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METHODS FOR SOLVING FIRST ORDER FUZZY DIFFERENTIAL EQUATIONS

Supervisor

Dr. Tofigh Allahviranloo

Prof. Saeid Abbasbandy

Advisers

Dr. Mohammad Ali Fariborzi Araghi

By

Nazanin Ahmady

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Research Supervisor: Dr. Tofigh Allahviranloo
Prof. Saeid Abbasbandy

External Examiner: Prof. Esmaeel Babolian

Examing Committee: Dr. Farhad Hosseinzadeh Lotfi

Dr. Alireza Vahidi

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Author: **Nazanin Ahmady**

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Signature of Author

[1] Poonak, Tehran

To my mother and father

and sisters

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Abstract

In this work, first, three numerical methods, namely the Adams-Bashforth, the Adams-Moulton and Predictor-Corrector methods, are discussed to solve "fuzzy initial value problem". The Predictor-Corrector method is obtained by combining the Adams-Bashforth and the Adams-Moulton methods. In addition, we propose an Improved Predictor-Corrector (IPC) method to solve the "fuzzy initial value problem". The IPC method is obtained by combining the explicit three-step method and the implicit two-step method. These methods, i.e. the IPC, the explicit and the implicit methods were compared with the Predictor-Corrector and the Adams-Bashforth methods, the former three proved to have more accuracy. The convergence and stability of the proposed methods are also presented in detail.

Also, in this thesis we propose numerical methods for hybrid fuzzy differential equations by an application of the Predictor-Corrector method .

Chapter 1

Preliminaries

1.1 Introduction

In the sciences and engineering, mathematical models are developed to aid in the understanding of physical phenomena. These models often an equation that contains some derivatives of an unknown function. Such an equations is called a differential equations.

After the above process and introducing a differential equation, we are now prepared to consider numerical methods for integrating differential equations.

1.2 Formulation of the Euler method

Consider a differential equation system

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{1.1}$$

where $f : [a, b] \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous and satisfies a Lipschitz condition $\|f(t, y) - f(t, z)\| \leq L\|y - z\|$, for all t in a neighborhood of t_0 and y and z in a neighborhood of y_0. For simplicity, we assume that the Lipschitz condition holds every where; this is not a serious loss of generality because the existence and uniqueness of a solution to (1.1) is known to hold in a suitable interval, containing t_0.

(1.1), is required to be approximated at a point $\bar{t}$, and that a number of intermediate step points are selected. Denote these by $t_1, t_2, \cdots, t_n = \underline{t}$. Define a function, $\hat{y}$, on $[t_0, \underline{t}]$ by the formula

$$\hat{y}(t) = \hat{y}(t_{k-1}) + (t - t_{k-1}) f(t_{k-1}, \hat{y}(t_{k-1})), \quad t \in (t_{k-1}, t_k], \tag{1.2}$$

for $k = 1, 2, \cdots n$. If it is assumed that $\hat{y}(t_0) = y(t_0) = y_0$, then $\hat{y}$ exactly agrees with the function computed using the Euler method at the points $t = t_k,\ k = 1, 2, \cdots, n$. The continuous function $\hat{y}$, on the interval $[t_0, \underline{t}]$, is a piece-wise linear interpolate of this Euler approximation.

$\hat{y}$, as an approximation to y, clearly depends on the values of the step points $t_1, t_2, \cdots$, and especially on the greatest of the distances between a point and the one preceding it.

1.3 Multi step method

The methods discussed in this chapter are called multi-step methods because these methods use the approximation at more than one of the previous mesh points to determine the approximation at the next mesh point.

An m-step method for solving the initial value problem

$$\frac{dy}{dt} = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha$$

is one whose difference equation for finding the approximation $y(t_{i+1})$ at the mesh point t_{i+1} can be represented by the following equation, where $m > 1$ is an integer:

$$y(t_{i+1}) = a_{m-1}y(t_i) + a_{m-2}y(t_{i-1}) + \ldots + a_0 y(t_{i+1-m}) \tag{1.3}$$

$$+h\{b_m f(t_{i+1}, y_{i+1}) + b_{m-1}f(t_i, y_i) + \ldots + b_0 f(t_{i+1-m}, y_{i+1-m})\},$$

for $i = m-1, m, \ldots, N-1$, such that $a = t_0 \leq t_1 \leq \ldots \leq t_N = b$, $h = \frac{(b-a)}{N} = t_{i+1} - t_i$ and $a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots, b_m$ are constants with the starting values

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad \ldots, \quad y_{m-1} = \alpha_{m-1}.$$

When $b_m = 0$, the method is known as explicit, since Eq.(1.3) gives y_{i+1} explicitly in terms of previously determined values. When $b_m \neq 0$, the method is known as implicit, since y_{i+1} occurs on both sides of Eq.(1.3) and is specified only implicitly.

Definition 1.3.1. Associated with the difference equation

$$y_{i+1} = a_{m-1}y_i + a_{m-2}y_{i-1} + \ldots + a_0 y_{i+1-m} + hF(t_i, h, y_{i+1}, y_i, \ldots, y_{i+1-m}), \tag{1.4}$$

$$y_0 = \alpha, y_1 = \alpha_1, \ldots, y_{m-1} = \alpha_{m-1},$$

the following relation is called the characteristic polynomial of the method is

$$p(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \ldots - a_1\lambda - a_0.$$

Assume $\lambda_i, i = 1, \cdots, m$ are root of $p(\lambda) = 0$.

If $|\lambda_i| \leq 1$ for each $i = 1, 2, \ldots, m$, and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Theorem 1.3.1. *A multi step method of the form (1.4) is stable if and only if it satisfies the root condition.*

Proof. See [17].

1.3.1 Explicit Multi-Step Methods

Some of the explicit multi-step methods together with their required starting values and local truncation errors are as follows.

- Adams-Bashforth two-step method:

$y_0 = \alpha_0, \quad y_1 = \alpha_1,$

$y_{i+1} = y_i + \frac{h}{2}[3f(t_i, y_i) - f(t_{i-1}, y_{i-1})], \quad$ where $i = 1, 2, \ldots, N-1$.

- Adams-Bashforth three-step method:

$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2,$

$y_{i+1} = y_i + \frac{h}{12}[23f(t_i, y_i) - 16f(t_{i-1}, y_{i-1}) + 5f(t_{i-2}, y_{i-2})], \quad$ where $i = 2, 3, \ldots, N-1$.

The following table shows the coefficients and the error constants for the Adams-Bashforth methods.

k	b_{m-1}	b_{m-2}	b_{m-3}	b_{m-4}	b_{m-5}	b_{m-6}	b_{m-7}	b_{m-8}	Error
1	1								$-\frac{1}{2}$
2	$\frac{3}{2}$	$-\frac{1}{2}$							$\frac{5}{12}$
3	$\frac{23}{12}$	$-\frac{4}{3}$	$\frac{5}{12}$						$-\frac{3}{8}$
4	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{3}{8}$					$\frac{251}{720}$
5	$\frac{1901}{720}$	$-\frac{1387}{360}$	$\frac{109}{30}$	$-\frac{637}{360}$	$\frac{251}{720}$				$-\frac{95}{288}$
6	$\frac{4277}{1440}$	$-\frac{2641}{480}$	$\frac{4991}{720}$	$-\frac{3649}{720}$	$\frac{959}{480}$	$-\frac{95}{288}$			$\frac{19087}{60480}$
7	$\frac{198721}{60480}$	$-\frac{18637}{2520}$	$\frac{235183}{20160}$	$-\frac{10754}{945}$	$\frac{135713}{20160}$	$-\frac{5603}{2520}$	$\frac{19087}{60480}$		$-\frac{5257}{17280}$
8	$\frac{16083}{4480}$	$-\frac{1152169}{120960}$	$\frac{242653}{13440}$	$-\frac{296053}{13440}$	$\frac{2102243}{120960}$	$-\frac{115747}{13440}$	$\frac{32863}{13440}$	$-\frac{5257}{17280}$	$\frac{1070017}{3628800}$

Table 1.1: Coefficients and error constants for Adams-Bashforth methods.

1.3.2 Implicit Multi-Step Methods

Implicit methods are derived by using $(t_{i+1}, f(t_{i+1}, y_{i+1}))$ as an additional interpolation node. Some of the more common implicit multi-step methods together with their required starting values and local truncation errors are as follows.

- Adams-Moulton two-step method:

$y_0 = \alpha_0, \quad y_1 = \alpha_1,$

$y_{i+1} = y_i + \frac{h}{12}[5f(t_{i+1}, y_{i+1}) + 8f(t_i, y_i) - f(t_{i-1}, y_{i-1})],$ where $i = 1, 2, \ldots, N-1.$

- Adams-Moulton three-step method:

$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2,$

$y_{i+1} = y_i + \frac{h}{24}[9f(t_{i+1}, y_{i+1}) + 19f(t_i, y_i) - 5f(t_{i-1}, y_{i-1}) + f(t_{i-2}, y_{i-2})],$ where $i = 2, 3, \ldots, N-1.$

The following table shows the coefficients and the constants error for the Adams-Moulton methods.

k	b_m	b_{m-1}	b_{m-2}	b_{m-3}	b_{m-4}	b_{m-5}	b_{m-6}	b_{m-7}	Error
0	1								$\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{1}{2}$							$-\frac{1}{12}$
2	$\frac{5}{12}$	$\frac{2}{3}$	$-\frac{1}{12}$						$\frac{1}{24}$
3	$\frac{3}{8}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$					$-\frac{19}{720}$
4	$\frac{251}{720}$	$\frac{323}{360}$	$-\frac{11}{30}$	$\frac{53}{360}$	$-\frac{19}{720}$				$\frac{3}{160}$
5	$\frac{95}{288}$	$\frac{1427}{1440}$	$-\frac{133}{240}$	$\frac{241}{720}$	$-\frac{173}{1440}$	$\frac{3}{160}$			$-\frac{863}{60480}$
6	$\frac{19087}{60480}$	$\frac{2713}{2520}$	$-\frac{15487}{20160}$	$\frac{586}{945}$	$-\frac{6737}{20160}$	$\frac{263}{2520}$	$-\frac{863}{60480}$		$-\frac{275}{24192}$
7	$\frac{5257}{17280}$	$\frac{139849}{120960}$	$-\frac{4511}{4480}$	$\frac{123133}{120960}$	$-\frac{88547}{120960}$	$\frac{1537}{4480}$	$-\frac{11351}{120960}$	$\frac{275}{24192}$	$-\frac{33953}{3628800}$

Table 1.2: Coefficients and error constants for Adams-Moulton methods.

1.4 Fuzzy Mathematics

Fuzziness is not a priori an obvious concept and demands some explanation. ”Fuzziness” is what Black calls ”vagueness” when he distinguishes it from ”generality” and from ”ambiguity”. Generalizing refers to the application of a symbol to a multiplicity of objects in the field of reference, ambiguity to the association of a finite number of alternative meanings having the same phonetic form. But, the fuzziness of a symbol lies in the lack of well-defined boundaries of the set of objects to which this symbol applies.

More specifically, let X be a field of reference, also called a universe of discourse or universe for short, covering a definite range of objects. Consider a subset $\tilde{A}$ where transition between membership and nonmembership is gradual rather than abrupt. This "fuzzy subset" obviously has no well-defined boundaries. Fuzzy classes of objects are often encountered in real life. For instance, $\tilde{A}$ may be the set of tall men in a community X. Usually, there are members of X who are definitely tall, others who are definitely not tall, but there exist also borderline cases. Traditionally, the grade of membership 1 is assigned to the objects that completely belong to $\tilde{A}$-here the men who are definitely tall, conversely the objects that do not belong to $\tilde{A}$ at all are assigned a membership value 0. Quite naturally, the grades of membership of the borderline cases lie between 0 and 1. The more an element or object x belongs to $\tilde{A}$, the closer to 1 is its grade of membership $\mu_{\tilde{A}}(x)$. The use of a numerical scale such as the interval [0, 1] allows a convenient representation of the gradation in membership. Precise membership values do not exist by themselves, they are tendency indices that are subjectively assigned by an individual or a group. Moreover, they are context-dependent. The grades of membership reflect an "ordering" of the objects in the universe, induced by the predicate associated with $\tilde{A}$; this "ordering", when it exists, is more important than the membership values themselves. The membership assessment of objects can sometimes be made easier by the use of a similarity measure with respect to an ideal element. Note that a membership value $\mu_{\tilde{A}}(x)$ can be interpreted as the degree of compatibility of the predicate associated with $\tilde{A}$ and the object x. For concepts such as "tallness", related to a physical measurement scale, the assignment of membership values will often be less controversial than for more complex and subjective concepts such as ”beauty”.

The above approach, developed by Zadeh (1964), provides a tool for modeling human-centered systems. As a matter of fact, fuzziness seems to pervade most human perception and thinking processes. Parikh (1977) has pointed out that no nontrivial first-order-logic-like observational predicate (i.e., one pertaining to perception) can be defined on an observationally connected space; the only possible observational predicates on such a space are not classical predicates but ”vague” ones. Moreover, according to Zadeh (1973), one of the most important facets of human thinking is the ability to summarize information ”into labels of fuzzy sets which bear an approximate relation to the primary data”. Linguistic descriptions, which are usually summary descriptions of complex situations, are fuzzy in essence.

It must be noticed that fuzziness differs from imprecision. In tolerance analysis imprecision refers to lack of knowledge about the value of a parameter and is thus expressed as a crisp tolerance interval. This interval is the set of possible values of the parameters. Fuzziness occurs when the interval has no sharp boundaries, i.e., is a fuzzy set $\tilde{A}$. Then, $\mu_{\tilde{A}}(x)$ is interpreted as the degree of possibility (Zadeh, 1978) that x is the value of the parameter fuzzily restricted by $\tilde{A}$.

The word fuzziness has also been used by Sugeno (1977) in a radically different context. Consider an arbitrary object x of the universe X; to each nonfuzzy subset A of X is assigned a value $g_x(A) \in [0,1]$ expressing the ”grade of fuzziness” of the statement ”x belongs to A”. In fact this grade of fuzziness must be understood as a grade of certainty: according to the mathematical definition of g, $g_x(A)$ can be interpreted as the probability, the degree of subjective belief, the possibility, that x belongs to A.

Generally, g is assumed increasing in the sense of set inclusion, but not necessarily additive as in the probabilistic case. The situation modeled by Sugeno is more a matter of guessing whether $x \in A$ rather than a problem of vagueness in the sense of Zadeh. The existence of two different points of view on "fuzziness" has been pointed out by MacVicar-Whelan (1977) and Skala. The monotonicity assumption for g seems to be more consistent with human guessing than does the additivity assumption. For instance, seeing a piece of Indian pottery in a shop, we may try to guess whether it is genuine or counterfeit; obviously, genuineness is a fuzzy concept. Hence x is the Indian pottery; A is the crisp set of genuine Indian artifacts; and $g_x(A)$ expresses, for instance, a subjective belief that the pottery is indeed genuine. The situation is slightly more complicated when we try to guess whether the pottery is old: actually, the set $\tilde{A}$ of old Indian pottery is fuzzy because "old" is a vague predicate.

1.5 Fuzzy Sets

Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X is often viewed as a characteristic function, μ_A from X to $\{0, 1\}$ such that

$$\mu_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases} \tag{1.5}$$

If the valuation set, i.e., 0 and 1, is allowed to be the real interval $[0,1]$, A is called a fuzzy set (Zadeh, 1965), and $\mu_A(x)$ is the grade of membership of x in A. Sometimes we denoted $\mu_A(x)$ by $A(x)$. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A. Clearly, A is a subset of X that has no sharp boundary. Hence A is completely characterized by the set of pairs

$$A = \{(x, \mu_A(x)) | x \in X\}.$$

A more convenient notation was proposed by Zadeh [36]. When X is a finite set $\{x_1, ..., x_n\}$, a fuzzy set on X is expressed as

$$A = \frac{\mu_A(x_1)}{x_1} + \cdots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}.$$

When X is not finite, we write

$$A = \int_x \mu_A(x)/x.$$

Definition 1.5.1. (Equality of fuzzy sets) Two fuzzy sets A and B are said to be equal (denoted $A = B$) iff

$$\forall\ x \in X,\ \ \mu_A(x) = \mu_B(x).$$

Definition 1.5.2. (Support) The support of a fuzzy set A is the ordinary subset of X:

$$supp\ A = \{x \in X,\ \ \mu_A(x) > 0\}.$$

Definition 1.5.3. (Core) The core of a fuzzy set A is the set of all points with the membership degree one in A:

$$coreA = \{x \in X|\ \mu_A(x) = 1\}.$$

Definition 1.5.4. (Height of a fuzzy set) The height of A is $hgt(A) = sup_{x\in X}\ \mu_A(x)$, i.e., the least upper bound of $\mu_A(x)$.

Definition 1.5.5. (Normal fuzzy set) A fuzzy set A is called normal if there exists a $x \in X$ such that $\mu_A(x) = 1$, otherwise A is subnormal.

Definition 1.5.6. (Empty fuzzy set) The empty set ϕ is defined as $\forall\ x \in X,\ \mu_\phi(x) = 0$.

Definition 1.5.7. ($\alpha - Cuts$) An α-level set of a fuzzy set A of X is a non-fuzzy set denoted by $[A]^\alpha$ and is defined by

$$[A]^\alpha = \begin{cases} \{\ x \in X|\ \mu_A(x) \geq \alpha\ \} & if \quad \alpha > 0 \\ cl(suppA) & if \quad \alpha = 0 \end{cases} \tag{1.6}$$

where $cl(suppA)$ denotes the closure of the support of A.

The membership function of a fuzzy set A can be expressed in terms of the characteristic functions of its $\alpha - cuts$ according to the formula [12]

$$\mu_A(x) = \sup_{\alpha\in]0,1]} \min(\alpha, \mu_{A^\alpha}(x)),$$

where

$$\mu_{A^\alpha}(x) = \begin{cases} 1, & x \in A^\alpha \\ 0, & otherwise. \end{cases} \tag{1.7}$$

It is easily checked that the following properties hold:

$$[A \cup B]^\alpha = [A]^\alpha \cup [B]^\alpha, \qquad [A \cap B]^\alpha = [A]^\alpha \cap [B]^\alpha. \tag{1.8}$$

Definition 1.5.8. (Convexity) A fuzzy set A of X is convex if and only if $\alpha-$cuts are convex. An equivalent definition of convexity is: A is convex if and only if $\forall x_1 \in X, \forall x_2 \in X, \forall \lambda \in [0,1]$,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\} \tag{1.9}$$

Definition 1.5.9. (Fuzzy number) The space E^n is all of fuzzy subsets U of $\mathbb{R}^n$ which satisfy the following conditions:

1. U is normal,
2. U is fuzzy convex,
3. U is upper semi-continuous,
4. $[U]^0$ is bounded subset of $\mathbb{R}^n$,

when $n = 1$, elements of E^1 are Fuzzy numbers.

In this thesis, we will write simply $A(x)$ instead of $\mu_A(x)$, also family of fuzzy numbers will be denoted E.

Definition 1.5.10. (Triangular fuzzy number)A fuzzy number A is called Triangular fuzzy number with center a, left width $m > 0$ and right width $n > 0$ if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{m} \quad if \quad a - m \leq t \leq a \\ 1 - \frac{(t-a)}{n} \quad if \quad a \leq t \leq a + n \\ 0 \quad otherwise \end{cases}$$

and we use for it the notation $A = (a, m, n)$.

Definition 1.5.11. (Trapezoidal fuzzy number) A fuzzy number A is called trapezoidal fuzzy number with tolerance interval $[a, b]$, left width $m > 0$ and right width

$n > 0$ if its membership function has the following form:

$$A(t) = \begin{cases} 1 - \frac{(a-t)}{m} & if \quad a - m \leq t \leq a \\ 1 & if \quad a \leq t \leq b \\ 1 - \frac{(t-b)}{n} & if \quad b \leq t \leq b + n \\ 0 & otherwise \end{cases}$$

and we use the notation $A = (a, b, m, n)$. It can easily be shown that

$$[A]^\alpha = [a - (1-\alpha)m, b + (1-\alpha)n], \forall \alpha \in [0, 1].$$

The support of A is $[a - m, b + n]$.

Definition 1.5.12. Any fuzzy number $A \in E$ can be described as

$$A(t) = \begin{cases} L(\frac{a-t}{m}) & if \quad t \in [a-m, a] \\ 1 & if \quad t \in [a, b] \\ R(\frac{t-b}{n}) & if \quad t \in [b, b+n] \\ 0 & otherwise \end{cases}$$

where $[a, b]$ is the core of A,

$$L : [0, 1] \to [0, 1], \quad R : [0, 1] \to [0, 1]$$

are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $R(1) = L(1) = 0$. We call this fuzzy interval of LR-type and refer to it by

$$A = (a, b, m, n)_{LR}.$$

The support of A is $(a - m, b + n)$.

Definition 1.5.13. (Fuzzy point) Let A be a fuzzy number. If supp(A)= $\{x_0\}$ then A is called a fuzzy point and we use the notation $A = \overline{x_0}$.

1.6 The extension principle

In order to use fuzzy numbers and relations in any intelligent system we must be able to perform arithmetic operations with these fuzzy quantities. In particular, we must be able to *add*, *subtract*, *multiply* and *divide* with fuzzy quantities. The process of doing these operations is called *fuzzy arithmetic*.

We shall first introduce an important concept from fuzzy set theory called the *extension principle*. We then use it to provide for these arithmetic operations on fuzzy numbers.

In general the extension principle plays a fundamental role in enabling us to extend any point operations to operations involving fuzzy sets. In the following, we define this principle.

Definition 1.6.1. (*extension principle*) Assume X and Y are crisp sets and let f be a mapping from X to Y,

$$f : X \to Y,$$

such that for each $x \in X$, $f(x) = y \in Y$. Assume A is a fuzzy subset of X, using extension principle, we can define $f(A)$ as a fuzzy subset of Y such that

$$f(A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $f^{-1}(y) = \{x \in X | f(x) = y\}$.

Definition 1.6.2. (*sup-min extension n-place functions*) Let X_1, X_2,...,X_n and Y be a family of sets. Assume f is a mapping from the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ into Y, that is, for each n-tuple $(x_1, x_2, \cdots, x_n)$, such that $x_i \in X_i$, we have $f(x_1, x_2, \cdots, x_n) = y \in Y$. Let A_1, A_2,...,A_n be fuzzy subsets of X_1, X_2,...,X_n, respectively, then we use the extension principle for the evaluation of $f(A_1, A_2, ..., A_n)$. In particular $f(A_1, A_2, ..., A_n) = B$ where B is a fuzzy subset of Y, such that

$$f(A_1, A_2, ..., A_n)(y) = \begin{cases} \sup\{\min\{A_1(x_1), A_2(x_2), \ldots, A_n(x_n)\} \mid x \in f^{-1}(y)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad f^{-1}(y) \neq \emptyset, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{otherwise,} \end{cases}$$

Definition 1.6.3. Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets and let f be a function from $E(X)$ to $E(Y)$. Then f is called a fuzzy function (or mapping) and we use the notation

$$f : E(X) \to E(Y).$$

Theorem 1.6.1. *Let $X \neq \emptyset$ and $Y \neq \emptyset$ be crisp sets. Then every fuzzy mapping $f : E(X) \to E(Y)$ defined by the extension principle is monotonic increasing.*

Proof. Let $A, A' \in E(X)$ such that $A \subset A'$. Then using the definition of sup-min extension principle we get

$$f(A)(y) = \sup_{x \in f^{-1}(y)} A(x) \leq \sup_{x \in f^{-1}(y)} A'(x) = f(A')(y)$$

for all $y \in Y$. $\square$

Let $A = (a_1, a_2, \alpha_1, \alpha_2)_{LR}$ and $B = (b_1, b_2, \beta_1, \beta_2)_{LR}$ be fuzzy numbers of $LR - type$. Using the (sup-min) extension principle, we can verify the following rules for addition and subtraction of fuzzy numbers of $LR - type$.

