuzzy coefficients in (3.22), $\widetilde{U}(t)$ is obtained, finally the solution of (3.2) will be obtained from $\widetilde{U}(t) = [\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n]^t = [\widetilde{y}, \widetilde{y'}, \ldots, \widetilde{y}^{(n-1)}]^t$.

Theorem 3.1.5. *The solution of fuzzy system (3.6) with multiple eigenvalues is a fuzzy number (3.22).*

Proof. It is shown that $\widetilde{U}(t) = \sum_{i=1}^{i=k} \widetilde{c}_i v_i'(t) + \sum_{i=k+1}^{i=n} \widetilde{c}_i v_i(t)$ is the solution of $\widetilde{U}' = A\widetilde{U}$. Let $\widetilde{C} = (\widetilde{c}_1, \widetilde{c}_2, \ldots, \widetilde{c}_n)$ with $\widetilde{c}_i$ which are fuzzy numbers and $[\widetilde{C}]_r = \prod_{i=1}^{n} [\widetilde{c}_i]_r$ then:

$$\begin{aligned}
\underline{U}(t,r) &= \min\{\textstyle\sum_{i=1}^{k} c_i e^{\lambda_i t}(\xi_i + t(A-\lambda_i I)\xi_i) + \sum_{i=k+1}^{n} c_i \xi_i e^{\lambda_i t} | c_i \in [\widetilde{C}]_r\} \\
&= \underline{\textstyle\sum_{i=1}^{k} c_i e^{\lambda_i t}(\xi_i + t(A-\lambda_i I)\xi_i) + \sum_{i=k+1}^{n} c_i \xi_i e^{\lambda_i t}} \\
&= \textstyle\sum_{i=1}^{k} \underline{c_i e^{\lambda_i t}(\xi_i + t(A-\lambda_i I)\xi_i)} + \sum_{i=k+1}^{n} \underline{c_i \xi_i e^{\lambda_i t}},
\end{aligned}$$

and

$$\begin{aligned}
\overline{U}(t,r) &= \max\{\textstyle\sum_{i=1}^{k} c_i e^{\lambda_i t}(\xi_i + t(A-\lambda_i I)\xi_i) + \sum_{i=k+1}^{n} c_i \xi_i e^{\lambda_i t} | c_i \in [\widetilde{C}]_r\} \\
&= \overline{\textstyle\sum_{i=1}^{k} c_i e^{\lambda_i t}(\xi_i + t(A-\lambda_i I)\xi_i) + \sum_{i=k+1}^{n} c_i \xi_i e^{\lambda_i t}} \\
&= \textstyle\sum_{i=1}^{k} \overline{c_i e^{\lambda_i t}(\xi_i + t(A-\lambda_i I)\xi_i)} + \sum_{i=k+1}^{n} \overline{c_i \xi_i e^{\lambda_i t}},
\end{aligned}$$

with differentiation of above equations are obtained:

$$\underline{U}'(t,r) = \textstyle\sum_{i=1}^{k} \underline{c_i[\lambda_i e^{\lambda_i t}(\xi_i + t(A-\lambda_i I)\xi_i) + e^{\lambda_i t}((A-\lambda_i I)\xi_i)]} + \sum_{i=k+1}^{n} \underline{c_i \xi_i \lambda_i e^{\lambda_i t}},$$

$$\overline{U}'(t,r) = \textstyle\sum_{i=1}^{k} \overline{c_i[\lambda_i e^{\lambda_i t}(\xi_i + t(A-\lambda_i I)\xi_i) + e^{\lambda_i t}((A-\lambda_i I)\xi_i)]} + \sum_{i=k+1}^{n} \overline{c_i \xi_i \lambda_i e^{\lambda_i t}},$$

since λ_i is an eigenvalue and ξ_i is its corresponding eigenvector of matrix A, then $A\xi_i = \lambda_i \xi_i$.

Therefore $\underline{U}' = A\underline{U}$ and $\overline{U}' = A\overline{U}$ then

$$(\underline{U}'(t,r), \overline{U}'(t,r)) = A(\underline{U}(t,r), \overline{U}(t,r)),$$

it means $\widetilde{U}' = A\widetilde{U}$. From Eqs. (3.1), (3.3) and $\widetilde{U}'(t) = [\widetilde{u}'_1, \widetilde{u}'_2, \ldots, \widetilde{u'}_n]^t$ it is clear that $\widetilde{U}(t)$ is a fuzzy number vector.