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)_{LR}$$

$$A - B = (a_1 - b_2, a_2 - b_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1)_{LR}$$

furthermore, if $\lambda \in \mathbb{R}$ is a real number then λA can be represented as

$$\lambda A = \begin{cases} (\lambda a_1, \lambda a_2, \lambda \alpha_1, \lambda \alpha_2)_{LR} & \text{if } \lambda \geq 0, \\ (\lambda a_2, \lambda a_1, \mid \lambda \mid \alpha_2, \mid \lambda \mid \alpha_1)_{LR} & \text{if } \lambda < 0. \end{cases}$$

In particular, if $A = (a_1, a_2, \alpha_1, \alpha_2)$ and $B = (b_1, b_2, \beta_1, \beta_2)$ are fuzzy numbers of trapezoidal form, then

$$A + B = (a_1 + b_1, a_2 + b_2, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$A - B = (a_1 - b_2, a_2 - b_1, \alpha_1 + \beta_2, \alpha_2 + \beta_1).$$

If $A = (a, \alpha_1, \alpha_2)$ and $B = (b, \beta_1, \beta_2)$ are fuzzy numbers of triangular form, then

$$A + B = (a + b, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$A - B = (a - b, \alpha_1 + \beta_2, \alpha_2 + \beta_1),$$

and if $A = (a, \alpha_1)$ and $B = (b, \beta_1)$ are fuzzy numbers of symmetrical triangular form, then

$$A + B = (a + b, \alpha_1 + \beta_1)$$

$$A - B = (a - b, \alpha_1 + \beta_1),$$

$$\lambda A = (\lambda a, \mid \lambda \mid \alpha_1).$$

The above results can be generalized to linear combination of fuzzy numbers.

Let A and B be fuzzy numbers with $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$, $0 \leq \alpha \leq 1$. Then it can easily be shown that

$$[A + B]^\alpha = [a_1(\alpha) + b_1(\alpha), a_2(\alpha) + b_2(\alpha)],$$

$$[-A]^\alpha = [-a_2(\alpha), -a_1(\alpha)],$$

$$[A - B]^\alpha = [a_1(\alpha) - b_2(\alpha), a_2(\alpha) - b_1(\alpha)],$$

$$[\lambda A]^\alpha = [\lambda a_1(\alpha), \lambda a_2(\alpha)] \quad if \quad \lambda \geq 0,$$

$$[\lambda A]^\alpha = [\lambda a_2(\alpha), \lambda a_1(\alpha)] \quad if \quad \lambda < 0,$$

for all $\alpha \in [0, 1]$. The following two theorems show that this property is valid for any continuous function.

Theorem 1.6.2. *[37] Let $f : X \to X$ be a continuous function and let A be a fuzzy number. Then,*

$$[f(A)]^\alpha = f([A]^\alpha),$$

where $f(A)$ is defined by the extension principle and

$$f([A]^\alpha) = \{f(x) \mid x \in [A]^\alpha\}.$$

If $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and f monotone increasing then from the above theorem we get

$$[f(A)]^\alpha = f([A]^\alpha) = f([a_1(\alpha), a_2(\alpha)]) = [f(a_1(\alpha)), f(a_2(\alpha))].$$

Theorem 1.6.3. *[37] Let $f : X \times X \to X$ be a continuous function and let A and B be fuzzy numbers. Then*

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha)$$

where,

$$f([A]^\alpha, [B]^\alpha) = \{f(x_1, x_2) \mid x_1 \in [A]^\alpha, x_2 \in [B]^\alpha\}.$$

Let $f(x,y) = xy$ and let $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$, $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ be the $\alpha-$level sets of two fuzzy numbers A and B. Applying above theorem we get

$$[f(A,B)]^\alpha = f([A]^\alpha, [B]^\alpha) = [A]^\alpha[B]^\alpha$$

However the equation

$$[AB]^\alpha = [A]^\alpha[B]^\alpha = [a_1(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)]$$

holds if and only if A and B are both nonnegative, i.e. $A(x) = B(x) = 0$ for $x \leq 0$.

If B is nonnegative then we have

$$[A]^\alpha[B]^\alpha = [\min\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha)\}, \max\{a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}].$$

In general case, we obtain a very complicated expression for the α-level sets of the product AB

$$[A]^\alpha[B]^\alpha = [\min\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\},$$

$$\max\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}].$$

1.7 Metric for fuzzy numbers

Let A and B be fuzzy numbers with $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$.

Definition 1.7.1. (Hausdorff distance) The Hausdorff distance between two (nonempty) sets $X,\ Y \subseteq \mathbb{R}$ is given as

$$d_H(X,Y) = \max\{\beta(X,Y), \beta(Y,X)\},$$

where $\beta(X,Y) = \sup_{x\in X} \rho(x,Y)$ and $\rho(x,Y) = \inf_{y\in Y} \mid x - y \mid$. The generalization

$$d_\infty(A,B) = \sup_{\alpha\in(0,1]} d_H([A]^\alpha, [B]^\alpha) \;\; \forall \; A,B \in E,$$

defines a distance measure. It is clear that

$$d_H(A,B) = \sup_{h\in[0,1]} \max\{\mid a_1(\alpha) - b_1(\alpha) \mid, \mid a_2(\alpha) - b_2(\alpha) \mid\},$$

i.e. $d_H(A,B)$ is the maximal distance between α-level sets of A and B.

Definition 1.7.2. For arbitrary fuzzy quantities $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$, the quantity

$$D_p(u,v) = [\int_0^1 (\underline{u}(\alpha) - \underline{v}(\alpha))^p dr + \int_0^1 (\overline{u}(\alpha) - \overline{v}(\alpha))^p dr]^{\frac{1}{p}} \tag{1.10}$$

is the distance between u and v.

1.8 Parametric and triple form of fuzzy number

Definition 1.8.1. An arbitrary fuzzy number is presented by an ordered pair of functions $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, which satisfies the following requirements:

- $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function over [0,1], with respect to any α.
- $\overline{u}(\alpha)$ is a bounded left continuous nonincreasing function over [0,1], with respect to any α.
- $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, $0 \leq \alpha \leq 1$.

A crisp number λ is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = \lambda, 0 \leq \alpha \leq 1$.

Lemma 1.8.1. *Let v and w be fuzzy numbers and s be a real number. Then for* $0 \leq \alpha \leq 1$

$$u = v \text{ if and only if } \underline{u}(\alpha) = \underline{v}(\alpha) \text{ and } \overline{u}(\alpha) = \overline{v}(\alpha),$$

$$v + w = (\underline{v}(\alpha) + \underline{w}(\alpha), \overline{v}(\alpha) + \overline{w}(\alpha)),$$

$$v - w = (\underline{v}(\alpha) - \overline{w}(\alpha), \overline{v}(\alpha) - \underline{w}(\alpha)),$$

$$v.w = (\min\{\underline{v}(\alpha).\underline{w}(\alpha), \underline{v}(\alpha).\overline{w}(\alpha), \overline{v}(\alpha).\underline{w}(\alpha), \overline{v}(\alpha).\overline{w}(\alpha)\},$$

$$\max\{\underline{v}(\alpha).\underline{w}(\alpha), \underline{v}(\alpha).\overline{w}(\alpha), \overline{v}(\alpha).\underline{w}(\alpha), \overline{v}(\alpha).\overline{w}(\alpha)\}),$$

$$sv = \begin{cases} s(\underline{v}(\alpha), \overline{v}(\alpha)) & s \geq 0 \\ s(\overline{v}(\alpha), \underline{v}(\alpha)) & s < 0 \end{cases}$$

Definition 1.8.2. Triangular fuzzy numbers are those fuzzy sets in E which are characterized by an ordered triple $(x_l, x_c, x_r) \in R^3$ with $x_l \leq x_c \leq x_r$ such that $[U]^0 = [x_l, x_r]$ and $[U]^1 = \{x_c\}$, then

$$[U]^\alpha = [x_c - (1-\alpha)(x_c - x_l), x_c + (1-\alpha)(x_r - x_c)], \tag{1.11}$$

for any $\alpha \in [0, 1]$.

1.9 Interpolation and approximation

The problem of interpolation for fuzzy sets is as follows:

suppose that at various time instant t information $f(t)$ is presented as fuzzy set. The aim is to approximate the function $\widetilde{f}(t)$, for all t in the domain of f. Let $t_0 < t_1 < \ldots < t_n$ be $n+1$ distinct points in R and let $\widetilde{u}_0, \widetilde{u}_1, \ldots, \widetilde{u}_n$ be $n+1$ fuzzy sets in E. A fuzzy polynomial interpolation of the data is a fuzzy-value continuous function $\widetilde{f}: R \to E$ satisfying:

- $\widetilde{f}(t_i) = \widetilde{u}_i,\ i = 1, \ldots, n,$
- If the data is crisp, then the interpolation polynomial f is a crisp polynomial.

A function $\widetilde{f}$ which fulfilling these conditions may be constructed as follows. Let $C^i_\alpha = [\widetilde{u}_i]^\alpha$ for any $\alpha \in [0,1],\ i = 0, 1, \ldots, n$. For each $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$, the unique polynomial of degree $\leq n$ denoted by P_X such that

$$P_X(t_i) = x_i, \quad i = 0, 1, \ldots, n,$$

$$P_X(t) = \sum_{i=0}^{n} x_i(\prod_{i \neq j} \frac{t - t_j}{t_i - t_j}).$$

Finally, for each $t \in \mathbb{R}$ and all $\xi \in \mathbb{R}$, $\widetilde{f}(t) \in E$ is defined by

$$(f(t))(\xi) = \sup\{\alpha \in [0,1] : \exists X \in C^0_\alpha \times \ldots \times C^n_\alpha \ \ \textit{such that}\ P_X(t) = \xi\}.$$

The interpolation polynomial can be written level set wise as

$$[f(t)]^\alpha = \{y \in \mathbb{R} : y = P_X(t),\ X \in [\widetilde{u}_i]^\alpha, i = 0, 1, 2, \ldots, n\},\ \textit{for}\ 0 \leq \alpha \leq 1.$$

When the data $\widetilde{u}_i$ presents as triangular fuzzy numbers, values of the interpolation polynomial are also triangular fuzzy numbers. Then $\widetilde{f}(t)$ has a particular simple form that is well suited to computation.

Theorem 1.9.1. *Let $(t_i, \widetilde{u}_i), i = 0, 1, 2, \ldots, n$ be the observed data and suppose that each of the $\widetilde{u}_i = (u^l_i, u^c_i, u^r_i)$ is an element of E. Then for each $t \in [t_0, t_n]$,*

$\widetilde{f}(t) = (f^l(t), f^c(t), f^r(t)) \in E,$

$f^l(t) = \sum_{\ell_i(t)\geq 0} \ell_i(t)u_i^l + \sum_{\ell_i(t)<0} \ell_i(t)u_i^r,$

$f^c(t) = \sum_{i=0}^{n} \ell_i(t)u_i^c,$

$f^r(t) = \sum_{\ell_i(t)\geq 0} \ell_i(t)u_i^r + \sum_{\ell_i(t)<0} \ell_i(t)u_i^l,$

such that $\ell_i(t) = \prod_{j\neq i}^{n} \frac{t-t_j}{t_i-t_j}.$

proof. see [18].

Let S_k denote the family of spline of order k with knots $t_i,\ i = 0, \ldots, n.$

That is $s \in S_k$ if

1- $s \in C^{k-1}$ on the interval $[t_0, t_n]$,

2- s is a polynomial of degree at most k on the interval $[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, n-1.$

Let $x = (x_0, x_1, \ldots, x_n)$ and let $S_x(t)$ be a spline in S_k which interpolates the data $(t_i, x_i),\quad i = 0, 1, \ldots, n.$ Then the interpolating fuzzy spline of order k which interpolates the data $(t_i, \widetilde{u}_i),\quad \widetilde{u}_i \in E\ i = 0, 1, \ldots, n$ is defined level setwise by $[\widetilde{f}(t)]^\alpha = \{y \in \mathbb{R} : y = P_x(t),\ x \in [\widetilde{u}_i]^\alpha\}$ for all $0 \leq \alpha \leq 1$.

If $s_i \in S_k$ interpolates the data $(t_j, f_j),\ j = 0, 1, \ldots, n$, where $f_j = 1$ if $j = i$ and zero otherwise, then $S_x(t) = \sum_{i=0}^{n} x_i s_i(t)$ and $f(t) = \sum_{i=0}^{n} s_i(t)u_i$. Consequently, $\widetilde{f}$ is continuous on $[t_0, t_n]$ and if all the $\widetilde{u}_i \in E$, then $\widetilde{f}(t) \in E$ for all $t \in [t_0, t_n]$.

Example 1.9.1. Consider the case when $k = 1$, which gives piecewise linear interpolation. Since $s_i(t) > 0$ if $t \in (t_{i-1}, t_{i+1})$ and $s_i(t) = 0$ for $t \notin [t_{i-1}, t_{i+1}]$, then

$$\widetilde{f}(t) = \frac{(t_{i+1} - t)}{(t_{i+1} - t_i)}\widetilde{u}_i + \frac{(t - t_i)}{(t_{i+1} - t_i)}\widetilde{u}_{i+1},$$

for all $t \in [t_i, t_{i+1}]$, $\quad i = 0, 1, \ldots, n-1$.

Hence, in terms of level sets, writing $[u_i]^\alpha = [a_i^\alpha, b_i^\alpha]$ and $[f(t)]^\alpha = [f^l(t,\alpha), f^r(t,\alpha)]$,

$$f^l(t,\alpha) = \frac{(t_{i+1} - t)}{(t_{i+1} - t_i)}a_i^\alpha + \frac{(t - t_i)}{(t_{i+1} - t_i)}a_{i+1}^\alpha$$

$$f^r(t,\alpha) = \frac{(t_{i+1} - t)}{(t_{i+1} - t_i)}b_i^\alpha + \frac{(t - t_i)}{(t_{i+1} - t_i)}b_{i+1}^\alpha$$

for all $t \in [t_i, t_{i+1}]$.

1.10 Fuzzy Derivatives

Let $\widetilde{X}(t)$ be a fuzzy number for each $t \in I$. Also, let $[\widetilde{X}(t)]^\alpha = [x_1(t,\alpha), x_2(t,\alpha)]$ and write $x_i'(t,\alpha)$ for the partial derivatives of $x_i(t,\alpha)$, with respect to t, $i = 1, 2$. We assume these partial derivatives always exist in this section. In what follows, a discussion of the Goetchel-Voxman derivative, the Seikkala derivative, the Dubois-Prade derivative, the Puri-Ralescu derivative, and the Kandel-Friedman-Ming derivative of $\widetilde{X}(t)$ is provided.

Goetschel-Voxman derivative

Let $\widetilde{X}(t)$ and $\widetilde{Z}(t)$ be two fuzzy functions for $t \in I = [0, T]$. Both $\widetilde{X}(t)$ and $\widetilde{Z}(t)$ are fuzzy numbers for each $t \in I$. Set $[\widetilde{X}(t)]^\alpha = [x_1(t, \alpha), x_2(t, \alpha)]$ and $[\widetilde{Z}(t)]^\alpha = [z_1(t, \alpha), z_2(t, \alpha)]$ for all t and α. Then metric d_H is

$$d_H(\widetilde{X}(t), \widetilde{Z}(t)) = \sup_\alpha \{\max[|x_1(t, \alpha) - z_1(t, \alpha)|, |x_2(t, \alpha) - z_2(t, \alpha)|]\}, \tag{1.12}$$

for all t in I.

The derivative of $\widetilde{X}(t)$ at t_0, $GVD\widetilde{X}(t)$ written by is defined as

$$GVD\widetilde{X}(t_0) = \lim_{h \to 0} (\frac{\widetilde{X}(t_0 + h) - \widetilde{X}(t_0)}{h}), \tag{1.13}$$

provided the limit exists with respect to metric d_H. However, the subtraction in Eq.(1.13) is not standard fuzzy subtraction because

$$[\widetilde{X}(t_0 + h) - \widetilde{X}(t_0)]^\alpha = [x_1(t_0 + h, \alpha) - x_1(t_0, \alpha), x_2(t_0 + h, \alpha) - x_2(t_0, \alpha)], \tag{1.14}$$

for all t, α. Standard fuzzy arithmetic would produce

$$[x_1(t_0 + h, \alpha) - x_2(t_0, \alpha), x_2(t_0 + h, \alpha) - x_1(t_0, \alpha)].$$

However, $GVD\widetilde{X}(t)$ may not be a fuzzy number for some t in I, and non-standard fuzzy subtraction is used in the definition of the derivative.

Seikkala derivative

The Seikkala derivative of $\widetilde{X}(t)$, written $SD\widetilde{X}(t)$, is defined as follows: if $[x'_1(t, \alpha), x'_2(t, \alpha)]$ are the α-cuts of a fuzzy number for each $t \in I$, then $SD\widetilde{X}(t)$ exists and $SD\widetilde{X}(t)(\alpha) = [x'_1(t, \alpha), x'_2(t, \alpha)]$.

$SD\widetilde{X}(t)$ is a fuzzy number for all $t \in I$.

Dubois-Prade derivative

The Dubois-Prade derivative of $\widetilde{X}(t)$, written $DPD\widetilde{X}(t)$, always exists and its membership function is given by

$$DPD\widetilde{X}(t)(x) = \sup\{\alpha | x = x_1'(t,\alpha), x = x_2'(t,\alpha)\}. \tag{1.15}$$

However, $DPD\widetilde{X}(t)$ may not be a fuzzy number.

Let us consider the situation where $DPD\widetilde{X}(t)$ can be a fuzzy number for t in I. Assume that $x_1'(t,\alpha)$ and $x_2'(t,\alpha)$ satisfy the sufficient conditions for $[x_1'(t,\alpha), x_2'(t,\alpha)]$ to define $\alpha-$cuts of a fuzzy number. If $x_1'(t,1) < x_2'(t,1)$ for some value of t, then we separately define $DPD\widetilde{X}(t) = 1$ for all x satisfying $x_1'(t,1) < x < x_2'(t,1)$. The $DPD\widetilde{X}(t)$ will be a fuzzy number.

Puri-Ralescu derivative

The Puri-Ralescu derivative of $\widetilde{X}(t)$ is written $PRD\widetilde{X}(t)$. Let $\widetilde{X}(t)$ and $\widetilde{Z}(t)$ be two fuzzy fuctions for $t \in I$. Both $\widetilde{X}(t)$ and $\widetilde{Z}(t)$ are fuzzy numbers for each $t \in I$. Set $[\widetilde{X}(t)]^\alpha = [x_1(t,\alpha), x_2(t,\alpha)]$ and $[\widetilde{Z}(t)]^\alpha = [z_1(t,\alpha), z_2(t,\alpha)]$ for all t and α. Then metric d_H is

$$d_H(\widetilde{X}(t), \widetilde{Z}(t)) = \sup_\alpha d_H([\widetilde{X}(t)]^\alpha, [\widetilde{Z}(t)]^\alpha), \tag{1.16}$$

for all t, where d_H is the Hausdorff metric on non-empty compact subsets of $\mathbb{R}$.

Next, the Hukuhara difference between two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ should be defined.

If there exists a fuzzy number $\widetilde{C}$ so that $\widetilde{C} + \widetilde{A} = \widetilde{B}$, then $\widetilde{C}$ is called the Hukuhara difference between $\widetilde{B}$ and $\widetilde{A}$ and is written as

$$\widetilde{B} \sim \widetilde{A} = \widetilde{C}.$$

$\widetilde{X}(t)$ is differentiable at t_0 in I if there exists a fuzzy number $PRD\widetilde{X}(t_0)$ so that

$$\lim_{h \to 0^+} (\frac{\widetilde{X}(t_0 + h) \sim \widetilde{X}(t_0)}{h}) = PRD\widetilde{X}(t_0), \tag{1.17}$$

and

$$\lim_{h \to 0^+} (\frac{\widetilde{X}(t_0) \sim \widetilde{X}(t_0 - h)}{h}) = PRD\widetilde{X}(t_0). \tag{1.18}$$

Both limits are taken with respect to metric d_H in Eq. (1.16).

If $PRD\widetilde{X}(t)$ exists, then

$$PRD\widetilde{X}(t)[\alpha] = [x_1'(t, \alpha), x_2'(t, \alpha)],$$

for all $t \in I$, $\alpha \in [0, 1]$. $PRD\widetilde{X}(t)$ is always a fuzzy number for each $t \in I$. Although, non-standard fuzzy subtraction is used, the Hukuhara difference of fuzzy sets is employed.

Kandel-Friedman-Ming derivative

The Kandel-Friedman-Ming derivative of $\widetilde{X}(t)$ is written as $KFMD\widetilde{X}(t)$.

First, fuzzy numbers now do not need to have compact support. The metric D used is

$$D(\widetilde{X}(t), \widetilde{Z}(t)) = \max\{[\int_0^1 |x_1(t, \alpha) - z_1(t, \alpha)|^p d\alpha]^{1/p}, [\int_0^1 |x_2(t, \alpha) - z_2(t, \alpha)|^p d\alpha]^{1/p}\},$$

for $x_1(t,\alpha), x_2(t,\alpha), z_1(t,\alpha)$ and $z_2(t,\alpha)$ all in $L_p[0,1]$ for all t in I.

$\widetilde{X}(t)$ is differentiable at $t_0 \in I$ if there is a fuzzy number $KFMD\widetilde{X}(t_0)$ so that

$$\lim_{h\to 0} D[\frac{\widetilde{X}(t_0+h) - \widetilde{X}(t_0)}{h}, KFMD\widetilde{X}(t_0)] = 0.$$

However, the subtraction $\widetilde{X}(t_0+h) - \widetilde{X}(t_0)$ in the above equation is not standard fuzzy subtraction since it is defined as in Eq. (1.14).

When this derivative exists

$$KFMD\widetilde{X}(t)(\alpha) = [x_1'(t,\alpha), x_2'(t,\alpha)],$$

for all $t \in I$,$\alpha \in [0,1]$. This derivative also equals a fuzzy number for all $t \in I$. Also, non-standard fuzzy subtraction is used.

Relationships

Some theorems in [4], which state the relationships among the above mentioned derivatives are presented here.

Theorem 1.10.1. *1. If $GVD\widetilde{X}(t)$ exists and is a fuzzy number for each $t \in I$, then $SD\widetilde{X}(t)$ exists and $GVD\widetilde{X}(t) = SD\widetilde{X}(t)$.*

2. If $PRD\widetilde{X}(t)$ exists, then $SD\widetilde{X}(t)$ exists and $PRD\widetilde{X}(t) = SD\widetilde{X}(t)$.

3. If $KFMD\widetilde{X}(t)$ exists, then so does $SD\widetilde{X}(t)$ and they are equal.

4. If $SD\widetilde{X}(t)$ exists and if $x_1'(t,\alpha)$ and $x_2'(t,\alpha)$ are both continuous in α for each t in I, then $SD\widetilde{X}(t) = DPD\widetilde{X}(t)$.

Proof. See [4].

Theorem 1.10.2. *Assume the continuity condition holds. If* $SD\widetilde{X}(t)$ *exists, then* $SD\widetilde{X}(t) = DPD\widetilde{X}(t) = GVD\widetilde{X}(t) = PRD\widetilde{X}(t) = KFMD\widetilde{X}(t)$.

Proof. See [4].

Theorem 1.10.3. *Assume the continuity condition holds. If one of the derivatives* SD *or* GVD *exists and it is a fuzzy number,* PRD*, or* $KMFD$ *exist, then so do the others and they are all equal.*

Proof. See [4].

1.11 Fuzzy initial value problem (FIVP)

Fuzzy initial value problem is defined as

$$\begin{cases} x'(t) & = f(t, x(t)); \quad t \in I = [t_0, T], \\ x(t_0) & = \widetilde{x}_0. \end{cases} \tag{1.19}$$

where x is a a fuzzy function of t, $f(t, x(t))$-a fuzzy function of the crisp variable t, and the fuzzy variable x, and x' is the fuzzy derivative of x.

If a fuzzy initial value $x(t_0) = \widetilde{x}_0$ is given, we obtain the fuzzy initial value problem.

S. Seikkala in [33], solved (1.19) as follows:

The extension principle of Zadeh leads to the following definition of $f(t, x)$ when $x(t)$ is a fuzzy number

$$f(t, x)(s) = \sup\{x(\tau) | \ s = f(t, \tau)\}, \quad s \in \mathbb{R}.$$

It follows that

$$[f(t, x)]^\alpha = [f_1(t, x; \alpha), f_2(t, x; \alpha)], \quad \alpha \in (0, 1].$$

where

$$\begin{aligned} f_1(t,x;\alpha) &= min\{f(t,u)|\ u \in [x_1(t;\alpha), x_2(t;\alpha)]\}, \\ f_2(t,x;\alpha) &= max\{f(t,u)|\ u \in [x_1(t;\alpha), x_2(t;\alpha)], \}. \end{aligned} \tag{1.20}$$

The function $x : R^+ \to E$ is a fuzzy solution of (1.19) on I, if

$$\begin{aligned} x_1'(t;\alpha) &= min\{f(t,u)|\ u \in [x_1(t,\alpha), x_2(t,\alpha)]\},\ x_1(0;\alpha) = x_{01}(\alpha), \\ x_2'(t;\alpha) &= max\{f(t,u)|\ u \in [x_1(t,\alpha), x_2(t,\alpha)]\},\ x_1(0;\alpha) = x_{01}(\alpha), \end{aligned} \tag{1.21}$$

for any $t \in I$ and $\alpha \in [0,1]$. By following theorem $[x_1(t;\alpha), x_2(t;\alpha)],\ \alpha \in [0,1]$, define a fuzzy number $\widetilde{x}(t)$ in E.

Theorem 1.11.1. *Let f satisfy*

$$|f(t,v) - f(t,\overline{v})| \leq g(t, |v - \overline{v}|), \quad t \geq 0, \quad v, \overline{v} \in R,$$

where g on $R^+ \times R^+$ is a continuous mapping such that $\alpha \to g(t,\alpha)$ is nondecreasing, the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \tag{1.22}$$

has a solution on R^+ for $u_0 > 0$ and that $u(t) = 0$ is the only solution of (1.22) for $u_0 = 0$. Then the fuzzy initial value problem (1.19) has a unique fuzzy solution.

Proof: see [33].

J. Buckley and T. Feuring in [4], solved the FIVP (1.19) by fuzzifying the crisp solution to obtain fuzzy solution using the extension principle.

In next section, we present two numerical methods for solving fuzzy initial value problem.

1.12 Numerical solution of fuzzy differential equations

Now we are going to discuss about numerical method for solving differential equations.

1.12.1 Euler method for solving fuzzy differential equations

M. Friedman, M. Ma, A. Kandel in [21] define a first-order fuzzy differential equation by $x'(t) = f(t, x(t))$ where x is a fuzzy function of t, $f(t, x(t))$ a fuzzy function of the crisp variable t and the fuzzy variable x, and x' is the fuzzy derivative of x. If an initial value $x(t_0) = \widetilde{x}_0$ is given, they obtain a fuzzy initial value problem:

$$x'(t) = f(t, x(t)), \qquad x(t_0) = \widetilde{x}_0. \tag{1.23}$$

Sufficient conditions for the existence of a unique solution to Eq.(1.23) are that f is continuous and that a Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad L > 0 \tag{1.24}$$

is fulfilled. They replace Eq. (1.23) by the equivalent system

$$\underline{x}'(t) = \underline{f}(t, x) = F(t, \underline{x}, \overline{x}), \quad \underline{x}(t_0) = \underline{x}_0 \tag{1.25}$$

$$\overline{x}'(t) = \overline{f}(t, x) = G(t, \underline{x}, \overline{x}), \quad \overline{x}(t_0) = \overline{x}_0$$

which possesses a unique solution $(\underline{x}, \overline{x})$ which is a fuzzy function, i.e. for each t, the pair $[\underline{x}(t; \alpha), \overline{x}(t; \alpha)]$ is a fuzzy number.

The parametric form of Eq. (1.25) is given by

$$\underline{x}'(t;\alpha) = F[t, \underline{x}(t;\alpha), \overline{x}(t;\alpha)], \quad \underline{x}(t_0;\alpha) = \underline{x}_0(\alpha), \tag{1.26}$$

$$\overline{x}'(t;\alpha) = F[t, \underline{x}(t;\alpha), \overline{x}(t;\alpha)], \quad \overline{x}(t_0;\alpha) = \overline{x}_0(\alpha),$$

for $\alpha \in [0,1]$. A solution to Eq. (1.26) must solve Eq. (1.25).

To integrate the system given in Eq. (1.26) from t_0 to a prefixed $T > t_0$, the interval $[t_0, T]$ is replaced by a set of discrete equally spaced grid points $t_0 < t_1 < t_2 < \ldots < t_N = T$ at which the exact solution $(\underline{Y}(t;\alpha), \overline{Y}(t;\alpha))$ is approximated by some $(\underline{y}(t;\alpha), \overline{y}(t;\alpha))$. (Note that throughout each integration, α is unchanged.) The exact and approximate solutions at t_n, $0 \leq n \leq N$ are denoted by $Y_n(r) = [\underline{Y}_n(\alpha), \overline{Y}_n(\alpha)]$ and $y_n(r) = [\underline{y}_n(\alpha), \overline{y}_n(\alpha)]$, respectively. The grid points at which the solution is calculated are

$$t_n = t_0 + nh, \qquad h = (T - t_0)/N; \quad 1 \leq n \leq N.$$

The Euler method is based on the first-order approximation of $\underline{Y}'(t;\alpha)$ and $\overline{Y}'(t;\alpha)$ and is given by

$$Z'(t;\alpha) \approx \frac{Z(t+h;\alpha) - Z(t;\alpha)}{h} \tag{1.27}$$

where $Z(t;\alpha)$ is $\underline{Y}(t;\alpha)$ and $\overline{Y}(t;\alpha)$ alternatively. By virtue of Eq. (1.27)

$$\underline{Y}_{n+1}(\alpha) \approx \underline{Y}_n(\alpha) + hF_n(\alpha), \tag{1.28}$$

$$\overline{Y}_{n+1}(\alpha) \approx \overline{Y}_n(\alpha) + hG_n(\alpha),$$

where

$$F_n(\alpha) \widehat{=} F[t_n, \underline{Y}_n(\alpha), \overline{Y}_n(\alpha)], \tag{1.29}$$

$$G_n(\alpha) \widehat{=} G[t_n, \underline{Y}_n(\alpha), \overline{Y}_n(\alpha)].$$

By Eq. (1.28) the following equations is obtained:

$$\underline{y}_{n+1}(\alpha) = \underline{y}_n(\alpha) + hF[t_n, \underline{y}_n(\alpha), \overline{y}_n(\alpha)], \tag{1.30}$$

$$\overline{y}_{n+1}(\alpha) = \overline{y}_n(\alpha) + hG[t_n, \underline{y}_n(\alpha), \overline{y}_n(\alpha)],$$

where $\underline{y}_0(\alpha) \widehat{=} \underline{x}_0(\alpha)$, $\overline{y}_0(\alpha) \widehat{=} \overline{x}_0(\alpha)$. The polygon curves

$$\underline{y}(t; h; \alpha) \widehat{=} [t_0, \underline{y}_0(\alpha)], [t_1, \underline{y}_1(\alpha)], \ldots, [t_N, \underline{y}_N(\alpha)]$$

$$\overline{y}(t; h; \alpha) \widehat{=} [t_0, \overline{y}_0(\alpha)], [t_1, \overline{y}_1(\alpha)], \ldots, [t_N, \overline{y}_N(\alpha)]$$

are the Euler approximates to $\underline{Y}(t; \alpha)$ and $\overline{Y}(t; \alpha)$, respectively, over the interval $t_0 \leq t \leq t_N$.

Theorem 1.12.1. *Let $F(t, u, v)$ and $G(t, u, v)$ belong to $C^1(K)$ and let the partial derivatives of F, G be bounded over K. Then, for arbitrary fixed $r: \ 0 \leq \alpha \leq 1$, the Euler approximates converge to the exact solution $\underline{Y}(t; \alpha)$ and $\overline{Y}(t; \alpha)$ uniformly in t.*

Proof: see [21].

1.12.2 Taylor method of order p

In this section we are going to study *Taylor method of order p* for solving fuzzy initial value problem proposed by S. Abbasbandy and T. Allahviranloo in [1].

Let the exact solution $[Y(t)]^\alpha = [Y_1(t;\alpha), Y_2(t;\alpha)]$ is approximated by some $[y(t)]^\alpha = [y_1(t;\alpha), y_2(t;\alpha)]$. The Taylor method of order p is based on the

$$y_{i+1} = y_i + hT(t_i, y_i), \quad i = 0, 1, \ldots, N-1, \tag{1.31}$$

and

$$T(t_i, y_i) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f^{(i)}(t_i, y_i), \tag{1.32}$$

where

$$a = t_0 \leq t_1 \leq \ldots \leq t_N = b \; and \; h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{1.33}$$

The following equations are defined:

$$\begin{aligned} F[t, x; \alpha] = & \; \textstyle\sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_1^{(i)}(t, x; \alpha), \\ G[t, x; \alpha] = & \; \textstyle\sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_2^{(i)}(t, x; \alpha). \end{aligned} \tag{1.34}$$

The exact and approximate solutions at $t_n, 0 \leq n \leq N$ are denoted by $[Y(t_n)]^\alpha = [Y_1(t_n;\alpha), Y_2(t_n;\alpha)]$ and $[y(t_n)]^\alpha = [y_1(t_n;\alpha), y_2(t_n;\alpha)]$, respectively. The solution is calculated at points t_n. By Taylor method of order p and substituting Y_1 and Y_2 in (1.31) and considering (1.34) they obtained

$$\begin{aligned} Y_1(t_{n+1};\alpha) \approx & \; Y_1(t_n;\alpha) + hF[t_n, Y(t_n);\alpha], \\ Y_2(t_{n+1};\alpha) \approx & \; Y_2(t_n;\alpha) + hG[t_n, Y(t_n);\alpha]. \end{aligned} \tag{1.35}$$

Hence

$$\begin{aligned} y_1(t_{n+1};\alpha) = & \; y_1(t_n;\alpha) + hF[t_n, y(t_n);\alpha], \\ y_2(t_{n+1};\alpha) = & \; y_2(t_n;\alpha) + hG[t_n, y(t_n);\alpha], \end{aligned} \tag{1.36}$$

where

$$y_1(0;\alpha) = x_1(0;\alpha) \quad , \quad y_2(0;\alpha) = x_2(0;\alpha).$$

Let $F^*(t,u,v)$ and $G^*(t,u,v)$ be the functions F and G in (1.34), where u and v are constants and $u \leq v$. In other words

$$F^*(t,u,v) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} min\{f_1^{(i)}(t,\tau)|\tau \in [u,v]\},$$

$$G^*(t,u,v) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} max\{f_1^{(i)}(t,\tau)|\tau \in [u,v]\},$$

i.e. $F^*(t,u,v)$ and $G^*(t,u,v)$ are obtained by substituting $[x(t)]^\alpha = [u,v]$ in (1.34).

Theorem 1.12.2. *Let $F^*(t,u,v)$ and $G^*(t,u,v)$ belong to $C^{p-1}(K)$, $K = \{(t,u,v)|t_0 \leq t \leq T, -\infty < v < \infty, -\infty < u < v\}$, and let the partial derivatives of F^* and G^* in terms of u and v be bounded over K. Then, for arbitrary fixed $\alpha : 0 \leq \alpha \leq 1$, the approximate solutions (1.19) converge to the exact solutions $Y_1(t;\alpha)$ and $Y_2(t;\alpha)$ uniformly in t.*

Proof: see [1].

1.13 Numerical solution to hybrid fuzzy systems

The problem of stabilizing a continuous plant governed by differential equation through the interaction with a discrete time controller has recently been investigated. This study leads to the consideration of hybrid systems. In this section, we study two numerical methods to hybrid fuzzy differential equations.

Consider the hybrid fuzzy differential system

$$\begin{cases} x'(t) = f(t, x(t), \lambda_k(x_k)), & t \in [t_k, t_{k+1}], \\ x(t_k) = x_k, \end{cases} \tag{1.37}$$

where $0 \leq t_0 < t_1 < \cdots < t_k < \cdots, t_k \to \infty$, $f \in C[R^+ \times E \times E, E]$, $\lambda_k \in C[E, E]$. It is assumed that the existence and uniqueness of solution of the hybrid system hold on each $[t_k, t_{k+1}]$. To be specific the system would look like:

$$x'(t) = \begin{cases} x_0'(t) = f(t, x_0(t), \lambda_0(x_0)), & x_0(t_0) = x_0, \quad t_0 \leq t \leq t_1, \\ x_1'(t) = f(t, x_1(t), \lambda_1(x_1)), & x_1(t_1) = x_1, \quad t_1 \leq t \leq t_2, \\ \cdots \\ x_k'(t) = f(t, x_k(t), \lambda_k(x_k)), & x_k(t_k) = x_k, \quad t_k \leq t \leq t_{k+1}, \\ \cdots \end{cases} \tag{1.38}$$

The following function is meant by the solution of (1.37),

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), & t_0 \leq t \leq t_1, \\ x_1(t), & t_1 \leq t \leq t_2, \\ \cdots \\ x_k(t), & t_k \leq t \leq t_{k+1}, \\ \cdots \end{cases} \tag{1.39}$$

The solution of (1.37) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E$ and $k = 0, 1, 2, \ldots$.

Similar to [21], system (1.37) can be replaced by an equivalent system:

$$\begin{cases} \underline{x}'(t) = \underline{f}(t, x, \lambda_k(x_k)) = F_k(t, \underline{x}, \overline{x}), & \underline{x}(t_k) = \underline{x}_k, \\ \overline{x}'(t) = \overline{f}(t, x, \lambda_k(x_k)) = G_k(t, \underline{x}, \overline{x}), & \overline{x}(t_k) = \overline{x}_k, \end{cases} \tag{1.40}$$

which possesses a unique solution $(\underline{x}, \overline{x})$ which is a fuzzy function. That is for each t, the pair $[\underline{x}(t, \alpha), \overline{x}(t, \alpha)]$ is a fuzzy number, where $\underline{x}(t, \alpha)$, $\overline{x}(t, \alpha)$ are respectively the solution of the parametric form given by:

$$\begin{cases} \underline{x}'(t, \alpha) = F_k(t, \underline{x}(t, \alpha), \overline{x}(t, \alpha)), & \underline{x}(t_k, \alpha) = \underline{x}_k(\alpha), \\ \overline{x}'(t, \alpha) = G_k(t, \underline{x}(t, \alpha), \overline{x}(t, \alpha)), & \overline{x}(t_k, \alpha) = \overline{x}_k(\alpha), \end{cases} \tag{1.41}$$

For $\alpha \in [0, 1]$.

For a fixed α, to integrate the system in (1.41) in $[t_0, t_1], [t_1, t_2], \cdots, [t_k, t_{k+1}], \cdots$, each interval is replaced by a $N_k + 1$ discrete equally spaced grid points (include the endpoints) at which the exact solution $(\underline{x}(t,\alpha), \overline{x}(t,\alpha))$ is approximated by some $(\underline{y}_k(t,\alpha), \overline{y}_k(t,\alpha))$. For the chosen grid points on $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k$, $h_k = \frac{t_{k+1}-t_k}{N_k}$, $0 \leq n \leq N_k$ let $(\underline{Y}_k(t,\alpha), \overline{Y}_k(t,\alpha)) \equiv (\underline{x}(t,\alpha), \overline{x}(t,\alpha))$. $(\underline{Y}_k(t,\alpha), \overline{Y}_k(t,\alpha))$ and $(\underline{y}_k(t,\alpha), \overline{y}_k(t,\alpha))$ may be denoted respectively by $(\underline{Y}_{k,n}(\alpha), \overline{Y}_{k,n}(\alpha))$ and $(\underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha))$.

1.13.1 The Euler method for hybrid fuzzy differential equations

The Euler method is the first order approximation of $\underline{Y}'_k(t,\alpha)$ and $\overline{Y}'_k(t,\alpha)$, which can be written as:

$$\begin{cases} \underline{Y}_{k,n+1}(\alpha) \approx \underline{Y}_{k,n}(\alpha) + h_k F_k[t_n, \underline{Y}_{k,n}(\alpha), \overline{Y}_{k,n}(\alpha)], \\ \overline{Y}_{k,n+1}(\alpha) \approx \overline{Y}_{k,n}(\alpha) + h_k F_k[t_n, \underline{Y}_{k,n}(\alpha), \overline{Y}_{k,n}(\alpha)]. \end{cases} \tag{1.42}$$

therefore

$$\begin{cases} \underline{y}_{k,n+1}(\alpha) = \underline{y}_{k,n}(\alpha) + h_k F_k[t_n, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)], \\ \overline{y}_{k,n+1}(\alpha) = \overline{y}_{k,n}(\alpha) + h_k F_k[t_n, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)]. \end{cases} \tag{1.43}$$

However, (1.43) will use $\underline{y}_{0,0}(\alpha) = \underline{x}_0(\alpha)$, $\overline{y}_{0,0}(\alpha) = \overline{x}_0(\alpha)$ and $\underline{y}_{k,0}(\alpha) = \underline{y}_{k-1,N_{k-1}}(\alpha)$, $\overline{y}_{k,0}(\alpha) = \overline{y}_{k-1,N_{k-1}}(\alpha)$ if $k \geq 1$. Then (1.43) represents an approximation of $\underline{Y}_k(t,\alpha)$ and $\overline{Y}_k(t,\alpha)$ for each of the intervals $t_0 \leq t \leq t_1$, $t_1 \leq t \leq t_2, \cdots, t_k \leq t \leq t_{k+1}, \cdots$.

Theorem 1.13.1. *Consider the systems (1.40) and (1.43), for a fixed $k \in Z^+$ and $\alpha \in [0,1]$,*
$\lim_{h_0,\cdots,h_k \to 0} \underline{y}_{k,N_k}(\alpha) = \underline{x}(t_{k+1},\alpha)$,
$\lim_{h_0,\cdots,h_k \to 0} \overline{y}_{k,N_k}(\alpha) = \overline{x}(t_{k+1},\alpha)$.

Proof: see [28].

1.13.2 The Runge-Kutta method for hybrid fuzzy differential equations

In this section, we describe how the Runge-Kutta method for a hybrid fuzzy differential equation (1.37) is developed via an application of the Runge-Kutta method for fuzzy differential equations when f and λ_k in (1.37) can be obtained via Zadeh is extension principle from $f \in C[R^+ \times R \times R, R]$ and $\lambda_k \in C[\mathbb{R}, \mathbb{R}]$.

For a fixed α, to integrate the system in (1.41) in $[t_0, t_1], [t_1, t_2], \cdots, [t_k, t_{k+1}], \cdots$, each interval is replaced by a $N_k + 1$ discrete equally spaced grid points (include the endpoints) at which the exact solution $(\underline{x}(t, \alpha), \overline{x}(t, \alpha))$ is approximated by some $(\underline{y}_k(t, \alpha), \overline{y}_k(t, \alpha))$. For the chosen grid points on $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k$, $h_k = \frac{t_{k+1} - t_k}{N_k}$, $0 \leq n \leq N_k$ let $(\underline{Y}_k(t, \alpha), \overline{Y}_k(t, \alpha)) \equiv (\underline{x}(t, \alpha), \overline{x}(t, \alpha))$. $(\underline{Y}_k(t, \alpha), \overline{Y}_k(t, \alpha))$ and $(\underline{y}_k(t, \alpha), \overline{y}_k(t, \alpha))$ may be denoted respectively by $(\underline{Y}_{k,n}(\alpha), \overline{Y}_{k,n}(\alpha))$ and $(\underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha))$. The runge-Kutta method is a fourth-order approximation of $\underline{Y}'_k(t, \alpha)$ and $\overline{Y}'_k(t, \alpha)$.