Remark 3.1.1. If in $\underline{U}' = A\underline{U}$, the matrix A has negative eigenvalue then the third condition of definition 2.1 for the solution's odd-order derivative do not hold, thus for solving this problem we can change two ends of ordered pair of solution's odd-order derivative.

3.2 Examples

Example 3.2.1. Consider the following fuzzy differential equation with fuzzy initial value

$$\widetilde{y}''' = 2\widetilde{y}'' + 3\widetilde{y}',$$

$$\widetilde{y}(0) = (3 + r, 5 - r),$$

$$\widetilde{y}'(0) = (-3 + r, -1 - r),$$

$$\widetilde{y}''(0) = (8 + r, 10 - r),$$

$\widetilde{U}' = A\widetilde{U}$ is obtained from above differential equation where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}$ and $\lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = -1$ and $\xi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\xi_2 = \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix}$, $\xi_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ are eigenvalues and eigenvectors of matrix A, respectively. Then

$$\begin{bmatrix} \widetilde{u}_1(t) \\ \widetilde{u}_2(t) \\ \widetilde{u}_3(t) \end{bmatrix} = \widetilde{c}_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \widetilde{c}_2 e^{3t} \begin{bmatrix} \frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} + \widetilde{c}_3 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

is obtained, by setting initial values in parametric form the following system is obtained:

$$\begin{bmatrix} 1 & \frac{1}{9} & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{9} & 1 \\ 0 & 0 & -1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \\ \overline{c}_1 \\ \overline{c}_2 \\ \overline{c}_3 \end{bmatrix} = \begin{bmatrix} 3+r \\ -3+r \\ 8+r \\ 5-r \\ -1-r \\ 10-r \end{bmatrix},$$

$\widetilde{c}_1 = (\frac{-1}{3}, \frac{-1}{3})$, $\widetilde{c}_2 = (\frac{21}{4}, \frac{21}{4})$ and $\widetilde{c}_3 = (\frac{11}{4} + r, \frac{19}{4} - r)$ are obtained from above system and are set in (3.7). Therefore the solution of fuzzy differential equation is as follows: $\widetilde{y}(t) = (\frac{-1}{3} + \frac{7}{12}e^{3t} + (\frac{11}{4} + r)e^{-t}, \frac{-1}{3} + (\frac{7}{12})e^{3t} + (\frac{19}{4} - r)e^{-t})$.

Example 3.2.2. Consider the following fuzzy differential equation with fuzzy initial value

$$\widetilde{y}''' = -\widetilde{y}'' - 3\widetilde{y}' + 5\widetilde{y}$$

$$\widetilde{y}(0) = (0.75 + 0.25r, 1.25 - 0.25r)$$

$$\widetilde{y}'(0) = (1.5 + 0.5r, 2.5 - 0.5r)$$

$$\widetilde{y}''(0) = (3.75 + 0.25r, 4.25 - 0.25r)$$

$\widetilde{U}' = A\widetilde{U}$ is obtained from above differential equation, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -3 & -1 \end{bmatrix}$

and $\lambda_1 = 1,\ \lambda_2 = -1 + 2i,\ \lambda_3 = -1 - 2i$ and $\xi_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\xi_2 = \begin{bmatrix} \frac{-3}{25} + \frac{4}{25}i \\ \frac{-1}{5} - \frac{2}{5}i \\ 1 \end{bmatrix}$,

$\xi_3 = \begin{bmatrix} \frac{-3}{25} - \frac{4}{25}i \\ \frac{-1}{5} + \frac{2}{5}i \\ 1 \end{bmatrix}$ are eigenvalues and eigenvectors of matrix A, respectively, and by setting initial values in parametric form the following system is obtained:

$$\begin{bmatrix} 1 & 0 & \frac{4}{25} & 0 & \frac{-3}{25} & 0 \\ 1 & 0 & 0 & 0 & \frac{-1}{5} & \frac{-2}{5} \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{-3}{25} & 0 & 1 & 0 & \frac{4}{25} \\ 0 & \frac{-1}{5} & \frac{-2}{5} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \\ \overline{c}_1 \\ \overline{c}_2 \\ \overline{c}_3 \end{bmatrix} = \begin{bmatrix} 0.75 + 0.25r \\ 1.5 + 0.5r \\ 3.75 + 0.25r \\ 1.25 - 0.25r \\ 2.5 - 0.5r \\ 4.25 - 0.25r \end{bmatrix},$$