To develop the Runge-Kutta method for (1.37),

$$\underline{y}_{k,n+1}(\alpha) - \underline{y}_{k,n}(\alpha) = \sum_{i=1}^{4} w_i \underline{k}_i(t_{k,n}, y_{k,n}(\alpha)),$$

$$\overline{y}_{k,n+1}(\alpha) - \overline{y}_{k,n}(\alpha) = \sum_{i=1}^{4} w_i \overline{k}_i(t_{k,n}, y_{k,n}(\alpha)),$$

where w_1, w_2, w_3, and w_4 are constants and

$$\underline{k}_1(t_{k,n}, y_{k,n}(\alpha)) = \min\{h_k f(t_{k,n}, u, \lambda_k(u_k)) | u \in [\underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)], u_k \in [\underline{y}_{k,0}(\alpha), \overline{y}_{k,0}(\alpha)]\}$$

$$\overline{k}_1(t_{k,n}, y_{k,n}(\alpha)) = \max\{h_k f(t_{k,n}, u, \lambda_k(u_k)) | u \in [\underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)], u_k \in [\underline{y}_{k,0}(\alpha), \overline{y}_{k,0}(\alpha)]\}$$

$$\underline{k}_2(t_{k,n}, y_{k,n}(\alpha)) = \min\{h_k f(t_{k,n} + h_k/2, u, \lambda_k(u_k)) |$$
$$u \in [\underline{z}_{k_1}(t_{k,n}, y_{k,n}(\alpha)), \overline{z}_{k_1}(t_{k,n}, y_{k,n}(\alpha))], u_k \in [\underline{y}_{k,0}(\alpha), \overline{y}_{k,0}(\alpha)]\}$$

$$\overline{k}_2(t_{k,n}, y_{k,n}(\alpha)) = \max\{h_k f(t_{k,n} + h_k/2, u, \lambda_k(u_k)) |$$
$$u \in [\underline{z}_{k_1}(t_{k,n}, y_{k,n}(\alpha)), \overline{z}_{k_1}(t_{k,n}, y_{k,n}(\alpha))], u_k \in [\underline{y}_{k,0}(\alpha), \overline{y}_{k,0}(\alpha)]\}$$

$$\underline{k}_3(t_{k,n}, y_{k,n}(\alpha)) = \min\{h_k f(t_{k,n} + h_k/2, u, \lambda_k(u_k)) |$$
$$u \in [\underline{z}_{k_2}(t_{k,n}, y_{k,n}(\alpha)), \overline{z}_{k_2}(t_{k,n}, y_{k,n}(\alpha))], u_k \in [\underline{y}_{k,0}(\alpha), \overline{y}_{k,0}(\alpha)]\}$$

$$\overline{k}_3(t_{k,n}, y_{k,n}(\alpha)) = \max\{h_k f(t_{k,n} + h_k/2, u, \lambda_k(u_k)) |$$
$$u \in [\underline{z}_{k_2}(t_{k,n}, y_{k,n}(\alpha)), \overline{z}_{k_2}(t_{k,n}, y_{k,n}(\alpha))], u_k \in [\underline{y}_{k,0}(\alpha), \overline{y}_{k,0}(\alpha)]\}$$

$$\underline{k}_4(t_{k,n}, y_{k,n}(\alpha)) = \min\{h_k f(t_{k,n} + h_k, u, \lambda_k(u_k)) |$$
$$u \in [\underline{z}_{k_3}(t_{k,n}, y_{k,n}(\alpha)), \overline{z}_{k_3}(t_{k,n}, y_{k,n}(\alpha))], u_k \in [\underline{y}_{k,0}(\alpha), \overline{y}_{k,0}(\alpha)]\}$$

$$\overline{k}_4(t_{k,n}, y_{k,n}(\alpha)) = \max\{h_k f(t_{k,n} + h_k, u, \lambda_k(u_k)) |$$
$$u \in [\underline{z}_{k_3}(t_{k,n}, y_{k,n}(\alpha)), \overline{z}_{k_3}(t_{k,n}, y_{k,n}(\alpha))], u_k \in [\underline{y}_{k,0}(\alpha), \overline{y}_{k,0}(\alpha)]\}$$

$$\underline{z}_{k_1}(t_{k,n}, y_{k,n}(\alpha)) = \underline{y}_{k,n}(\alpha) + 0.5.\underline{k}_1(t_{k,n}, y_{k,n}(\alpha)),$$

$$\overline{z}_{k_1}(t_{k,n}, y_{k,n}(\alpha)) = \overline{y}_{k,n}(\alpha) + 0.5.\overline{k}_1(t_{k,n}, y_{k,n}(\alpha)),$$

$$\underline{z}_{k_2}(t_{k,n}, y_{k,n}(\alpha)) = \underline{y}_{k,n}(\alpha) + 0.5.\underline{k}_2(t_{k,n}, y_{k,n}(\alpha)),$$

$$\overline{z}_{k_2}(t_{k,n}, y_{k,n}(\alpha)) = \overline{y}_{k,n}(\alpha) + 0.5.\overline{k}_2(t_{k,n}, y_{k,n}(\alpha)),$$

$$\underline{z}_{k_3}(t_{k,n}, y_{k,n}(\alpha)) = \underline{y}_{k,n}(\alpha) + \underline{k}_3(t_{k,n}, y_{k,n}(\alpha)),$$

$$\overline{z}_{k_3}(t_{k,n}, y_{k,n}(\alpha)) = \overline{y}_{k,n}(\alpha) + \overline{k}_3(t_{k,n}, y_{k,n}(\alpha)).$$

Next

$$S_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)] = \underline{k}_1(t_{k,n}, y_{k,n}(\alpha)) + 2\underline{k}_2(t_{k,n}, y_{k,n}(\alpha)) + 2\underline{k}_3(t_{k,n}, y_{k,n}(\alpha)) + \underline{k}_4(t_{k,n}, y_{k,n}(\alpha)),$$

$T_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)] = \overline{k}_1(t_{k,n}, y_{k,n}(\alpha)) + 2\overline{k}_2(t_{k,n}, y_{k,n}(\alpha)) + 2\overline{k}_3(t_{k,n}, y_{k,n}(\alpha)) + \overline{k}_4(t_{k,n}, y_{k,n}(\alpha))$,

The exact solution at $t_{k,n+1}$ is given by

$$\begin{cases} \underline{Y}_{k,n+1}(\alpha) \approx \underline{Y}_{k,n}(\alpha) + \frac{1}{6}S_k[t_{k,n}, \underline{Y}_{k,n}(\alpha), \overline{Y}_{k,n}(\alpha)], \\ \overline{Y}_{k,n+1}(\alpha) \approx \overline{Y}_{k,n}(\alpha) + \frac{1}{6}T_k[t_{k,n}, \underline{Y}_{k,n}(\alpha), \overline{Y}_{k,n}(\alpha)]. \end{cases} \tag{1.44}$$

The approximate solution is given by

$$\begin{cases} \underline{y}_{k,n+1}(\alpha) = \underline{y}_{k,n}(\alpha) + \frac{1}{6}S_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)], \\ \overline{y}_{k,n+1}(\alpha) = \overline{y}_{k,n}(\alpha) + \frac{1}{6}T_k[t_{k,n}, \underline{y}_{k,n}(\alpha), \overline{y}_{k,n}(\alpha)]. \end{cases} \tag{1.45}$$

Theorem 1.13.2. *Consider the systems (1.40) and (1.45), for a fixed $k \in Z^+$ and $\alpha \in [0, 1]$,*

$\lim_{h_0,\cdots,h_k \to 0} \underline{y}_{k,N_k}(\alpha) = \underline{x}(t_{k+1}, \alpha)$,

$\lim_{h_0,\cdots,h_k \to 0} \overline{y}_{k,N_k}(\alpha) = \overline{x}(t_{k+1}, \alpha)$.

Proof: see [29].

Chapter 2

Predictor-Corrector method

2.1 Introduction

In this chapter, three numerical methods to solve "The Fuzzy initial value problem" are discussed. The three methods are the Adams-Bashforth, the Adams-Moulton and the Predictor-Corrector methods. The Predictor-Corrector method is obtained by combining the Adams-Bashforth and the Adams-Moulton methods. The convergence and stability of the proposed methods are also proved in detail. In addition, these methods are illustrated by solving two examples.

2.2 Adams-Bashforth Methods

Consider the first-order fuzzy differential equation $y' = f(t, y)$ where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and fuzzy variable y, and y' is the Seikkala fuzzy derivative of y. If an initial value $\widetilde{y}(t_0) = \widetilde{\alpha}_0$ is given, a fuzzy initial value problem will be obtained as follows:

$$\begin{cases} y'(t) & = f(t, y(t)), \quad t_0 \leq t \leq T, \\ \widetilde{y}(t_0) & = \widetilde{\alpha}_0. \end{cases} \tag{2.1}$$

2.2.1 Adams-Bashforth two-step methods

Now we are going to solve fuzzy initial value problem $y'(t) = f(t, y(t))$ by Adams-Bashforth two-step method. Let the fuzzy initial values be $\widetilde{y}(t_i), \widetilde{y}(t_{i-1})$, i.e

$$\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i))$$

which are triangular fuzzy numbers and are shown as

$$(f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))), (f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))).$$

Consider the following fuzzy equation

$$\widetilde{y}(t_{i+1}) = \widetilde{y}(t_i) + \int_{t_i}^{t_{i+1}} \widetilde{f}(t, y(t)) \, dt \tag{2.2}$$

By fuzzy interpolation for $\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i))$ we have

$$f^l(t, y(t)) = \sum_{j=i-1_{\ell_j(t) \geq 0}}^{i} \ell_j(t) f^l(t_j, y(t_j)) + \sum_{j=i-1_{\ell_j(t) < 0}}^{i} \ell_j(t) f^r(t_j, y(t_j))$$

$$f^c(t, y(t)) = \sum_{j=i-1}^{i} \ell_j(t) f^c(t_j, y(t_j))$$

$$f^r(t, y(t)) = \sum_{j=i-1_{\ell_j(t) \geq 0}}^{i} \ell_j(t) f^r(t_j, y(t_j)) + \sum_{j=i-1_{\ell_j(t) < 0}}^{i} \ell_j(t) f^l(t_j, y(t_j))$$

for $t_i \leq t \leq t_{i+1}$ we have :

$\ell_{i-1}(t) = \frac{(t-t_i)}{(t_{i-1}-t_i)} \leq 0$, $\ell_i(t) = \frac{(t-t_{i-1})}{(t_i-t_{i-1})} \geq 0$

therefore the following result will be obtain:

$$f^l(t, y(t)) = \ell_{i-1}(t) f^r(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^l(t_i, y(t_i)) \tag{2.3}$$

$$f^c(t, y(t)) = \ell_{i-1}(t) f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^c(t_i, y(t_i)) \tag{2.4}$$

$$f^r(t, y(t)) = \ell_{i-1}(t) f^l(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^r(t_i, y(t_i)) \tag{2.5}$$

By using (1.11) and (2.2) we have :

$$\widetilde{y}^\alpha(t_{i+1}) = [\underline{y}^\alpha(t_{i+1}), \overline{y}^\alpha(t_{i+1})]$$

with

$$\underline{y}^\alpha(t_{i+1}) = \underline{y}^\alpha(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha f^c(t, y(t)) + (1-\alpha) f^l(t, y(t)) dt\} \tag{2.6}$$

and

$$\overline{y}^\alpha(t_{i+1}) = \overline{y}^\alpha(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha f^c(t, y(t)) + (1-\alpha) f^r(t, y(t)) dt\} \tag{2.7}$$

If (2.3), (2.4) are set in (2.6), and (2.4), (2.5) in (2.7) then we get :

$$\underline{y}^\alpha(t_{i+1}) = \underline{y}^\alpha(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha(\ell_{i-1}(t) f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^c(t_i, y(t_i))) +$$

$$(1-\alpha)(\ell_{i-1}(t) f^r(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^l(t_i, y(t_i)))\} dt$$

$$\overline{y}^{\alpha}(t_{i+1}) = \overline{y}^{\alpha}(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha(\ell_{i-1}(t) f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^c(t_i, y(t_i)))+$$

$$(1-\alpha)(\ell_{i-1}(t) f^l(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^r(t_i, y(t_i)))\} dt$$

By integration we have :

$$\underline{y}^{\alpha}(t_{i+1}) = \underline{y}^{\alpha}(t_i) - (\frac{h}{2})(\alpha f^c(t_{i-1}, y(t_{i-1})) + (1-\alpha) f^r(t_{i-1}, y(t_{i-1}))) + (\frac{3h}{2})(\alpha f^c(t_i, y(t_i))$$

$$+(1-\alpha) f^l(t_i, y(t_i)))$$

$$\overline{y}^{\alpha}(t_{i+1}) = \overline{y}^{\alpha}(t_i) - (\frac{h}{2})(\alpha f^c(t_{i-1}, y(t_{i-1})) + (1-\alpha) f^l(t_{i-1}, y(t_{i-1}))) + (\frac{3h}{2})(\alpha f^c(t_i, y(t_i)))$$

$$+(1-\alpha) f^r(t_i, y(t_i)))$$

thus:

$$\underline{y}^{\alpha}(t_{i+1}) = \underline{y}^{\alpha}(t_i) - (\frac{h}{2}) \overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + (\frac{3h}{2}) \underline{f}^{\alpha}(t_i, y(t_i)) \tag{2.8}$$

$$\overline{y}^{\alpha}(t_{i+1}) = \overline{y}^{\alpha}(t_i) - (\frac{h}{2}) \underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + (\frac{3h}{2}) \overline{f}^{\alpha}(t_i, y(t_i)) \tag{2.9}$$

By (2.8) and (2.9) we have :

$$\begin{cases} \underline{y}^{\alpha}(t_{i+1}) = \underline{y}^{\alpha}(t_i) - (\frac{h}{2}) \overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + (\frac{3h}{2}) \underline{f}^{\alpha}(t_i, y(t_i)) \\ \overline{y}^{\alpha}(t_{i+1}) = \overline{y}^{\alpha}(t_i) - (\frac{h}{2}) \underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + (\frac{3h}{2}) \overline{f}^{\alpha}(t_i, y(t_i)) \\ \underline{y}^{\alpha}(t_{i-1}) = \alpha_0, \underline{y}^{\alpha}(t_i) = \alpha_1 \\ \overline{y}^{\alpha}(t_{i-1}) = \alpha_2, \overline{y}^{\alpha}(t_i) = \alpha_3 \end{cases} \tag{2.10}$$

2.2.2 Adams-Bashforth three-step methods

Now we are going to solve fuzzy initial value problem $y'(t) = f(t, y(t))$ by Adams-Bashforth three-step method. Let the fuzzy initial values be $\widetilde{y}(t_{i-1}), \widetilde{y}(t_i), \widetilde{y}(t_{i+1})$, i.e

$$\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i))\, \widetilde{f}(t_{i+1}, y(t_{i+1})),$$

which are triangular fuzzy numbers and are shown as

$$\{f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))\}, \{f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))\}$$

$$\{f^l(t_{i+1}, y(t_{i+1})), f^c(t_{i+1}, y(t_{i+1})), f^r(t_{i+1}, y(t_{i+1}))\}.$$

Consider the following fuzzy equation

$$\widetilde{y}(t_{i+2}) = \widetilde{y}(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \widetilde{f}(t, y(t))\, dt. \tag{2.11}$$

By fuzzy interpolation for $\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i))$, and $\widetilde{f}(t_{i+1}, y(t_{i+1}))$ we have:

$f^l(t, y(t)) = \sum_{j=i-1}^{i+1}{}_{\ell_j(t)\geq 0} \ell_j(t) f^l(t_j, y(t_j)) + \sum_{j=i-1}^{i+1}{}_{\ell_j(t)<0} \ell_j(t) f^r(t_j, y(t_j))$,

$f^c(t, y(t)) = \sum_{j=i-1}^{i+1} \ell_j(t) f^c(t_j, y(t_j))$,

$f^r(t, y(t)) = \sum_{j=i-1}^{i+1}{}_{\ell_j(t)\geq 0} \ell_j(t) f^r(t_j, y(t_j)) + \sum_{j=i-1}^{i+1}{}_{\ell_j(t)<0} \ell_j(t) f^l(t_j, y(t_j))$,

for $t_{i+1} \leq t \leq t_{i+2}$:

$\ell_{i-1}(t) = \frac{(t-t_i)(t-t_{i+1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})} \geq 0$,

$\ell_i(t) = \frac{(t-t_{i-1})(t-t_{i+1})}{(t_i-t_{i-1})(t_i-t_{i+1})} \leq 0,$

$\ell_{i+1}(t) = \frac{(t-t_{i-1})(t-t_i)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} \geq 0,$

therefore the following results will be obtained:

$$f^l(t, y(t)) = \ell_{i-1}(t) f^l(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^r(t_i, y(t_i)) + \ell_{i+1}(t) f^l(t_{i+1}, y(t_{i+1})), \quad (2.12)$$

$$f^c(t, y(t)) = \ell_{i-1}(t) f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^c(t_i, y(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, y(t_{i+1})), \quad (2.13)$$

$$f^r(t, y(t)) = \ell_{i-1}(t) f^r(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^l(t_i, y(t_i)) + \ell_{i+1}(t) f^r(t_{i+1}, y(t_{i+1})). \quad (2.14)$$

From (1.11) and (2.11) it follows that:

$$\widetilde{y}^\alpha(t_{i+2}) = [\underline{y}^\alpha(t_{i+2}), \overline{y}^\alpha(t_{i+2})],$$

where

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t, y(t)) + (1-\alpha) f^l(t, y(t))\} dt, \quad (2.15)$$

and

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t, y(t)) + (1-\alpha) f^r(t, y(t))\} dt. \quad (2.16)$$

If (2.12), (2.13) are set in (2.15) and (2.13), (2.14) in (2.16), then we have:

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha(\ell_{i-1}(t)f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t)f^c(t_i, y(t_i)) + \ell_{i+1}(t)f^c(t_{i+1}, y(t_{i+1}))),$$

$$+(1-\alpha)(\ell_{i-1}(t)f^l(t_{i-1}, y(t_{i-1})) + \ell_i(t)f^r(t_i, y(t_i)) + \ell_{i+1}(t)f^l(t_{i+1}, y(t_{i+1})))\}dt,$$

and

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha(\ell_{i-1}(t)f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t)f^c(t_i, y(t_i)) + \ell_{i+1}(t)f^c(t_{i+1}, y(t_{i+1})))$$

$$+(1-\alpha)(\ell_{i-1}(t)f^r(t_{i-1}, y(t_{i-1})) + \ell_i(t)f^l(t_i, y(t_i)) + \ell_{i+1}(t)f^r(t_{i+1}, y(t_{i+1})))\}dt.$$

The following results will be obtained by integration:

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i+1}) + \frac{5h}{12}[\alpha f^c(t_{i-1}, y(t_{i-1})) + (1-\alpha)f^l(t_{i-1}, y(t_{i-1}))] - \frac{16h}{12}[\alpha f^c(t_i, y(t_i))$$

$$+(1-\alpha)f^r(t_i, y(t_i))] + \frac{23h}{12}[\alpha f^c(t_{i+1}, y(t_{i+1})) + (1-\alpha)f^l(t_{i+1}, y(t_{i+1}))],$$

and

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i+1}) + \frac{5h}{12}[\alpha f^c(t_{i-1}, y(t_{i-1})) + (1-\alpha)f^r(t_{i-1}, y(t_{i-1}))] - \frac{16h}{12}[\alpha f^c(t_i, y(t_i))$$

$$+(1-\alpha)f^l(t_i, y(t_i))] + \frac{23h}{12}[\alpha f^c(t_{i+1}, y(t_{i+1})) + (1-\alpha)f^r(t_{i+1}, y(t_{i+1}))].$$

Thus

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i+1}) + \frac{h}{12}[5\underline{f}^\alpha(t_{i-1}, y(t_{i-1}) - 16\overline{f}^\alpha(t_i, y(t_i)) + 23\underline{f}^\alpha(t_{i+1}, y(t_{i+1}))], \quad (2.17)$$

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i+1}) + \frac{h}{12}[5\overline{f}^\alpha(t_{i-1}, y(t_{i-1}) - 16\underline{f}^\alpha(t_i, y(t_i)) + 23\overline{f}^\alpha(t_{i+1}, y(t_{i+1}))]. \quad (2.18)$$

Therefore Adams-Bashforth three-step method is obtained as follows:

$$\begin{cases} \underline{y}^{\alpha}(t_{i+2}) = \underline{y}^{\alpha}(t_{i+1}) + \frac{h}{12}[5\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1}) - 16\overline{f}^{\alpha}(t_i, y(t_i)) + 23\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))], \\ \overline{y}^{\alpha}(t_{i+2}) = \overline{y}^{\alpha}(t_{i+1}) + \frac{h}{12}[5\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1}) - 16\underline{f}^{\alpha}(t_i, y(t_i)) + 23\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))], \\ \underline{y}^{\alpha}(t_{i-1}) = \alpha_0, \underline{y}^{\alpha}(t_i) = \alpha_1, \underline{y}^{\alpha}(t_{i+1}) = \alpha_2, \overline{y}^{\alpha}(t_{i-1}) = \alpha_3, \overline{y}^{\alpha}(t_i) = \alpha_4, \overline{y}^{\alpha}(t_{i+1}) = \alpha_5. \end{cases}$$

2.3 Adams-Moulton Methods

2.3.1 Adams-Moulton two-step method

Now we would like to solve fuzzy initial value problem $y'(t) = f(t, y(t))$ by Adams-Moulton two-step method. Let the fuzzy initial values be $\widetilde{y}(t_{i-1}), \widetilde{y}(t_i)$ i.e

$$\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i))$$

which are triangular fuzzy numbers and are shown as

$$\{f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))\}, \{f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))\},$$

Consider the following fuzzy equation

$$\widetilde{y}(t_{i+1}) = \widetilde{y}(t_i) + \int_{t_i}^{t_{i+1}} \widetilde{f}(t, y(t))\ dt, \tag{2.19}$$

By fuzzy interpolation for $\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i))$ and

$\widetilde{f}(t_{i+1}, y(t_{i+1}))$ we have:

$f^l(t, y(t)) = \sum_{j=i-1}^{i+1}{}_{\ell_j(t)\geq 0} \ell_j(t) f^l(t_j, y(t_j)) + \sum_{j=i-1}^{i+1}{}_{\ell_j(t)<0} \ell_j(t) f^r(t_j, y(t_j)),$

$f^c(t,y(t)) = \sum_{j=i-1}^{i+1} \ell_j(t) f^c(t_j, y(t_j)),$

$f^r(t,y(t)) = \sum_{j=i-1}^{i+1}{}_{\ell_j(t)\geq 0} \ell_j(t) f^r(t_j, y(t_j)) + \sum_{j=i-1}^{i+1}{}_{\ell_j(t)<0} \ell_j(t) f^l(t_j, y(t_j)).$

for $t_i \leq t \leq t_{i+1}$:

$\ell_{i-1}(t) = \frac{(t-t_i)(t-t_{i+1})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})} \leq 0,$

$\ell_i(t) = \frac{(t-t_{i-1})(t-t_{i+1})}{(t_i-t_{i-1})(t_i-t_{i+1})} \geq 0,$

$\ell_{i+1}(t) = \frac{(t-t_{i-1})(t-t_i)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} \geq 0,$

therefore the following results will be obtained:

$$f^l(t,y(t)) = \ell_{i-1}(t) f^r(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^l(t_i, y(t_i)) + \ell_{i+1}(t) f^l(t_{i+1}, y(t_{i+1})), \quad (2.20)$$

$$f^c(t,y(t)) = \ell_{i-1}(t) f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^c(t_i, y(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, y(t_{i+1})), \quad (2.21)$$

$$f^r(t,y(t)) = \ell_{i-1}(t) f^l(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^r(t_i, y(t_i)) + \ell_{i+1}(t) f^r(t_{i+1}, y(t_{i+1})). \quad (2.22)$$

From (1.11) and (2.19) it follows that:

$$\widetilde{y}^\alpha(t_{i+1}) = [\underline{y}^\alpha(t_{i+1}), \overline{y}^\alpha(t_{i+1})],$$

where

$$\underline{y}^{\alpha}(t_{i+1}) = \underline{y}^{\alpha}(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha f^c(t, y(t)) + (1-\alpha) f^l(t, y(t))\} dt, \tag{2.23}$$

and

$$\overline{y}^{\alpha}(t_{i+1}) = \overline{y}^{\alpha}(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha f^c(t, y(t)) + (1-\alpha) f^r(t, y(t))\} dt, \tag{2.24}$$

If (2.20), (2.21) are set in (2.23) and (2.21), (2.22) in (2.24), then we get:

$$\underline{y}^{\alpha}(t_{i+1}) = \underline{y}^{\alpha}(t_i) + \int_{t_i}^{t_{i+1}} \{\alpha(\ell_{i-1}(t) f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^c(t_i, y(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, y(t_{i+1})) +$$

$$+ (1-\alpha)(\ell_{i-1}(t) f^r(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^l(t_i, y(t_i)) + \ell_{i+1}(t) f^l(t_{i+1}, y(t_{i+1})))\} dt,$$

and

$$\overline{y}^{\alpha}(t_{i+2}) = \overline{y}^{\alpha}(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha(\ell_{i-1}(t) f^c(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^c(t_i, y(t_i)) + \ell_{i+1}(t) f^c(t_{i+1}, y(t_{i+1})) +$$

$$+ (1-\alpha)(\ell_{i-1}(t) f^l(t_{i-1}, y(t_{i-1})) + \ell_i(t) f^r(t_i, y(t_i)) + \ell_{i+1}(t) f^r(t_{i+1}, y(t_{i+1})))\} dt.$$

Therefore by integration Adams-Moulton two-step method is obtained as follows:

$$\begin{cases} \underline{y}^{\alpha}(t_{i+1}) = \underline{y}^{\alpha}(t_i) - (\frac{h}{12})\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + (\frac{2h}{3})\underline{f}^{\alpha}(t_i, y(t_i)) + (\frac{5h}{12})\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1})), \\ \overline{y}^{\alpha}(t_{i+1}) = \overline{y}^{\alpha}(t_i) - (\frac{h}{12})\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + (\frac{2h}{3})\overline{f}^{\alpha}(t_i, y(t_i)) + (\frac{5h}{12})\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1})), \\ \underline{y}^{\alpha}(t_{i-1}) = \alpha_0, \underline{y}^{\alpha}(t_i) = \alpha_1 \ \overline{y}^{\alpha}(t_{i-1}) = \alpha_2, \overline{y}^{\alpha}(t_i) = \alpha_3. \end{cases} \tag{2.25}$$

2.3.2 Adams-Moulton three-step method

Now we are going to solve fuzzy initial value problem $y'(t) = f(t, y(t))$ by Adams-Moulton three-step method. Let the fuzzy initial values be $\widetilde{y}(t_{i-1}), \widetilde{y}(t_i), \widetilde{y}(t_{i+1})$, i.e

$$\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i)), \widetilde{f}(t_{i+1}, y(t_{i+1})),$$

which are triangular fuzzy numbers and are shown by

$$\{f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))\}, \{f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))\},$$

$$\{f^l(t_{i+1}, y(t_{i+1})), f^c(t_{i+1}, y(t_{i+1})), f^r(t_{i+1}, y(t_{i+1}))\}.$$

Also

$$\widetilde{y}(t_{i+2}) = \widetilde{y}(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \widetilde{f}(t, y(t))\ dt, \tag{2.26}$$

By fuzzy interpolation for $\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i)), \widetilde{f}(t_{i+1}, y(t_{i+1}))$ and $\widetilde{f}(t_{i+2}, y(t_{i+2}))$ we have

$$f^l(t, y(t)) = \sum_{j=i-1}^{i+2}{}_{\ell_j(t)\geq 0} \ell_j(t) f^l(t_j, y(t_j)) + \sum_{j=i-1}^{i+2}{}_{\ell_j(t)<0} \ell_j(t) f^r(t_j, y(t_j)),$$

$$f^c(t, y(t)) = \sum_{j=i-1}^{i+2} \ell_j(t) f^c(t_j, y(t_j)),$$

$$f^r(t, y(t)) = \sum_{j=i-1}^{i+2}{}_{\ell_j(t)\geq 0} \ell_j(t) f^r(t_j, y(t_j)) + \sum_{j=i-1}^{i+2}{}_{\ell_j(t)<0} \ell_j(t) f^l(t_j, y(t_j)).$$

for $t_{i+1} \leq t \leq t_{i+2}$:

$$\ell_{i-1}(t) = \frac{(t-t_i)(t-t_{i+1})(t-t_{i+2})}{(t_{i-1}-t_i)(t_{i-1}-t_{i+1})(t_{i-1}-t_{i+2})} \geq 0,$$

$$\ell_i(t) = \frac{(t-t_{i-1})(t-t_{i+1})(t-t_{i+2})}{(t_i-t_{i-1})(t_i-t_{i+1})(t_i-t_{i+2})} \leq 0,$$

$\ell_{i+1}(t) = \frac{(t-t_{i-1})(t-t_i)(t-t_{i+2})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_{i+1}-t_{i+2})} \geq 0,$

$\ell_{i+2}(t) = \frac{(t-t_{i-1})(t-t_i)(t-t_{i+1})}{(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} \geq 0,$

therefore the following results will be obtained:

$$f^l(t,y(t)) = \ell_{i-1}(t)f^l(t_{i-1},y(t_{i-1})) + \ell_i(t)f^r(t_i,y(t_i)) + \ell_{i+1}(t)f^l(t_{i+1},y(t_{i+1})) + \ell_{i+2}(t)f^l(t_{i+2},y(t_{i+2})), \quad (2.27)$$

$$f^c(t,y(t)) = \ell_{i-1}(t)f^c(t_{i-1},y(t_{i-1})) + \ell_i(t)f^c(t_i,y(t_i)) + \ell_{i+1}(t)f^c(t_{i+1},y(t_{i+1})) + \ell_{i+2}(t)f^c(t_{i+2},y(t_{i+2})), \quad (2.28)$$

$$f^r(t,y(t)) = \ell_{i-1}(t)f^r(t_{i-1},y(t_{i-1})) + \ell_i(t)f^l(t_i,y(t_i)) + \ell_{i+1}(t)f^r(t_{i+1},y(t_{i+1})) + \ell_{i+2}(t)f^r(t_{i+2},y(t_{i+2})). \quad (2.29)$$

From (1.11) and (2.26) it follows that:

$$\widetilde{y}^{\alpha}(t_{i+2}) = [\underline{y}^{\alpha}(t_{i+2}), \overline{y}^{\alpha}(t_{i+2})],$$

where

$$\underline{y}^{\alpha}(t_{i+2}) = \underline{y}^{\alpha}(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t,y(t)) + (1-\alpha)f^l(t,y(t))\}dt, \quad (2.30)$$

and

$$\overline{y}^{\alpha}(t_{i+2}) = \overline{y}^{\alpha}(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha f^c(t,y(t)) + (1-\alpha)f^r(t,y(t))\}dt, \quad (2.31)$$

If (2.27), (2.28) are situated in (2.30) and (2.28), (2.29) in (2.31):

$$\underline{y}^{\alpha}(t_{i+2}) = \underline{y}^{\alpha}(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha(\ell_{i-1}(t)f^{c}(t_{i-1}, y(t_{i-1})) + \ell_{i}(t)f^{c}(t_{i}, y(t_{i})) + \ell_{i+1}(t)f^{c}(t_{i+1}, y(t_{i+1}))$$

$$+\ell_{i+2}(t)f^{c}(t_{i+2}, y(t_{i+2}))) + (1-\alpha)(\ell_{i-1}(t)f^{l}(t_{i-1}, y(t_{i-1})) + \ell_{i}(t)f^{r}(t_{i}, y(t_{i})) + \ell_{i+1}(t)f^{l}(t_{i+1}, y(t_{i+1})))$$

$$+ \ell_{i+2}(t)f^{l}(t_{i+2}, y(t_{i+2})))\}dt,$$

and

$$\overline{y}^{\alpha}(t_{i+2}) = \overline{y}^{\alpha}(t_{i+1}) + \int_{t_{i+1}}^{t_{i+2}} \{\alpha(\ell_{i-1}(t)f^{c}(t_{i-1}, y(t_{i-1})) + \ell_{i}(t)f^{c}(t_{i}, y(t_{i})) + \ell_{i+1}(t)f^{c}(t_{i+1}, y(t_{i+1}))$$

$$\ell_{i+2}(t)f^{c}(t_{i+2}, y(t_{i+2}))) + (1-\alpha)(\ell_{i-1}(t)f^{r}(t_{i-1}, y(t_{i-1})) + \ell_{i}(t)f^{l}(t_{i}, y(t_{i})) + \ell_{i+1}(t)f^{r}(t_{i+1}, y(t_{i+1})))$$

$$+ \ell_{i+2}(t)f^{r}(t_{i+2}, y(t_{i+2})))\}dt.$$

The following results will be obtained by integration:

$$\underline{y}^{\alpha}(t_{i+2}) = \underline{y}^{\alpha}(t_{i+1}) + \frac{9h}{24}[\alpha f^{c}(t_{i+2}, y(t_{i+2})) + (1-\alpha)f^{l}(t_{i+2}, y(t_{i+2}))] + \frac{19h}{24}[\alpha f^{c}(t_{i+1}, y(t_{i+1}))$$

$$+(1-\alpha)f^{l}(t_{i+1}, y(t_{i+1}))] - \tfrac{5h}{24}[\alpha f^{c}(t_{i-1}, y(t_{i-1})) + (1-\alpha)f^{r}(t_{i}, y(t_{i}))]$$

$$+\frac{h}{24}[\alpha f^{c}(t_{i+1}, y(t_{i+1})) + (1-\alpha)f^{l}(t_{i-1}, y(t_{i-1}))],$$

and

$$\overline{y}^{\alpha}(t_{i+2}) = \overline{y}^{\alpha}(t_{i+1}) + \frac{9h}{24}[\alpha f^{c}(t_{i+2}, y(t_{i+2})) + (1-\alpha)f^{r}(t_{i+2}, y(t_{i+2}))] + \frac{19h}{24}[\alpha f^{c}(t_{i+1}, y(t_{i+1})),$$

$$+(1-\alpha)f^{r}(t_{i+1}, y(t_{i+1}))] - \tfrac{5h}{24}[\alpha f^{c}(t_{i-1}, y(t_{i-1})) + (1-\alpha)f^{l}(t_{i}, y(t_{i}))]$$

$$+\frac{h}{24}[\alpha f^{c}(t_{i+1}, y(t_{i+1})) + (1-\alpha)f^{r}(t_{i-1}, y(t_{i-1}))],$$

thus

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i+1}) + \frac{h}{24}[\underline{f}^\alpha(t_{i-1}, y(t_{i-1})) - 5\overline{f}^\alpha(t_i, y(t_i)) + 19\underline{f}^\alpha(t_{i+1}, y(t_{i+1})) + 9\underline{f}^\alpha(t_{i+2}, y(t_{i+2}))], \tag{2.32}$$

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i+1}) + \frac{h}{24}[\overline{f}^\alpha(t_{i-1}, y(t_{i-1})) - 5\underline{f}^\alpha(t_i, y(t_i)) + 19\overline{f}^\alpha(t_{i+1}, y(t_{i+1})) + 9\overline{f}^\alpha(t_{i+2}, y(t_{i+2}))]. \tag{2.33}$$

Therefore Adams-Moulton three-step method is obtained as follows:

$$\begin{cases} \underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i+1}) + \frac{h}{24}[\underline{f}^\alpha(t_{i-1}, y(t_{i-1})) - 5\overline{f}^\alpha(t_i, y(t_i)) + 19\underline{f}^\alpha(t_{i+1}, y(t_{i+1})) + \\ \qquad\qquad 9\underline{f}^\alpha(t_{i+2}, y(t_{i+2}))], \\ \overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i+1}) + \frac{h}{24}[\overline{f}^\alpha(t_{i-1}, y(t_{i-1})) - 5\underline{f}^\alpha(t_i, y(t_i)) + 19\overline{f}^\alpha(t_{i+1}, y(t_{i+1})) + \\ \qquad\qquad 9\overline{f}^\alpha(t_{i+2}, y(t_{i+2}))], \\ \underline{y}^\alpha(t_{i-1}) = \alpha_0, \underline{y}^\alpha(t_i) = \alpha_1, \underline{y}^\alpha(t_{i+1}) = \alpha_2\ \overline{y}^\alpha(t_{i-1}) = \alpha_3, \overline{y}^\alpha(t_i) = \alpha_4, \overline{y}^\alpha(t_{i+1}) = \alpha_5. \end{cases}$$

2.4 Predictor-Corrector Three-step method

The following algorithm is based on Adams-Bashforth three-step method as a predictor and also an iteration of Adams-Moulton two-step method as a corrector.

ALGORITHM (Predictor-Corrector three-step method)

To approximate the solution of following fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t_0 \le t \le T, \\ \underline{y}^\alpha(t_0) = \alpha_0, \underline{y}^\alpha(t_1) = \alpha_1, \underline{y}^\alpha(t_2) = \alpha_2, \\ \overline{y}^\alpha(t_0) = \alpha_3, \overline{y}^\alpha(t_1) = \alpha_4, \overline{y}^\alpha(t_2) = \alpha_5, \end{cases}$$

positive integer N is chosen.

step 1. Let $h = \frac{T-t_0}{N}$,

$$\underline{w}^{\alpha}(t_0) = \alpha_0, \underline{w}^{\alpha}(t_1) = \alpha_1, \underline{w}^{\alpha}(t_2) = \alpha_2,$$

$$\overline{w}^{\alpha}(t_0) = \alpha_3, \overline{w}^{\alpha}(t_1) = \alpha_4, \overline{w}^{\alpha}(t_2) = \alpha_5.$$

step 2. Let $i = 1$.

step 3. Let

$$\begin{cases} \underline{w}^{(0)\alpha}(t_{i+2}) = \underline{w}^{\alpha}(t_{i+1}) + \frac{h}{12}[5\underline{f}^{\alpha}(t_{i-1}, w(t_{i-1})) - 16\overline{f}^{\alpha}(t_i, w(t_i)) + 23\underline{f}^{\alpha}(t_{i+1}, w(t_{i+1}))], \\ \overline{w}^{(0)\alpha}(t_{i+2}) = \overline{w}^{\alpha}(t_{i+1}) + \frac{h}{12}[5\overline{f}^{\alpha}(t_{i-1}, w(t_{i-1})) - 16\underline{f}^{\alpha}(t_i, w(t_i)) + 23\overline{f}^{\alpha}(t_{i+1}, w(t_{i+1}))], \end{cases}$$

step 4. Let $t_{i+2} = t_0 + (i+2)h$,

step 5. Let

$$\begin{cases} \underline{w}^{\alpha}(t_{i+2}) = \underline{w}^{\alpha}(t_{i+1}) - (\frac{h}{12})\overline{f}^{\alpha}(t_i, w(t_i)) + (\frac{2h}{3})\underline{f}^{\alpha}(t_{i+1}, w(t_{i+1})) + (\frac{5h}{12})\underline{f}^{\alpha}(t_{i+2}, w^{(0)}(t_{i+2})), \\ \overline{w}^{\alpha}(t_{i+2}) = \overline{w}^{\alpha}(t_{i+1}) - (\frac{h}{12})\underline{f}^{\alpha}(t_i, w(t_i)) + (\frac{2h}{3})\overline{f}^{\alpha}(t_{i+1}, w(t_{i+1})) + (\frac{5h}{12})\overline{f}^{\alpha}(t_{i+2}, w^{(0)}(t_{i+2})), \end{cases}$$

step 6. $i = i + 1$.

step 7. if $i \leq N - 2$ go to step 3.

step 8. algorithm will be completed and $(\underline{w}^{\alpha}(T), \overline{w}^{\alpha}(T))$ approximates real value of $(\underline{Y}^{\alpha}(T), \overline{Y}^{\alpha}(T))$.

2.5 Convergence and Stability

To integrate the system given in Eq. (2.1) from t_0 a prefixed $T > t_0$, the interval $[t_0, T]$ will be replaced by a set of discrete equally spaced grid points $t_0 < t_1 < t_2 < \ldots < t_N = T$ which the exact solution $(\underline{Y}(t,\alpha), \overline{Y}(t,\alpha))$ is approximated by some $(\underline{y}(t,\alpha), \overline{y}(t,\alpha))$. The exact and approximate solutions at t_n, $0 \leq n \leq N$ are denoted by $Y_n(\alpha) = [\underline{Y}_n(\alpha), \overline{Y}_n(\alpha)]$, and $y_n(\alpha) = [\underline{y}_n(\alpha), \overline{y}_n(\alpha)]$, respectively. The grid points which the solution is calculated are $t_n = t_0 + nh, \quad h = (T - t_0)/N, \ 1 \leq n \leq N$.

From Eq. (2.1), the polygon curves

$$\underline{y}(t, h, \alpha) = \{[t_0, \underline{y}_0(\alpha)], [t_1, \underline{y}_1(\alpha)], \ldots, [t_N, \underline{y}_N(\alpha)]\},$$

$$\overline{y}(t, h, \alpha) = \{[t_0, \overline{y}_0(\alpha)], [t_1, \overline{y}_1(\alpha)], \ldots, [t_N, \overline{y}_N(\alpha)]\},$$

are the approximates to $\underline{Y}(t, \alpha)$ and $\overline{Y}(t, \alpha)$, over the interval $t_0 \leq t \leq t_N$.

The following lemmas will be applied to show convergence of these approximates, i.e.,

$$lim_{h \to 0} \underline{y}(t, h, \alpha) = \underline{Y}(t, \alpha), \quad lim_{h \to 0} \overline{y}(t, h, \alpha) = \overline{Y}(t, \alpha).$$

Lemma 2.5.1. *Let a sequence of numbers* $\{w_n\}_{n=0}^{N}$ *satisfy:*

$$|w_{n+1}| \leq A|w_n| + B|w_{n-1}| + C, \qquad 0 \leq n \leq N - 1$$

for some given positive constants A *and* B *,*C*. Then*

$|w_n| \leq (A^{n-1} + \beta_1 A^{n-3} B + \beta_2 A^{n-5} B^2 + \ldots + \beta_s B^{[\frac{n}{2}]})|w_1| + (A^{n-2} B + \gamma_1 A^{n-4} B^2 + \ldots + \gamma_t A B^{[\frac{n}{2}]})|w_0| + (A^{n-2} + A^{n-3} + \ldots + 1)C + (\delta_1 A^{n-4} + \delta_2 A^{n-5} + \ldots + \delta_m A + 1)BC + (\zeta_1 A^{n-6} + \zeta_2 A^{n-7} + \ldots + \zeta_l A + 1)B^2 C + (\lambda_1 A^{n-8} + \lambda_2 A^{n-9} + \ldots + \lambda_p A + 1)B^3 C + \ldots, \qquad n \ \ odd$

and

$|w_n| \leq (A^{n-1} + \beta_1 A^{n-3} B + \beta_2 A^{n-5} B^2 + \ldots + \beta_s A B^{[\frac{n}{2}]-1})|w_1| + (A^{n-2} B + \gamma_1 A^{n-4} B^2 + \ldots + \gamma_t B^{[\frac{n}{2}]})|w_0| + (A^{n-2} + A^{n-3} + \ldots + 1)C + (\delta_1 A^{n-4} + \delta_2 A^{n-5} + \ldots + \delta_m A + 1)BC + (\zeta_1 A^{n-6} + \zeta_2 A^{n-7} + \ldots + \zeta_l A + 1)B^2 C + (\lambda_1 A^{n-8} + \lambda_2 A^{n-9} + \ldots + \lambda_p A + 1)B^3 C + \ldots, \qquad n \ \ even$

where $\beta_s, \gamma_t, \delta_m, \zeta_l, \lambda_p,$ *are constants for all* s, t, m, l *and* p.

The proof, by using mathematical induction is straightforward.

Theorem 2.5.2. *For arbitrary fixed* $\alpha : \ 0 \le \alpha \le 1$, *the Adams-Bashforth two-step approximates of Eq. (2.10) converge to the exact solutions* $\underline{Y}(t,\alpha), \overline{Y}(t,\alpha)$ *for* $\underline{Y}, \overline{Y} \in C^4[t_0, T]$.

Proof. As in ordinary differential equation, it is sufficient to show :

$$lim_{h\to 0}\underline{y}_N(\alpha) = \underline{Y}(T,\alpha), \quad lim_{h\to 0}\overline{y}_N(\alpha) = \overline{Y}(T,\alpha)$$

By using exact value we get :

$$\underline{Y}_{n+1}(\alpha) = \underline{Y}_n(\alpha) - \frac{h}{2} f(t_{n-1}, \overline{Y}_{n-1}(\alpha)) + \frac{3h}{2} f(t_n, \underline{Y}_n(\alpha)) + \frac{5}{12} h^3 \underline{Y}'''(\xi_n)$$

$$\overline{Y}_{n+1}(\alpha) = \overline{Y}_n(\alpha) - \frac{h}{2} f(t_{n-1}, \underline{Y}_{n-1}(\alpha)) + \frac{3h}{2} f(t_n, \overline{Y}_n(\alpha)) + \frac{5}{12} h^3 \overline{Y}'''(\xi_n)$$

where $t_n < \underline{\xi}_n, \overline{\xi}_n < t_{n+1}$. Consequently

$$\underline{Y}_{n+1}(\alpha) - \underline{y}_{n+1}(\alpha) = \underline{Y}_n(\alpha) - \underline{y}_n(\alpha) - \tfrac{h}{2}\{f(t_{n-1}, \overline{Y}_{n-1}(\alpha)) - f(t_{n-1}, \overline{y}_{n-1}(\alpha))\} +$$

$$\frac{3h}{2}\{f(t_n, \underline{Y}_n(\alpha)) - f(t_n, \underline{y}_n(\alpha))\} + \frac{5}{12} h^3 \underline{Y}'''(\xi_n)$$

$$\overline{Y}_{n+1}(\alpha) - \overline{y}_{n+1}(\alpha) = \overline{Y}_n(\alpha) - \overline{y}_n(\alpha) - \tfrac{h}{2}\{f(t_{n-1}, \underline{Y}_{n-1}(\alpha)) - f(t_{n-1}, \underline{y}_{n-1}(\alpha))\} +$$

$$\frac{3h}{2}\{f(t_n, \overline{Y}_n(\alpha)) - f(t_n, \overline{y}_n(\alpha))\} + \frac{5}{12} h^3 \overline{Y}'''(\xi_n)$$

Denote $w_n = \underline{Y}_n(\alpha) - \underline{y}_n(\alpha)$, $v_n = \overline{Y}_n(\alpha) - \overline{y}_n(\alpha)$. Then

$$|w_{n+1}| \leq (1 + \frac{3hL_1}{2})|w_n| + \frac{hL_2}{2}|v_{n-1}| + \frac{5}{12}h^3\underline{M}$$
$$|v_{n+1}| \leq (1 + \frac{3hL_3}{2})|v_n| + \frac{hL_4}{2}|w_{n-1}| + \frac{5}{12}h^3\overline{M}$$

where $\underline{M} = max_{t_0 \leq t \leq T}\underline{Y}'''(t,\alpha)$ and $\overline{M} = max_{t_0 \leq t \leq T}\overline{Y}'''(t,\alpha)$ and we put $L = max\{L_1, L_2, L_3, L_4\}$ then we have :

$$|w_{n+1}| \leq (1 + \frac{3hL}{2})|w_n| + \frac{hL}{2}|v_{n-1}| + \frac{5}{12}h^3\underline{M}$$
$$|v_{n+1}| \leq (1 + \frac{3hL}{2})|v_n| + \frac{hL}{2}|w_{n-1}| + \frac{5}{12}h^3\overline{M}$$

thus

$$|w_{n+1}|, |v_{n+1}| \leq |u_{n+1}| \leq (1 + \frac{3hL}{2})|u_n| + \frac{hL}{2}|u_{n-1}| + \frac{5}{12}h^3[\underline{M} + \overline{M}]$$

where $|u_n| = |w_n| + |v_n|$. By lemma (2.5.1) and $w_0 = v_0 = 0$ we obtain :

$|w_n|, |v_n| \leq \frac{(1+\frac{3hL}{2})^{n-1}-1}{\frac{3hL}{2}} \times \frac{5}{12}h^3[\underline{M} + \overline{M}] + \{\delta_1(1 + \frac{3hL}{2})^{n-4} + \delta_2(1 + \frac{3hL}{2})^{n-5} + \ldots + \delta_m(1 + \frac{3hL}{2}) + 1\}(\frac{hL}{2})^2\frac{5}{12}h^3[\underline{M} + \overline{M}] + \{\zeta_1(1 + \frac{3hL}{2})^{n-6} + \zeta_2(1 + \frac{3hL}{2})^{n-7} + \ldots + \zeta_l(1 + \frac{3hL}{2}) + 1\}(\frac{hL}{2}^3)\frac{5}{12}h^3[\underline{M} + \overline{M}] + \{\lambda_1(1 + \frac{3hL}{2})^{n-8} + \lambda_2(1 + \frac{3hL}{2})^{n-9} + \ldots + \lambda_p(1 + \frac{3hL}{2}) + 1\}(\frac{hL}{2})^4\frac{5}{12}h^3[\underline{M} + \overline{M}] + \ldots$

therefore

$|w_n|, |v_n| \leq \frac{5e^{\frac{3L(T-t_1)}{2}}}{18L} \times h^2[\underline{M} + \overline{M}] + \{\delta_1(1 + \frac{3hL}{2})^{n-4} + \delta_2(1 + \frac{3hL}{2})^{n-5} + \ldots + \delta_m(1 + \frac{3hL}{2}) + 1\}(\frac{hL}{2})^2\frac{5}{12}h^3[\underline{M} + \overline{M}] + \{\zeta_1(1 + \frac{3hL}{2})^{n-6} + \zeta_2(1 + \frac{3hL}{2})^{n-7} + \ldots + \zeta_l(1 + \frac{3hL}{2}) +$

$1\}(\frac{hL}{2}^3)\frac{5}{12}h^3[\underline{M}+\overline{M}] + \{\lambda_1(1+\frac{3hL}{2})^{n-8}+\lambda_2(1+\frac{3hL}{2})^{n-9}+\ldots+\lambda_p(1+\frac{3hL}{2})+$
$1\}(\frac{hL}{2})^4\frac{5}{12}h^3[\underline{M}+\overline{M}]+\ldots$

and if $h \to 0$ we get $w_n \to 0$,$v_n \to 0$ which concludes the proof.