$\widetilde{c}_1 = (1.553571428 + 0.0714285714r, 1.696428572 - 0.0714285714r)$,

$\widetilde{c}_2 = (2.196428572 + 0.1785714286r, 2.553571428 - 0.1785714286r)$,

$\widetilde{c}_3 = (-3.107142858 + 0.9821428567r, -1.142857142 - 0.9821428572r)$, are obtained from above system and are set in (3.7). Therefore the solution of fuzzy differential equation is as follows:

$\underline{y}(t) = (1.553571428 + 0.0714285714r)e^t + (2.553571428 - 0.1785714286r)e^{-t}(\frac{-3}{25}cos(2t) - \frac{4}{25}sin(2t)) + (-3.107142858 + 0.9821428567r)e^{-t}(\frac{-3}{25}sin(2t) + \frac{4}{25}cos(2t)),$

$\overline{y}(t) = (1.696428572 - 0.0714285714r)e^t + (2.196428572 + 0.1785714286r)e^{-t}(\frac{-3}{25}cos(2t) - \frac{4}{25}sin(2t)) + (-1.142857142 - 0.9821428572r)e^{-t}(\frac{-3}{25}sin(2t) + \frac{4}{25}cos(2t)).$

3.2.1 Comparative example

In this section we compare this method with Buckely and Feuring method [12].

Example 3.2.3. Consider the following fuzzy differential equation with fuzzy initial value

$$\widetilde{y}'' = 15\widetilde{y}' - 50\widetilde{y}$$

$$\widetilde{y}(0) = (0.75 + 0.25r, 1.25 - 0.25r)$$

$$\widetilde{y}'(0) = (1.5 + 0.5r, 2.5 - 0.5r)$$

$\widetilde{U}' = A\widetilde{U}$ is obtained from above differential equation, where $A = \begin{bmatrix} 0 & 1 \\ -50 & 15 \end{bmatrix}$ and $\lambda_1 = 5$, $\lambda_2 = 10$, and $\xi_1 = \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}$, $\xi_2 = \begin{bmatrix} \frac{1}{10} \\ 1 \end{bmatrix}$, are eigenvalues and eigenvectors of matrix A, respectively. Then

$$\begin{bmatrix} \widetilde{u}_1(t) \\ \widetilde{u}_2(t) \end{bmatrix} = \widetilde{c}_1 e^{5t} \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} + \widetilde{c}_2 e^{10t} \begin{bmatrix} \frac{1}{10} \\ 1 \end{bmatrix},$$

is obtained, by setting initial values in parametric form the following system is obtained:

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{10} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{10} \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \\ \overline{c}_1 \\ \overline{c}_2 \end{bmatrix} = \begin{bmatrix} 0.75 + 0.25r \\ 1.5 + 0.5r \\ 1.25 - 0.25r \\ 2.5 - 0.5r \end{bmatrix},$$

$\widetilde{c}_1 = (6 + 2r, 10 - 2r)$ and $\widetilde{c}_2 = (-4.5 - 1.5r, -7.5 + 1.5r)$ are obtained from above system and are set in (3.7). Therefore the solution of fuzzy differential equation is as follows:

$\widetilde{y}(t) = (\frac{1}{5}(6 + 2r)e^{5t} + \frac{1}{10}(-4.5 - 1.5r)e^{10t}, \frac{1}{5}(10 - 2r)e^{5t} + \frac{1}{10}(-7.5 + 1.5r)e^{10t})$.

And Buckley and Feuring solution is

$\widetilde{y}(t) = ((1.2 + 0.4r)e^{5t} + (-0.45 - 0.15r)e^{10t}, (2 - 0.4r)e^{5t} + (-0.75 + 0.15r)e^{10t})$.

Conclusions

In this thesis a numerical method similar to the collocation method, based on a positive basis for solving fuzzy differential equations was discussed. Three cases were considered. Following each case, fuzzy approximate solutions were obtained by solving an extended system of linear equations. Also an analytic method for solving n-th order fuzzy linear differential equations with fuzzy initial conditions is presented. In this method a n-th order fuzzy linear differential equation is converted to a fuzzy system which will be solved with eigenvalue-eigenvector method. It is shown that the solution of differential equation is a fuzzy number.

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