Theorem 2.5.3. *For arbitrary fixed r : $0 \le r \le 1$, the Adams-Moulton two-step approximates of Eq. (2.25) converge to the exact solutions $\underline{Y}(t,\alpha)$,$\overline{Y}(t,\alpha)$ for $\underline{Y},\overline{Y} \in C^4[t_0,T]$.*

Proof. It is sufficient to show

$$lim_{h\to 0}\underline{y}_N(\alpha) = \underline{Y}(T,\alpha), \quad lim_{h\to 0}\overline{y}_N(\alpha) = \overline{Y}(T,\alpha).$$

By using exact value the following results will be obtained:

$$\underline{Y}_{n+1}(\alpha) = \underline{Y}_n(\alpha) - \frac{h}{12}f(t_{n-1},\overline{Y}_{n-1}(\alpha)) + \frac{2h}{3}f(t_n,\underline{Y}_n(\alpha)) + \frac{5h}{12}f(t_{n+1},\underline{Y}_{n+1}(\alpha)) + \frac{1}{24}h^4\underline{Y}''''(\xi_n),$$
$$\overline{Y}_{n+1}(\alpha) = \overline{Y}_n(\alpha) - \frac{h}{12}f(t_{n-1},\underline{Y}_{n-1}(\alpha)) + \frac{2h}{3}f(t_n,\overline{Y}_n(\alpha)) + \frac{5h}{12}f(t_{n+1},\overline{Y}_{n+1}(\alpha)) + \frac{1}{24}h^4\overline{Y}''''(\xi_n),$$

where $t_n < \underline{\xi}_n,\overline{\xi}_n < t_{n+1}$. Consequently

$$\underline{Y}_{n+1}(\alpha) - \underline{y}_{n+1}(\alpha) = \underline{Y}_n(\alpha) - \underline{y}_n(\alpha) - \tfrac{h}{12}\{f(t_{n-1},\overline{Y}_{n-1}(\alpha)) - f(t_{n-1},\overline{y}_{n-1}(\alpha))\} +$$
$$\frac{2h}{3}\{f(t_n,\underline{Y}_n(\alpha)) - f(t_n,\underline{y}_n(\alpha))\} + \frac{5h}{12}\{f(t_{n+1},\underline{Y}_{n+1}(\alpha)) - f(t_{n+1},\underline{y}_{n+1}(\alpha))\} + \frac{1}{24}h^4\underline{Y}''''(\xi_n),$$

$$\overline{Y}_{n+1}(\alpha) - \overline{y}_{n+1}(\alpha) = \overline{Y}_n(\alpha) - \overline{y}_n(\alpha) - \tfrac{h}{12}\{f(t_{n-1},\underline{Y}_{n-1}(\alpha)) - f(t_{n-1},\underline{y}_{n-1}(\alpha))\} +$$
$$\frac{2h}{3}\{f(t_n,\overline{Y}_n(\alpha)) - f(t_n,\overline{y}_n(\alpha))\} + \frac{5h}{12}\{f(t_{n+1},\overline{Y}_{n+1}(\alpha)) - f(t_{n+1},\overline{y}_{n+1}(\alpha))\} + \frac{1}{24}h^4\overline{Y}''''(\xi_n).$$

Denote $w_n = \underline{Y}_n(\alpha) - \underline{y}_n(\alpha)$, $v_n = \overline{Y}_n(\alpha) - \overline{y}_n(\alpha)$. Then

$$|w_{n+1}| \le (1 + \frac{2hL_1}{3})|w_n| + (\frac{hL_2}{12})|v_{n-1}| + (\frac{5hL_3}{12})|w_{n+1}| + \frac{1}{24}h^4\underline{M},$$
$$|v_{n+1}| \le (1 + \frac{2hL_4}{3})|v_n| + (\frac{hL_5}{12})|w_{n-1}| + (\frac{5hL_6}{12})|v_{n+1}| + \frac{1}{24}h^4\overline{M},$$

where $\underline{M} = max_{t_0 \le t \le T}|\underline{Y}''''(t,\alpha)|$ and $\overline{M} = max_{t_0 \le t \le T}|\overline{Y}''''(t,\alpha)|$ and is put

$$L = max\{L_1, L_2, L_3, L_4, L_5, L_6\} < \frac{12}{5h},$$

then

$$|w_{n+1}| \le (1 + \frac{13hL}{12 - 5hL})|w_n| + (\frac{hL}{12 - 5hL})|v_{n-1}| + (\frac{1}{24 - 10hL})h^4\underline{M},$$
$$|v_{n+1}| \le (1 + \frac{13hL}{12 - 5hL})|v_n| + (\frac{hL}{12 - 5hL})|w_{n-1}| + (\frac{1}{24 - 10hL})h^4\overline{M},$$

are resulted, where $|u_n| = |w_n| + |v_n|$, then by lemma (2.5.1) and $w_0 = v_0 = 0$ (also with $w_1 = v_1 = 0$):

$$|u_n| \le \frac{(1+\frac{13hL}{12-5hL})^{n-1}-1}{\frac{13hL}{12-5hL}} \times \frac{1}{24-10hL}h^4(\underline{M}+\overline{M}) + \{\delta_1(1+\frac{13hL}{12-5hL})^{n-4} + \delta_2(1+\frac{13hL}{12-5hL})^{n-5} + \ldots + \delta_m(1+\frac{13hL}{12-5hL}) + 1\}(\frac{hL}{12-5hL})^2(\frac{1}{24-10hL}h^4(\underline{M}+\overline{M})) + \{\zeta_1(1+\frac{13hL}{12-5hL})^{n-6} + \zeta_2(1+\frac{13hL}{12-5hL})^{n-7} + \ldots + \zeta_n(1+\frac{13hL}{12-5hL}) + 1\}((\frac{hL}{12-5hL})^3)(\frac{1}{24-10hL}h^4(\underline{M}+\overline{M})) + \{\lambda_1(1+\frac{13hL}{12-5hL})^{n-8} + \lambda_2(1+\frac{13hL}{12-5hL})^{n-9} + \ldots + \lambda_p(1+\frac{13hL}{12-5hL}) + 1\}(\frac{hL}{12-5hL})^4(\frac{1}{24-10hL}h^4(\underline{M}+\overline{M})) + \ldots$$

are obtained. If $h \to 0$ then $w_n \to 0$,$v_n \to 0$ which concludes the proof.

Remark 2.5.1. Above theorem results that convergence order is $O(h^3)$.

Remark 2.5.2. It is easy to show that convergence order of Adams-Bashforth two-step method is $O(h^2)$.

Theorem 2.5.4. *Adams-Bashforth two-step and three-step methods are stable.*

Proof. For Adams-Bashforth two-step method exist only one characteristic polynomial $p(\lambda) = \lambda^2 - \lambda$ and it is clear that satisfies the root condition then by theorem (1.3.1), the Adams-Bashforth two-step method is stable.
Also, for Adams-Bashforth three-step method, there exist only one characteristic polynomial $p(\lambda) = \lambda^3 - \lambda^2$ then it satisfies the root condition, therefore it is a stable method.

Theorem 2.5.5. *Adams-Moulton two-step and three-step methods are stable.*

Proof. Similar to theorem 2.5.4.

The reason of choosing the Adams-Bashforth and Adams-Moulton as a Predictor-Corrector technique is that both of them are stable.

2.6 Numerical examples

Example 2.6.1. Consider the initial value problem
$\widetilde{y}'(t) = -\widetilde{y}(t) + t + 1$,
$\widetilde{y}(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha)$,
$\widetilde{y}(0.01) = (0.01+(0.985+0.015\alpha)e^{-0.01}-(1-\alpha)0.025e^{0.01}, 0.01+(0.985+0.015\alpha)e^{-0.01}+(1-\alpha)0.025e^{0.01})$,
$\widetilde{y}(0.02) = (0.02+(0.985+0.015\alpha)e^{-0.02}-(1-\alpha)0.025e^{0.02}, 0.02+(0.985+0.015\alpha)e^{-0.02}+(1-\alpha)0.025e^{0.02})$.

The exact solution in at $t = 0.1$ is given by

$$\widetilde{Y}(0.1, \alpha) = (0.1 + (0.985 + 0.015\alpha)e^{-0.1} - (1-\alpha)0.025e^{0.1},$$

$$0.1 + (0.985 + 0.015\alpha)e^{-0.1} + (1-\alpha)0.025e^{0.1}).$$

By using the Adams-Bashforth two-step method and Predictor-Corrector three-step method with $N = 10$, tables 2.1 an 2.2, are obtained. Also results of example 2.6.1, are shown in fig 2.1.

In the figures 2.2 and 2.3, we compare real value and predictor-corrector method in 0-level and 1-level respectively.

α	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	.9611151980	.9636355838	-0.25203858e-2	1.021421245	1.018894130	0.2527115e-2
0.1	.9654877615	.9677557672	-0.22680057e-2	1.019763204	1.017488459	0.2274745e-2
0.2	.9698603251	.9718759506	-0.20156255e-2	1.018105163	1.016082788	0.2022375e-2
0.3	.9742328886	.9759961341	-0.17632455e-2	1.016447122	1.014677116	0.1770006e-2
0.4	.9786054521	.9801163175	-0.15108654e-2	1.014789081	1.013271445	0.1517636e-2
0.5	.9829780157	.9842365009	-0.12584852e-2	1.013131040	1.011865774	0.1265266e-2
0.6	.9873505792	.9883566843	-0.10061051e-2	1.011472999	1.010460103	0.1012896e-2
0.7	.9917231427	.9924768677	-0.7537250e-3	1.009814958	1.009054432	0.760526e-3
0.8	.9960957062	.9965970512	-0.5013450e-3	1.008156917	1.007648760	0.508157e-3
0.9	1.000468270	1.000717235	-0.248965e-3	1.006498876	1.006243089	0.255787e-3
1	1.004840833	1.004837418	0.3415e-5	1.004840835	1.004837418	0.3417e-5

Table. 2.1. Adams-Bashforth two-step method with $N = 10$ for example 2.6.1.

α	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	.9632483676	.9636355838	-0.3872162e-3	1.019281352	1.018894130	0.387222e-3
0.1	.9674072729	.9677557672	-0.3484943e-3	1.017836959	1.017488459	0.348500e-3
0.2	.9715661783	.9718759506	-0.3097723e-3	1.016392566	1.016082788	0.309778e-3
0.3	.9757250836	.9759961341	-0.2710505e-3	1.014948173	1.014677116	0.271057e-3
0.4	.9798839889	.9801163175	-0.2323286e-3	1.013503780	1.013271445	0.232335e-3
0.5	.9840428943	.9842365009	-0.1936066e-3	1.012059387	1.011865774	0.193613e-3
0.6	.9882017996	.9883566843	-0.1548847e-3	1.010614994	1.010460103	0.154891e-3
0.7	.9923607049	.9924768677	-0.1161628e-3	1.009170601	1.009054432	0.116169e-3
0.8	.9965196102	.9965970512	-0.774410e-4	1.007726207	1.007648760	0.77447e-4
0.9	1.000678516	1.000717235	-0.38719e-4	1.006281814	1.006243089	0.38725e-4
1	1.004837421	1.004837418	0.3e-8	1.004837421	1.004837418	0.3e-8

Table. 2.2. Predictor-Corrector three-step method with $N = 10$, for example 2.6.1.

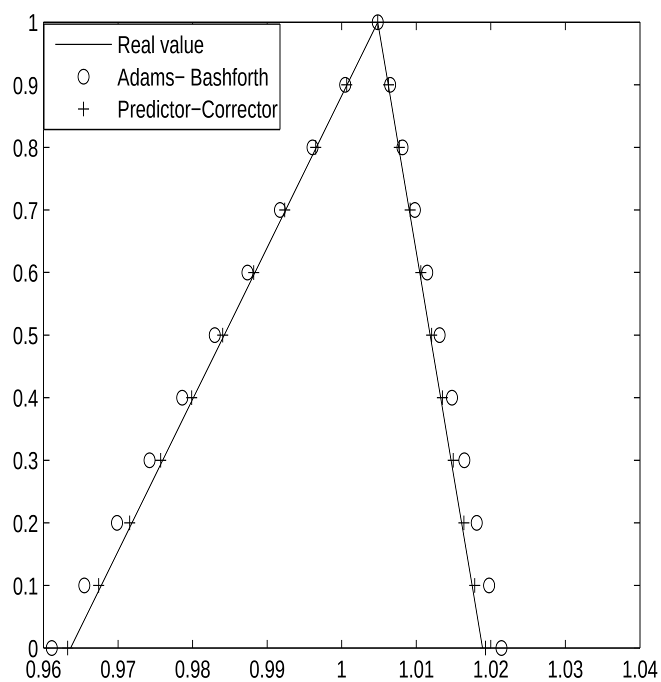

Fig. 2.1. The results for example 2.6.1.

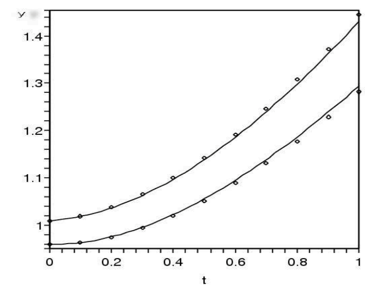

Fig. 2.2. The Real value and its approximation by

Predictor-Corrector method in 0-level for example 2.6.1,

solid curve: Real value; dotted curve: Predictor-Corrector method

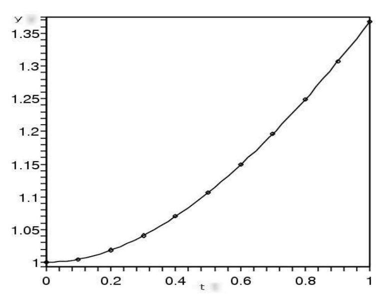

Fig. 2.3. The Real value and its approximation by

Predictor-Corrector method in 1-level for example 2.6.1,

solid curve: Real value; dotted curve: Predictor-Corrector method

Example 2.6.2. Consider the initial value problem

$\widetilde{y'}(t) = -\widetilde{y}(t)$,

$\widetilde{y}(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha)$,

$\widetilde{y}(0.01) = (0.01+(0.985+0.015\alpha)e^{-0.01}-(1-\alpha)0.025e^{0.01}, 0.01+(0.985+0.015\alpha)e^{-0.01}+ (1-\alpha)0.025e^{0.01})$,

$\widetilde{y}(0.02) = (0.02+(0.985+0.015\alpha)e^{-0.02}-(1-\alpha)0.025e^{0.02}, 0.02+(0.985+0.015\alpha)e^{-0.02}+ (1-\alpha)0.025e^{0.02})$,

The exact solution in at $t = 0.1$ is given by $\widetilde{Y}(0.1, \alpha) = ((0.985 + 0.015\alpha)e^{-0.1} - (1 - \alpha)0.025e^{0.1}, (0.985 + 0.015\alpha)e^{-0.1} + (1 - \alpha)0.025e^{0.1})$.

By using the Adams-Bashforth two-step method and Predictor-Corrector three-step method with $N = 10$, tables 2.3 an 2.4 are obtained. Also results of example 2.6.2 are shown in fig 2.4.

In the figures 2.5 and 2.6, we compare real value and predictor-corrector method in 0-level and 1-level respectively.

α	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	.8612677593	.8686439213	0.73761620e-2	.9204077866	.9138857922	0.65219944e-2
0.1	.8655594514	.8722632710	0.67038196e-2	.9188168404	.9129809548	0.58358856e-2
0.2	.8698511436	.8758826206	0.60314770e-2	.9172258943	.9120761173	0.51497770e-2
0.3	.8741428358	.8795019703	0.53591345e-2	.9156349481	.9111712799	0.44636682e-2
0.4	.8784345279	.8831213200	0.46867921e-2	.9140440019	.9102664425	0.37775594e-2
0.5	.8827262200	.8867406696	0.40144496e-2	.9124530558	.9093616051	0.30914507e-2
0.6	.8870179122	.8903600193	0.33421071e-2	.9108621096	.9084567677	0.24053419e-2
0.7	.8913096044	.8939793690	0.26697646e-2	.9092711634	.9075519303	0.17192331e-2
0.8	.8956012965	.8975987187	0.19974222e-2	.9076802173	.9066470928	0.10331245e-2
0.9	.8998929886	.9012180683	0.13250797e-2	.9060892711	.9057422554	0.3470157e-3
1	.9041846808	.9048374180	0.6527372e-3	.9044983249	.9048374180	-0.3390931e-3

Table. 2.3. Adams-Bashforth two-step method with $N = 10$ for example 2.6.2

and by using the Predictor-Corrector three-step method with $N = 10$ the following results are obtained:

α	$\underline{y}$	$\underline{Y}$	ERROR	$\overline{y}$	$\overline{Y}$	ERROR
0	.8643464102	.8686439213	0.42975111e-2	.9181833096	.9138857922	0.42975174e-2
0.1	.8683955113	.8722632710	0.38677597e-2	.9168487208	.9129809548	0.38677660e-2
0.2	.8724446124	.8758826206	0.34380082e-2	.9155141319	.9120761173	0.34380146e-2
0.3	.8764937135	.8795019703	0.30082568e-2	.9141795431	.9111712799	0.30082632e-2
0.4	.8805428146	.8831213200	0.25785054e-2	.9128449542	.9102664425	0.25785117e-2
0.5	.8845919157	.8867406696	0.21487539e-2	.9115103654	.9093616051	0.21487603e-2
0.6	.8886410168	.8903600193	0.17190025e-2	.9101757765	.9084567677	0.17190088e-2
0.7	.8926901179	.8939793690	0.12892511e-2	.9088411877	.9075519303	0.12892574e-2
0.8	.8967392190	.8975987187	0.8594997e-3	.9075065989	.9066470928	0.8595061e-3
0.9	.9007883201	.9012180683	0.4297482e-3	.9061720100	.9057422554	0.4297546e-3
1	.9048374212	.9048374180	-0.32e-8	.9048374212	.9048374180	0.32e-8

Table. 2.4. Predictor-Corrector three-step method with $N = 10$, for example 2.6.2.

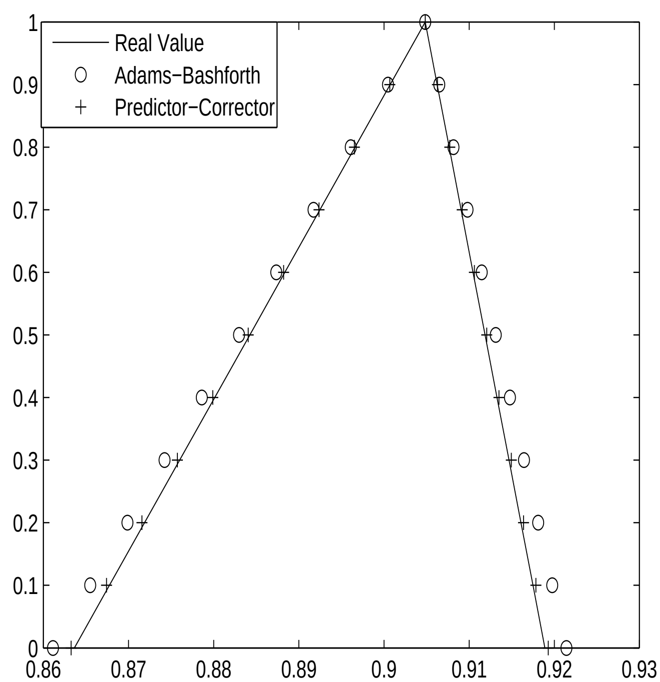

Fig. 2.4. The results for example 2.6.2.

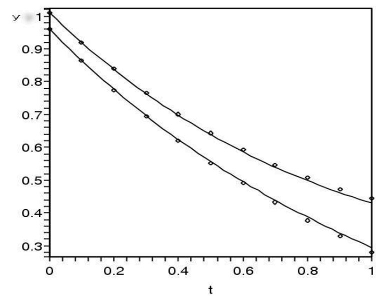

Fig. 2.5. The Real value and its approximation by

Predictor-Corrector method in 0-level for example 2.6.2,

solid curve: Real value; dotted curve: Predictor-Corrector method

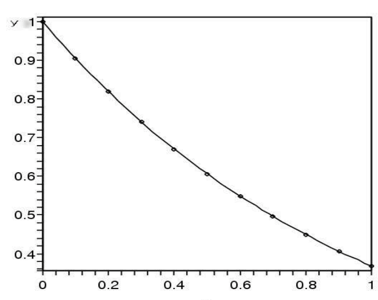

Fig. 2.6. The Real value and its approximation by

Predictor-Corrector method in 1-level for example 2.6.2,

solid curve: Real value; dotted curve: Predictor-Corrector method

Chapter 3

Improved Predictor Corrector

3.1 Introduction

In this chapter, an Improved Predictor-Corrector (IPC) method to solve the "fuzzy initial value problem" is proposed. The IPC method is obtained by combining an explicit three-step method and an implicit two-step method. These methods are compared with the methods discussed in the previous chapter, and they proved to have more accuracy. The convergence and stability of the proposed methods are also presented in detail. In addition, these methods are illustrated by solving some examples.

3.2 Explicit three-step method

In this section we would like to solve the fuzzy initial value problem $y'(t) = f(t, y(t))$ by the explicit three-step method. Let the fuzzy initial values be $\widetilde{y}(t_{i-1}), \widetilde{y}(t_i), \widetilde{y}(t_{i+1})$, i.e.,

$$\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i))\, \widetilde{f}(t_{i+1}, y(t_{i+1})),$$

which are triangular fuzzy numbers and are shown as

$$\{f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))\}, \{f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))\}$$

$$\{f^l(t_{i+1}, y(t_{i+1})), f^c(t_{i+1}, y(t_{i+1})), f^r(t_{i+1}, y(t_{i+1}))\}.$$

Consider the following fuzzy equation

$$\widetilde{y}(t_{i+2}) = \widetilde{y}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+2}} \widetilde{f}(t, y(t))\ dt. \tag{3.1}$$

By fuzzy linear spline interpolation for $\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i)), \widetilde{f}(t_{i+1}, y(t_{i+1}))$ we have:

$$\widetilde{f}_1(t, y(t)) = \frac{t_i - t}{t_i - t_{i-1}} \widetilde{f}(t_{i-1}, y(t_{i-1})) + \frac{t - t_{i-1}}{t_i - t_{i-1}} \widetilde{f}(t_i, y(t_i)) \quad t \in [t_{i-1}, t_i]$$

$$\widetilde{f}_2(t, y(t)) = \frac{t_{i+1} - t}{t_{i+1} - t_i} \widetilde{f}(t_i, y(t_i)) + \frac{t - t_i}{t_{i+1} - t_i} \widetilde{f}(t_{i+1}, y(t_{i+1})) \quad t \in [t_i, t_{i+1}],$$

therefore the following results will be obtained:

$$f_1^l(t, y(t)) = \frac{t_i - t}{t_i - t_{i-1}} f_1^l(t_{i-1}, y(t_{i-1})) + \frac{t - t_{i-1}}{t_i - t_{i-1}} f_1^l(t_i, y(t_i)), \quad t \in [t_{i-1}, t_i], \tag{3.2}$$

$$f_1^c(t, y(t)) = \frac{t_i - t}{t_i - t_{i-1}} f_1^c(t_{i-1}, y(t_{i-1})) + \frac{t - t_{i-1}}{t_i - t_{i-1}} f_1^c(t_i, y(t_i)), \quad t \in [t_{i-1}, t_i], \tag{3.3}$$

$$f_1^r(t, y(t)) = \frac{t_i - t}{t_i - t_{i-1}} f_1^r(t_{i-1}, y(t_{i-1})) + \frac{t - t_{i-1}}{t_i - t_{i-1}} f_1^r(t_i, y(t_i)), \quad t \in [t_{i-1}, t_i], \tag{3.4}$$

and

$$f_2^l(t,y(t)) = \frac{t_{i+1}-t}{t_{i+1}-t_i} f_2^l(t_i,y(t_i)) + \frac{t-t_i}{t_{i+1}-t_i} f_2^l(t_{i+1},y(t_{i+1})), \quad t \in [t_i,t_{i+1}], \tag{3.5}$$

$$f_2^c(t,y(t)) = \frac{t_{i+1}-t}{t_{i+1}-t_i} f_2^c(t_i,y(t_i)) + \frac{t-t_i}{t_{i+1}-t_i} f_2^c(t_{i+1},y(t_{i+1})), \quad t \in [t_i,t_{i+1}], \tag{3.6}$$

$$f_2^r(t,y(t)) = \frac{t_{i+1}-t}{t_{i+1}-t_i} f_2^r(t_i,y(t_i)) + \frac{t-t_i}{t_{i+1}-t_i} f_2^r(t_{i+1},y(t_{i+1})), \quad t \in [t_i,t_{i+1}]. \tag{3.7}$$

From (4.8) it follows that:

$$\widetilde{y}^\alpha(t_{i+2}) = [\underline{y}^\alpha(t_{i+2}), \overline{y}^\alpha(t_{i+2})],$$

where

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i-1}) + \int_{t_{i-1}}^{t_i} \{\alpha f_1^c(t,y(t)) + (1-\alpha) f_1^l(t,y(t))\}dt + \tag{3.8}$$
$$\int_{t_i}^{t_{i+2}} \{\alpha f_2^c(t,y(t)) + (1-\alpha) f_2^l(t,y(t))\}dt,$$

and

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i-1}) + \int_{t_{i-1}}^{t_i} \{\alpha f_1^c(t,y(t)) + (1-\alpha) f_1^r(t,y(t))\}dt + \tag{3.9}$$
$$\int_{t_i}^{t_{i+2}} \{\alpha f_2^c(t,y(t)) + (1-\alpha) f_2^r(t,y(t))\}dt.$$

If (3.2), (3.3),(3.5) and (3.6) are set in (3.8), and (3.3), (3.4), (3.6) and (3.7) in (3.9), then we get:

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i-1}) + \int_{t_{i-1}}^{t_i} \{\alpha\{\tfrac{t_i-t}{t_i-t_{i-1}} f_1^c(t_{i-1},y(t_{i-1})) + \tfrac{t-t_{i-1}}{t_i-t_{i-1}} f_1^c(t_i,y(t_i))\} + (1-\alpha)\{\tfrac{t_i-t}{t_i-t_{i-1}}$$

$$f_1^l(t_{i-1}, y(t_{i-1})) + \tfrac{t-t_{i-1}}{t_i-t_{i-1}} f_1^l(t_i, y(t_i))\}\}dt + \int_{t_i}^{t_{i+2}} \{\alpha\{\tfrac{t_{i+1}-t}{t_{i+1}-t_i} f_2^c(t_i, y(t_i)) + \tfrac{t-t_i}{t_{i+1}-t_i}$$

$$f_2^c(t_{i+1}, y(t_{i+1}))\} + (1-\alpha)\{\tfrac{t_{i+1}-t}{t_{i+1}-t_i} f_2^l(t_i, y(t_i)) + \tfrac{t-t_i}{t_{i+1}-t_i} f_2^l(t_{i+1}, y(t_{i+1}))\}\}dt,$$

and

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i-1}) + \int_{t_{i-1}}^{t_i} \{\alpha\{\tfrac{t_i-t}{t_i-t_{i-1}} f_1^c(t_{i-1}, y(t_{i-1})) + \tfrac{t-t_{i-1}}{t_i-t_{i-1}} f_1^c(t_i, y(t_i))\} + (1-\alpha)\{\tfrac{t_i-t}{t_i-t_{i-1}}$$

$$f_1^r(t_{i-1}, y(t_{i-1})) + \tfrac{t-t_{i-1}}{t_i-t_{i-1}} f_1^r(t_i, y(t_i))\}\}dt + \int_{t_i}^{t_{i+2}} \{\alpha\{\tfrac{t_{i+1}-t}{t_{i+1}-t_i} f_2^c(t_i, y(t_i)) + \tfrac{t-t_i}{t_{i+1}-t_i}$$

$$f_2^c(t_{i+1}, y(t_{i+1}))\} + (1-\alpha)\{\tfrac{t_{i+1}-t}{t_{i+1}-t_i} f_2^r(t_i, y(t_i)) + \tfrac{t-t_i}{t_{i+1}-t_i} f_2^r(t_{i+1}, y(t_{i+1}))\}\}dt.$$

By integration, the following results will be obtained:

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i-1}) + \frac{h}{2}[\alpha f^c(t_{i-1}, y(t_{i-1})) + (1-\alpha) f^l(t_{i-1}, y(t_{i-1}))] + \frac{h}{2}[\alpha f^c(t_i, y(t_i))$$

$$+(1-\alpha) f^l(t_i, y(t_i))] + 2h[\alpha f^c(t_{i+1}, y(t_{i+1})) + (1-\alpha) f^l(t_{i+1}, y(t_{i+1}))],$$

and

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i-1}) + \frac{h}{2}[\alpha f^c(t_{i-1}, y(t_{i-1})) + (1-\alpha) f^r(t_{i-1}, y(t_{i-1}))] + \frac{h}{2}[\alpha f^c(t_i, y(t_i))$$

$$+(1-\alpha) f^r(t_i, y(t_i))] + 2h[\alpha f^c(t_{i+1}, y(t_{i+1})) + (1-\alpha) f^r(t_{i+1}, y(t_{i+1}))],$$

thus

$$\underline{y}^\alpha(t_{i+2}) = \underline{y}^\alpha(t_{i-1}) + \frac{h}{2}[\underline{f}^\alpha(t_{i-1}, y(t_{i-1})) + \underline{f}^\alpha(t_i, y(t_i)) + 4\underline{f}^\alpha(t_{i+1}, y(t_{i+1}))], \quad (3.10)$$

$$\overline{y}^\alpha(t_{i+2}) = \overline{y}^\alpha(t_{i-1}) + \frac{h}{2}[\overline{f}^\alpha(t_{i-1}, y(t_{i-1})) + \overline{f}^\alpha(t_i, y(t_i)) + 4\overline{f}^\alpha(t_{i+1}, y(t_{i+1}))]. \quad (3.11)$$

Therefore, the explicit three-step method is obtained as follows:

$$\begin{cases} \underline{y}^{\alpha}(t_{i+2}) = \underline{y}^{\alpha}(t_{i-1}) + \frac{h}{2}[\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + \underline{f}^{\alpha}(t_i, y(t_i)) + 4\underline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))], \\ \overline{y}^{\alpha}(t_{i+2}) = \overline{y}^{\alpha}(t_{i-1}) + \frac{h}{2}[\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + \overline{f}^{\alpha}(t_i, y(t_i)) + 4\overline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))], \\ \underline{y}^{\alpha}(t_{i-1}) = \alpha_0, \underline{y}^{\alpha}(t_i) = \alpha_1, \underline{y}^{\alpha}(t_{i+1}) = \alpha_2, \overline{y}^{\alpha}(t_{i-1}) = \alpha_3, \overline{y}^{\alpha}(t_i) = \alpha_4, \overline{y}^{\alpha}(t_{i+1}) = \alpha_5. \end{cases} \tag{3.12}$$

3.3 Implicit two-step method

Now we are going to solve the fuzzy initial value problem $y'(t) = f(t, y(t))$ by the implicit two-step method. Let the fuzzy initial values be $\widetilde{y}(t_{i-1}), \widetilde{y}(t_i)$, i.e.,

$$\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i)),$$

which are triangular fuzzy numbers and are shown as

$$\{f^l(t_{i-1}, y(t_{i-1})), f^c(t_{i-1}, y(t_{i-1})), f^r(t_{i-1}, y(t_{i-1}))\}, \{f^l(t_i, y(t_i)), f^c(t_i, y(t_i)), f^r(t_i, y(t_i))\}.$$

Consider the following fuzzy equation,

$$\widetilde{y}(t_{i+1}) = \widetilde{y}(t_{i-1}) + \int_{t_{i-1}}^{t_{i+1}} \widetilde{f}(t, y(t))\ dt. \tag{3.13}$$

By fuzzy linear spline interpolation for $\widetilde{f}(t_{i-1}, y(t_{i-1})), \widetilde{f}(t_i, y(t_i))$, and $\widetilde{f}(t_{i+1}, y(t_{i+1}))$ and from (3.13) we have:

$$\widetilde{y}^{\alpha}(t_{i+1}) = [\underline{y}^{\alpha}(t_{i+1}), \overline{y}^{\alpha}(t_{i+1})],$$

where

$$\underline{y}^{\alpha}(t_{i+1}) = \underline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_i} \{\alpha f_1^c(t, y(t)) + (1-\alpha) f_1^l(t, y(t))\}dt + \tag{3.14}$$

$$\int_{t_i}^{t_{i+1}} \{\alpha f_2^c(t, y(t)) + (1-\alpha) f_2^l(t, y(t))\}dt,$$

and

$$\overline{y}^{\alpha}(t_{i+1}) = \overline{y}^{\alpha}(t_{i-1}) + \int_{t_{i-1}}^{t_i} \{\alpha f_1^c(t, y(t)) + (1-\alpha) f_1^r(t, y(t))\}dt + \tag{3.15}$$

$$\int_{t_i}^{t_{i+1}} \{\alpha f_2^c(t, y(t)) + (1-\alpha) f_2^r(t, y(t))\}dt.$$

If (3.2), (3.3),(3.5)and (3.6) are set in (3.14), and (3.3), (3.4), (3.6) and (3.7) in (3.15), and in a way similar to that in section 3, we obtain the implicit two-step method as follows:

$$\begin{cases} \underline{y}^{\alpha}(t_{i+1}) = \underline{y}^{\alpha}(t_{i-1}) + \frac{h}{2}[\underline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + 2\underline{f}^{\alpha}(t_i, y(t_i)) + \underline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))], \\ \overline{y}^{\alpha}(t_{i+1}) = \overline{y}^{\alpha}(t_{i-1}) + \frac{h}{2}[\overline{f}^{\alpha}(t_{i-1}, y(t_{i-1})) + 2\overline{f}^{\alpha}(t_i, y(t_i)) + \overline{f}^{\alpha}(t_{i+1}, y(t_{i+1}))], \\ \underline{y}^{\alpha}(t_{i-1}) = \alpha_0, \underline{y}^{\alpha}(t_i) = \alpha_1, \overline{y}^{\alpha}(t_{i-1}) = \alpha_2, \overline{y}^{\alpha}(t_i) = \alpha_3. \end{cases} \tag{3.16}$$

3.4 Improved Predictor Corrector three-step method

The following algorithm is based on the explicit three-step method as a predictor and also an iteration of the implicit two-step method as a corrector.

ALGORITHM (*IPC three-step method*)

To approximate the solution of the following fuzzy initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t_0 \le t \le T, \\ \underline{y}^{\alpha}(t_0) = \alpha_0, \underline{y}^{\alpha}(t_1) = \alpha_1, \underline{y}^{\alpha}(t_2) = \alpha_2, \\ \overline{y}^{\alpha}(t_0) = \alpha_3, \overline{y}^{\alpha}(t_1) = \alpha_4, \overline{y}^{\alpha}(t_2) = \alpha_5, \end{cases}$$

an arbitrary positive integer N is chosen.

Step 1. Let $h = \frac{T-t_0}{N}$,

$$\underline{w}^{\alpha}(t_0) = \alpha_0, \underline{w}^{\alpha}(t_1) = \alpha_1, \underline{w}^{\alpha}(t_2) = \alpha_2,$$

$$\overline{w}^{\alpha}(t_0) = \alpha_3, \overline{w}^{\alpha}(t_1) = \alpha_4, \overline{w}^{\alpha}(t_2) = \alpha_5.$$

Step 2. Let $i = 1$.

Step 3. Let

$$\begin{cases} \underline{w}^{(0)\alpha}(t_{i+2}) = \underline{w}^{\alpha}(t_{i-1}) + \frac{h}{2}[\underline{f}^{\alpha}(t_{i-1}, w(t_{i-1})) + \underline{f}^{\alpha}(t_i, w(t_i)) + 4\underline{f}^{\alpha}(t_{i+1}, w(t_{i+1}))], \\ \overline{w}^{(0)\alpha}(t_{i+2}) = \overline{w}^{\alpha}(t_{i-1}) + \frac{h}{2}[\overline{f}^{\alpha}(t_{i-1}, w(t_{i-1})) + \overline{f}^{\alpha}(t_i, w(t_i)) + 4\overline{f}^{\alpha}(t_{i+1}, w(t_{i+1}))], \end{cases}$$

Step 4. Let $t_{i+2} = t_0 + (i+2)h$,

Step 5. Let

$$\begin{cases} \underline{w}^{\alpha}(t_{i+2}) = \underline{w}^{\alpha}(t_i) + (\frac{h}{2})\underline{f}^{\alpha}(t_i, w(t_i)) + h\underline{f}^{\alpha}(t_{i+1}, w(t_{i+1})) + (\frac{h}{2})\underline{f}^{\alpha}(t_{i+2}, w^{(0)}(t_{i+2})), \\ \overline{w}^{\alpha}(t_{i+2}) = \overline{w}^{\alpha}(t_i) + (\frac{h}{2})\overline{f}^{\alpha}(t_i, w(t_i)) + h\overline{f}^{\alpha}(t_{i+1}, w(t_{i+1})) + (\frac{h}{2})\overline{f}^{\alpha}(t_{i+2}, w^{(0)}(t_{i+2})), \end{cases}$$

Step 6. $i = i + 1$.

Step 7. If $i \leq N - 2$, go to step 3.

Step 8. The algorithm will end and $(\underline{w}^{\alpha}(T), \overline{w}^{\alpha}(T))$ approximates the real value of $(\underline{Y}^{\alpha}(T), \overline{Y}^{\alpha}(T))$.

3.5 Convergence

To integrate the system given in Eq.(2.1) from t_0 to a prefixed $T > t_0$, the interval $[t_0, T]$ will be replaced by a set of discrete equally spaced grid points $t_0 < t_1 < t_2 < \ldots < t_N = T$, and the exact solution $(\underline{Y}(t, \alpha), \overline{Y}(t, \alpha))$ is approximated by some

$(\underline{y}(t,\alpha),\overline{y}(t,\alpha))$. The exact and approximate solutions at $t_n,\ 0 \le n \le N$ are denoted by $Y_n(\alpha) = [\underline{Y}_n(\alpha), \overline{Y}_n(\alpha)]$, and $y_n(\alpha) = [\underline{y}_n(\alpha), \overline{y}_n(\alpha)]$, respectively. The grid points at which the solution is calculated are $t_n = t_0 + nh, \quad h = (T - t_0)/N,\ 1 \le n \le N$. From Eq. (3.16), the polygon curves

$$\underline{y}(t,h,\alpha) = \{[t_0, \underline{y}_0(\alpha)], [t_1, \underline{y}_1(\alpha)], \ldots, [t_N, \underline{y}_N(\alpha)]\},$$

$$\overline{y}(t,h,\alpha) = \{[t_0, \overline{y}_0(\alpha)], [t_1, \overline{y}_1(\alpha)], \ldots, [t_N, \overline{y}_N(\alpha)]\},$$

are the approximates to $\underline{Y}(t,\alpha)$ and $\overline{Y}(t,\alpha)$, respectively, over the interval $t_0 \le t \le t_N$. Now we are going to prove

$$lim_{h\to 0}\underline{y}(t,h,\alpha) = \underline{Y}(t,\alpha), \quad lim_{h\to 0}\overline{y}(t,h,\alpha) = \overline{Y}(t,\alpha).$$

Theorem 3.5.1. *For any arbitrary fixed $r: \ 0 \le r \le 1$, the implicit two-step approximates of Eq. (3.16) converge to the exact solutions $\underline{Y}(t,\alpha)$,$\overline{Y}(t,\alpha)$ for $\underline{Y}, \overline{Y} \in C^3[t_0, T]$.*

Proof. It is sufficient to show

$$lim_{h\to 0}\underline{y}_N(\alpha) = \underline{Y}(T,\alpha), \quad lim_{h\to 0}\overline{y}_N(\alpha) = \overline{Y}(T,\alpha).$$

By using Taylor's theorem we have:

$$\underline{Y}_{n+1}(\alpha) = \underline{Y}_{n-1}(\alpha) + \frac{h}{2} f(t_{n-1}, \underline{Y}_{n-1}(\alpha)) + hf(t_n, \underline{Y}_n(\alpha)) + \frac{h}{2} f(t_{n+1}, \underline{Y}_{n+1}(\alpha)) + \frac{1}{6} h^3 \underline{Y}'''(\xi_n),$$

$$\overline{Y}_{n+1}(\alpha) = \overline{Y}_{n-1}(\alpha) + \frac{h}{2} f(t_{n-1}, \overline{Y}_{n-1}(\alpha)) + hf(t_n, \overline{Y}_n(\alpha)) + \frac{h}{2} f(t_{n+1}, \overline{Y}_{n+1}(\alpha)) + \frac{1}{6} h^3 \overline{Y}'''(\xi_n),$$

where $t_n < \underline{\xi}_n, \overline{\xi}_n < t_{n+1}$. Consequently

$$\underline{Y}_{n+1}(\alpha) - \underline{y}_{n+1}(\alpha) = \underline{Y}_{n-1}(\alpha) - \underline{y}_{n-1}(\alpha) + \tfrac{h}{2}\{f(t_{n-1}, \underline{Y}_{n-1}(\alpha)) - f(t_{n-1}, \underline{y}_{n-1}(\alpha))\}+$$

$$h\{f(t_n, \underline{Y}_n(\alpha)) - f(t_n, \underline{y}_n(\alpha))\} + \frac{h}{2}\{f(t_{n+1}, \underline{Y}_{n+1}(\alpha)) - f(t_{n+1}, \underline{y}_{n+1}(\alpha))\} + \frac{1}{6}h^3\underline{Y}'''(\xi_n),$$

$$\overline{Y}_{n+1}(\alpha) - \overline{y}_{n+1}(\alpha) = \overline{Y}_{n-1}(\alpha) - \overline{y}_{n-1}(\alpha) + \tfrac{h}{2}\{f(t_{n-1}, \overline{Y}_{n-1}(\alpha)) - f(t_{n-1}, \overline{y}_{n-1}(\alpha))\}+$$

$$h\{f(t_n, \overline{Y}_n(\alpha)) - f(t_n, \overline{y}_n(\alpha))\} + \frac{h}{2}\{f(t_{n+1}, \overline{Y}_{n+1}(\alpha)) - f(t_{n+1}, \overline{y}_{n+1}(\alpha))\} + \frac{1}{6}h^3\overline{Y}'''(\xi_n).$$

Denote $w_n = \underline{Y}_n(\alpha) - \underline{y}_n(\alpha)$, $v_n = \overline{Y}_n(\alpha) - \overline{y}_n(\alpha)$. Then

$$|w_{n+1}| \leq hL_1|w_n| + (1 + \frac{hL_2}{2})|w_{n-1}| + (\frac{hL_3}{2})|w_{n+1}| + \frac{1}{6}h^3\underline{M},$$
$$|v_{n+1}| \leq hL_4|v_n| + (1 + \frac{hL_5}{2})|v_{n-1}| + (\frac{hL_6}{2})|v_{n+1}| + \frac{1}{6}h^3\overline{M},$$

where $\underline{M} = max_{t_0 \leq t \leq T}|\underline{Y}'''(t, \alpha)|$ and $\overline{M} = max_{t_0 \leq t \leq T}|\overline{Y}'''(t, \alpha)|$.

Set

$$L = max\{L_1, L_2, L_3, L_4, L_5, L_6\} < \frac{2}{h},$$

then

$$|w_{n+1}| \leq (\frac{2hL}{2 - hL})|w_n| + (\frac{2 + hL}{2 - hL})|w_{n-1}| + (\frac{1}{6 - 3hL})h^3\underline{M},$$
$$|v_{n+1}| \leq (\frac{2hL}{2 - hL})|v_n| + (\frac{2 + hL}{2 - hL})|v_{n-1}| + (\frac{1}{6 - 3hL})h^3\overline{M},$$

are obtained, where $|u_n| = |w_n| + |v_n|$, then by Lemma (2.5.1), and also by $w_0 = v_0 = 0$ and $w_1 = v_1 = 0$) we have:

$|u_n| \leq \frac{(\frac{2hL}{2-hL})^{n-1}-1}{\frac{3hL-2}{2-hL}} \times \frac{1}{6-3hL}h^3(\underline{M}+\overline{M})+\{\delta_1(\frac{2hL}{2-hL})^{n-4}+\delta_2(\frac{2hL}{2-hL})^{n-5}+\ldots+\delta_m(\frac{2hL}{2-hL})+1\}(\frac{2+hL}{2-hL})(\frac{1}{6-3hL}h^3(\underline{M}+\overline{M}))+\{\zeta_1(\frac{2hL}{2-hL})^{n-6}+\zeta_2(\frac{2hL}{2-hL})^{n-7}+\ldots+\zeta_n(\frac{2hL}{2-hL})+1\}(\frac{2+hL}{2-hL})^2(\frac{1}{6-3hL}h^3(\underline{M}+\overline{M}))+\{\lambda_1(\frac{2hL}{2-hL})^{n-8}+\lambda_2(\frac{2hL}{2-hL})^{n-9}+\ldots+\lambda_p(\frac{2hL}{2-hL})+1\}(\frac{2+hL}{2-hL})^3(\frac{1}{6-3hL}h^3(\underline{M}+\overline{M}))+\ldots$

If $h \to 0$ then $w_n \to 0$,$v_n \to 0$, which concludes the proof.

Theorem 3.5.2. *For any arbitrary fixed α : $0 \leq \alpha \leq 1$, the explicit three-step approximates of Eq. (3.16) converge to the exact solution $\underline{Y}(t,\alpha)$,$\overline{Y}(t,\alpha)$ for $\underline{Y},\overline{Y} \in C^3[t_0,T]$.*

Proof. Similar to Theorem 3.5.1.

Theorem 3.5.3. *The explicit three-step method is stable.*

Proof. For the explicit three-step method, there exists only one characteristic polynomial $p(\lambda) = \lambda^3 - \lambda$, then it satisfies the root condition and, therefore, it is a stable method.

Theorem 3.5.4. *The implicit two-step method is stable.*

Proof. Similar to Theorem 3.5.3.

Regarding the above-mentioned theorems, it is obvious that the IPC Three-step method is convergent and stable.

3.6 Numerical examples

Example 3.6.1. Consider the initial value problem

$\widetilde{y'}(t) = -\widetilde{y}(t) + t + 1,$

$\widetilde{y}(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha),$

$\widetilde{y}(0.01) = (0.01+(0.985+0.015\alpha)e^{-0.01}-(1-\alpha)0.025e^{0.01}, 0.01+(0.985+0.015\alpha)e^{-0.01}+ (1-\alpha)0.025e^{0.01}),$

$\widetilde{y}(0.02) = (0.02+(0.985+0.015\alpha)e^{-0.02}-(1-\alpha)0.025e^{0.02}, 0.02+(0.985+0.015\alpha)e^{-0.02}+ (1-\alpha)0.025e^{0.02}),$

The exact solution in at $t = 0.1$ is given by

$\widetilde{Y}(0.1, \alpha) = (0.1+(0.985+0.015\alpha)e^{-0.1}-(1-\alpha)0.025e^{0.1}, 0.1+(0.985+0.015\alpha)e^{-0.1}+ (1-\alpha)0.025e^{0.1}).$

The results of the implementation of the numerical method, the explicit three-step method, and the IPC three-step method, with $N = 10$, on the mentioned differential equation are shown in Tables 3.1 and 3.2.

Table 3.1, contains real values, the error of Adams-Bashforth three-step method denoted by error 1, and the error of the explicit three-step method, denoted by error 2. Also, Table 3.2, contains real values, the error of the Predictor-Corrector Three-step method, denoted by error 1 and the error of the IPC three-step method denoted by error 2. The comparison of the exact solution with the approximate one at the

0−level and 1−level is depicted in Figures 3.2, 3.3.

α	$\underline{Y}$	Error1	Error2	$\overline{Y}$	Error1	Error2
0	.9636355838	-0.62158736e-2	0.43803e-5	1.018894130	0.6215822e-2	0.4339e-5
0.1	.9677557672	-0.55942890e-2	0.41948e-5	1.017488459	0.5594237e-2	0.4158e-5
0.2	.9718759506	-0.49727043e-2	0.40093e-5	1.016082788	0.4972653e-2	0.3977e-5
0.3	.9759961341	-0.43511197e-2	0.38238e-5	1.014677116	0.4351068e-2	0.3796e-5
0.4	.9801163175	-0.37295351e-2	0.36384e-5	1.013271445	0.3729484e-2	0.3614e-5
0.5	.9842365009	-0.31079505e-2	0.34528e-5	1.011865774	0.3107899e-2	0.3433e-5
0.6	.9883566843	-0.24863659e-2	0.32673e-5	1.010460103	0.2486314e-2	0.3251e-5
0.7	.9924768677	-0.18647817e-2	0.30814e-5	1.009054432	0.1864729e-2	0.3070e-5
0.8	.9965970512	-0.12431969e-2	0.28960e-5	1.007648760	0.1243144e-2	0.2888e-5
0.9	1.000717235	-0.621613e-3	0.2710e-5	1.006243089	0.621560e-3	0.2708e-5
1	1.004837418	-0.27e-7	0.2525e-5	1.004837418	-0.25e-7	0.2526e-5

Table. 3.1.

α	$\underline{Y}$	Error1	Error2	$\overline{Y}$	Error1	Error2
0	.9636355838	-0.3872162e-3	-0.6166e-6	1.018894130	0.387222e-3	-0.580e-6
0.1	.9677557672	-0.3484943e-3	-0.6157e-6	1.017488459	0.348500e-3	-0.583e-6
0.2	.9718759506	-0.3097723e-3	-0.6147e-6	1.016082788	0.309778e-3	-0.585e-6
0.3	.9759961341	-0.2710505e-3	-0.6138e-6	1.014677116	0.271057e-3	-0.588e-6
0.4	.9801163175	-0.2323286e-3	-0.6128e-6	1.013271445	0.232335e-3	-0.591e-6
0.5	.9842365009	-0.1936066e-3	-0.6120e-6	1.011865774	0.193613e-3	-0.594e-6
0.6	.9883566843	-0.1548847e-3	-0.6111e-6	1.010460103	0.154891e-3	-0.596e-6
0.7	.9924768677	-0.1161628e-3	-0.6105e-6	1.009054432	0.116169e-3	-0.599e-6
0.8	.9965970512	-0.774410e-4	-0.6095e-6	1.007648760	0.77447e-4	-0.602e-6
0.9	1.000717235	-0.38719e-4	-0.609e-6	1.006243089	0.38725e-4	-0.604e-6
1	1.004837418	0.3e-8	-0.607e-6	1.004837418	0.3e-8	-0.604e-6

Table. 3.2.

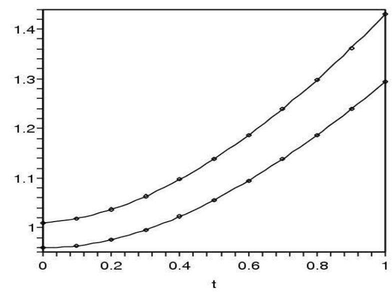

Fig. 3.1. Comparison of exact and approximate solutions obtained by the use of the IPC at 0-level in example 3.6.1.

Solid curve: Real value; Dotted curve: IPC.

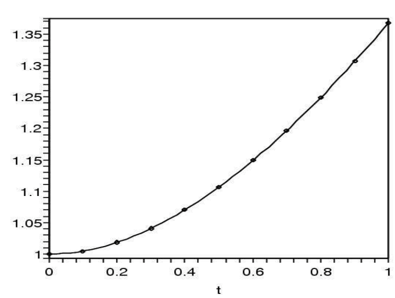

Fig. 3.2. Comparison of exact and approximate solutions obtained by the use of the IPC at 1-level in example 3.6.1.

Solid curve: Real value; Dotted curve: IPC.

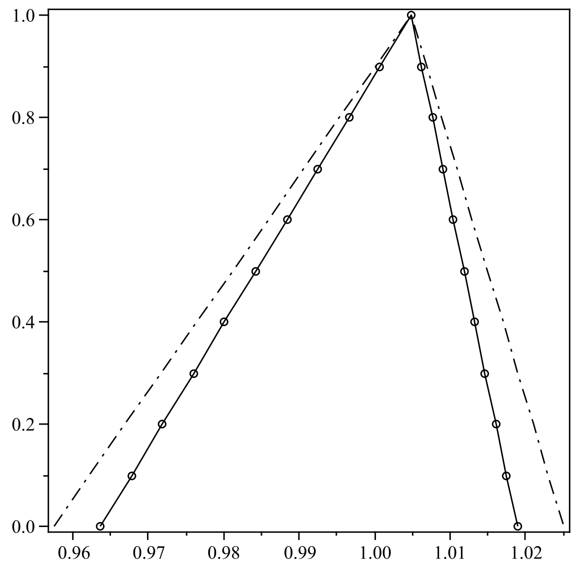

Fig. 3.3. Comparison of exact solution, Adams-Bashforth three-step method and explicit three-step method in example 3.6.1.

Solid curve: Real value; Dash Dotted curve: Adams-Bashforth three-step method and Dotted curve: explicit three-step method.

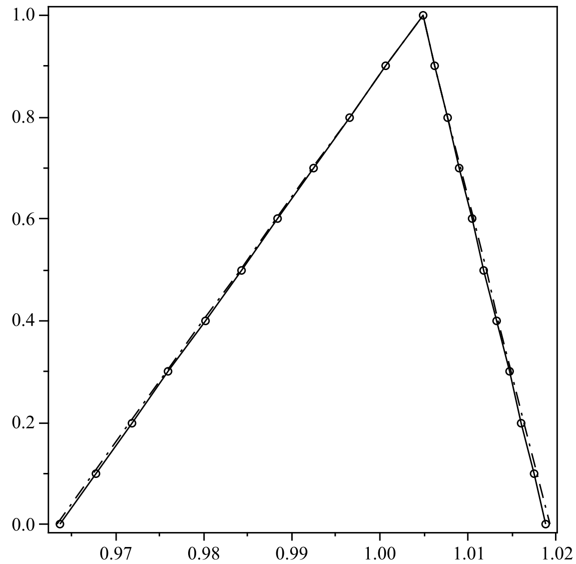

Fig. 3.4. Comparison of exact solution, Predictor-Corrector method and IPC method in example 3.6.1.

Solid curve: Real value; Dotted curve: IPC method and

Dash Dotted curve: Predictor-Corrector method.

Example 3.6.2. Consider the initial value problem

$\widetilde{y'}(t) = -\widetilde{y}(t)$,

$\widetilde{y}(0) = (0.96 + 0.04\alpha, 1.01 - 0.01\alpha)$,

$\widetilde{y}(0.01) = (0.01+(0.985+0.015\alpha)e^{-0.01}-(1-\alpha)0.025e^{0.01}, 0.01+(0.985+0.015\alpha)e^{-0.01}+(1-\alpha)0.025e^{0.01})$,

$\widetilde{y}(0.02) = (0.02+(0.985+0.015\alpha)e^{-0.02}-(1-\alpha)0.025e^{0.02}, 0.02+(0.985+0.015\alpha)e^{-0.02}+(1-\alpha)0.025e^{0.02})$,

The exact solution at $t = 0.1$ is given by

$\widetilde{Y}(0.1, \alpha) = ((0.985 + 0.015\alpha)e^{-0.1} - (1 - \alpha)0.025e^{0.1}, (0.985 + 0.015\alpha)e^{-0.1} + (1 - \alpha)0.025e^{0.1})$.

The results of the implementation of the numerical method, the explicit three-step method, and the IPC three-step method, with $N = 10$, on the mentioned differential equation are shown in Tables 3.3 and 3.4.

Table 3.3, contains real values, the error of Adams-Bashforth three-step method denoted by error 1 and the error of the explicit three-step method denoted by error 2. Also, Table 3.4, contains real values, the error of the Predictor-Corrector Three-step method, denoted by error 1 and the error of the IPC three-step method denoted by error 2. The comparison of the exact solution with the approximate one at the 0−level and 1−level is depicted in Figures 3.6, 3.7.

α	$\underline{Y}$	Error1	Error2	$\overline{Y}$	Error1	Error2
0	.8636355838	-0.62158733e-2	0.6788e-6	.9188941296	0.62158200e-2	0.6366e-6
0.1	.8677557672	-0.55942887e-2	0.6777e-6	.9174884584	0.55942351e-2	0.6395e-6
0.2	.8718759506	-0.49727040e-2	0.6766e-6	.9160827873	0.49726505e-2	0.6427e-6
0.3	.8759961341	-0.43511194e-2	0.6756e-6	.9146771161	0.43510658e-2	0.6458e-6
0.4	.8801163175	-0.37295348e-2	0.6745e-6	.9132714450	0.37294812e-2	0.6489e-6
0.5	.8842365009	-0.31079502e-2	0.6733e-6	.9118657738	0.31078964e-2	0.6520e-6
0.6	.8883566843	-0.24863656e-2	0.6722e-6	.9104601026	0.24863117e-2	0.6551e-6
0.7	.8924768677	-0.18647810e-2	0.6712e-6	.9090544315	0.18647271e-2	0.6582e-6
0.8	.8965970512	-0.12431963e-2	0.6701e-6	.9076487603	0.12431424e-2	0.6614e-6
0.9	.9007172346	-0.6216118e-3	0.6689e-6	.9062430892	0.6215577e-3	0.6644e-6
1	.9048374180	-0.272e-7	0.6678e-6	.9048374180	-0.270e-7	0.6676e-6

Table. 3.3.

α	$\underline{Y}$	Error1	Error2	$\overline{Y}$	Error1	Error2
0	.8636355838	-0.3872159e-3	-0.6167e-6	.9188941296	0.3872223e-3	-0.5801e-6
0.1	.8677557672	-0.3484940e-3	-0.6158e-6	.9174884584	0.3485004e-3	-0.5830e-6
0.2	.8718759506	-0.3097720e-3	-0.6148e-6	.9160827873	0.3097785e-3	-0.5857e-6
0.3	.8759961341	-0.2710502e-3	-0.6139e-6	.9146771161	0.2710566e-3	-0.5857e-6
0.4	.8801163175	-0.2323283e-3	-0.6129e-6	.9132714450	0.2323346e-3	-0.5911e-6
0.5	.8842365009	-0.1936063e-3	-0.6121e-6	.9118657738	0.1936128e-3	-0.5939e-6
0.6	.8883566843	-0.1548844e-3	-0.6112e-6	.9104601026	0.1548909e-3	-0.5966e-6
0.7	.8924768677	-0.1161625e-3	-0.6102e-6	.9090544315	0.1161689e-3	-0.5993e-6
0.8	.8965970512	-0.774406e-4	-0.6093e-6	.9076487603	0.774471e-4	-0.6021e-6
0.9	.9007172346	-0.387187e-4	-0.6084e-6	.9062430892	0.387251e-4	-0.6048e-6
1	.9048374180	0.32e-8	-0.6084e-6	.9048374180	0.33e-8	-0.6076e-6

Table. 3.4.

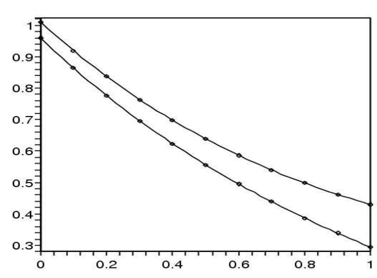

Fig. 3.5. Comparison of exact and approximate solutions obtained

by the use of the IPC at 0-level in example 3.6.2.

Solid curve: Real value; Dotted curve: IPC.

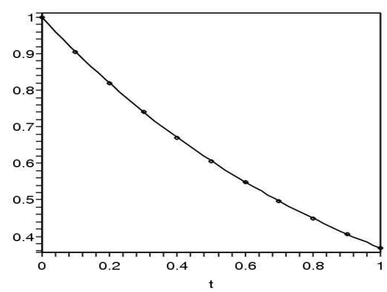

Fig. 3.6. Comparison of exact and approximate solutions obtained by the use of the IPC at 1-level in example 3.6.2.

Solid curve: Real value; Dotted curve: IPC.

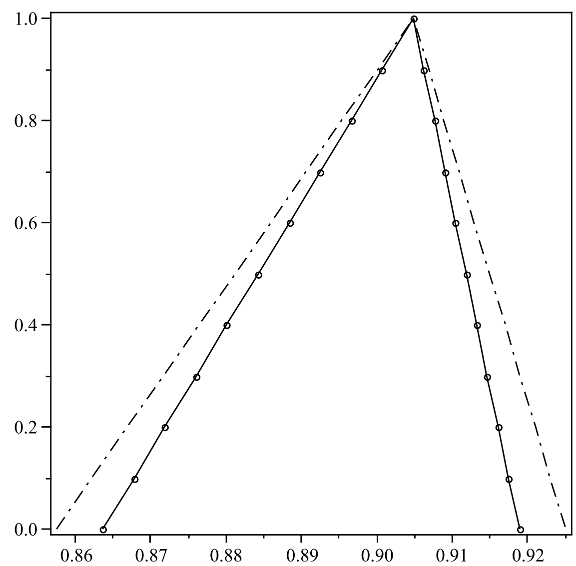

Fig. 3.7. Comparison of exact solution, Adams-Bashforth three-step method and explicit three-step method in example 3.6.2.

Solid curve: Real value; Dash Dotted curve: Adams-Bashforth three-step method and Dotted curve: explicit three-step method.

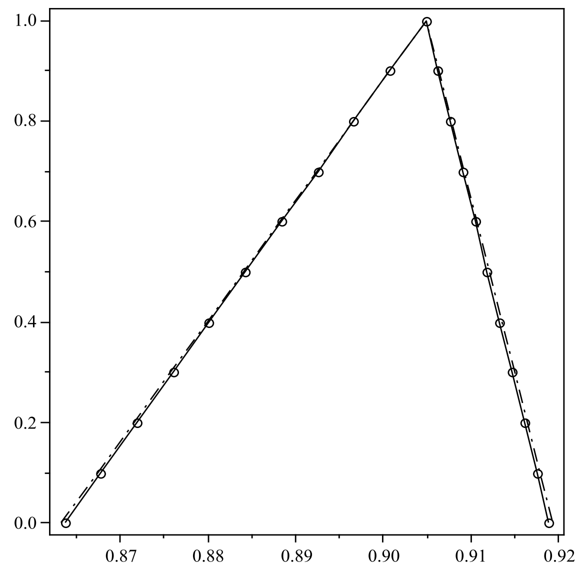

Fig. 3.8. Comparison of exact solution, Predictor-Corrector method and IPC method in example 3.6.2.

Solid curve: Real value; Dotted curve: IPC method and

Dash Dotted curve: Predictor-Corrector method.

Chapter 4

Hybrid fuzzy differential equations

4.1 Introduction

In this chapter, some numerical methods for solving hybrid fuzzy differential equations are proposed. We develop a numerical method for solving hybrid fuzzy differential equations by an application of the multi-step methods for fuzzy differential equations. The convergence theorem and a numerical example are also provided.

4.2 Multi-step methods for solving hybrid fuzzy differential equations

Consider the hybrid fuzzy differential system

$$\begin{cases} x'(t) = f(t, x(t), \lambda_k(x_k)), & t \in I_k, \\ x(t_{k,1}) = \widetilde{\alpha}_{k,1}, \quad x(t_{k,2}) = \widetilde{\alpha}_{k,2} & \end{cases} \tag{4.1}$$

where " $'$ " denotes the Seikkala derivative, and I_k is an interval such that $I_k \cap I_{k-1} \neq \emptyset$, $f \in C[R^+ \times E \times E, E]$, $\lambda_k \in C[E, E]$.

To be specific, the system would look like

$$x'(t) = \begin{cases} x_0'(t) = f(t, x_0(t), \lambda_0(x_0)), \quad x_0(t_{0,0}) = \widetilde{\alpha}_{0,1} \quad x_0(t_{0,1}) = \widetilde{\alpha}_{0,2} \quad t, t_{0,0}, t_{0,1} \in I_0, \\ x_1'(t) = f(t, x_1(t), \lambda_1(x_1)), \quad x_1(t_{1,0}) = \widetilde{\alpha}_{1,1} \quad x_1(t_{1,1}) = \widetilde{\alpha}_{1,2} \quad t, t_{1,0}, t_{1,1} \in I_1, \\ \cdots \\ x_k'(t) = f(t, x_k(t), \lambda_k(x_k)), \quad x_k(t_{k,0}) = \widetilde{\alpha}_{k,1} \quad x_k(t_{k,1}) = \widetilde{\alpha}_{k,2} \quad t, t_{k,0}, t_{k,1} \in I_k, \\ \cdots \end{cases} \tag{4.2}$$

For solving system (4.2) by the two-step method, first we replace each interval $I_0, I_1, \cdots, I_k, \ldots,$ by set of $N+1$ discrete equally spaced points (incuding the end-points) at which the exact solution $x(t;\alpha) = (\underline{x}(t;\alpha), \overline{x}(t;\alpha))$ is approximated by some $(\underline{y}_k(t;\alpha), \overline{y}_k(t;\alpha))$. For the chosen grid points on I_k at $t_{k,n} = t_k + nh, \quad h = \frac{t_{k+1}-t_k}{N}, \; 0 \leq n \leq N$, let $(\underline{Y}_k(t;\alpha), \overline{Y}_k(t;\alpha)) \equiv (\underline{x}(t,\alpha), \overline{x}(t,\alpha))$ where $I_k \cap I_{k-1} = \{t_{k-1,N-1}, t_{k-1,N}, t_{k,0}, t_{k,1}\}$, such that $t_{k-1,N-1} = t_{k,0}, t_{k-1,N} = t_{k,1}$.

Then we have the following system:

$$x'(t) = \begin{cases} x_0'(t) = f(t, x_0(t), \lambda_0(x_0)), \quad x_0(t_{0,0}) = \alpha_{0,1}, \quad x_0(t_{0,1}) = \alpha_{0,2} \quad , t \in \{t_{0,0}, t_{0,1}, \cdots, t_{0,N-1}, t_{0,N}\}, \\ x_1'(t) = f(t, x_1(t), \lambda_1(x_1)), \quad x_1(t_{1,0}) = \alpha_{1,1}, \quad x_1(t_{1,1}) = \alpha_{1,2} \quad , t \in \{t_{1,0}, t_{1,1}, \cdots, t_{1,N-1}, t_{1,N}\}, \\ \cdots \\ x_k'(t) = f(t, x_k(t), \lambda_k(x_k)), \quad x_k(t_{k,0}) = \alpha_{k,1}, \quad x_k(t_{k,1}) = \alpha_{k,2} \quad , t \in \{t_{k,0}, t_{k,1}, \cdots, t_{k,N-1}, t_{k,N}\}, \\ \cdots \end{cases} \tag{4.3}$$

Then we solve the following system (4.3) by the Adams Bashforth two-step method, which can be written as:

$$\begin{cases} \underline{Y}_{k,n+1}(\alpha) \approx \underline{Y}_{k,n}(\alpha) - (\frac{h}{2})\overline{f}(t_{k,n-1}, Y_{k,n-1}(\alpha)) + (\frac{3h}{2})\underline{f}(t_{k,n}, Y_{k,n}(\alpha)), \\ \overline{Y}_{k,n+1}(\alpha) \approx \overline{Y}_{k,n}(\alpha) - (\frac{h}{2})\underline{f}(t_{k,n-1}, Y_{k,n-1}(\alpha)) + (\frac{3h}{2})\overline{f}(t_{k,n}, Y_{k,n}(\alpha)), \end{cases} \tag{4.4}$$

By (4.4) we define:

$$\begin{cases} \underline{y}_{k,n+1}(\alpha) = \underline{y}_{k,n}(\alpha) - (\frac{h}{2})\overline{f}(t_{k,n-1}, y_{k,n-1}(\alpha)) + (\frac{3h}{2})\underline{f}(t_{k,n}, y_{k,n}(\alpha)), \\ \overline{y}_{k,n+1}(\alpha) = \overline{y}_{k,n}(\alpha) - (\frac{h}{2})\underline{f}(t_{k,n-1}, y_{k,n-1}(\alpha)) + (\frac{3h}{2})\overline{f}(t_{k,n}, y_{k,n}(\alpha)), \end{cases} \tag{4.5}$$

However, (4.5) will use

$$\begin{cases} \underline{y}_{0,0}(\alpha) = \underline{x}_{0,0}(\alpha),\ \overline{y}_{0,0}(\alpha) = \overline{x}_{0,0}(\alpha),\ \underline{y}_{0,1}(\alpha) = \underline{x}_{0,1}(\alpha),\ \overline{y}_{0,1}(\alpha) = \overline{x}_{0,1}(\alpha), \\ \underline{y}_{k,0}(\alpha) = \underline{y}_{k-1,N-1}(\alpha),\ \overline{y}_{k,0}(\alpha) = \overline{y}_{k-1,N-1}(\alpha),\ \underline{y}_{k,1}(\alpha) = \underline{y}_{k-1,N}(\alpha), \overline{y}_{k,1}(\alpha) = \overline{y}_{k-1,N}(\alpha). \end{cases} \tag{4.6}$$

if $k \geq 1$. Then (4.5) yields an approximation of $\underline{Y}_k(t;\alpha)$ and $\overline{Y}_k(t;\alpha)$ for each of the intervals $I_0, I_1, \cdots, I_k, \cdots$

In a similar way, the hybrid fuzzy differential system $x'(t) = f(t, x(t), \lambda_k(x_k))$ can be solved by the Adams-Bashforth three-step and the Adams-Moulton two-step method as follows:

∘Adams-Bashforth three-step method:

$$\underline{y}_{k,n+2}(\alpha) = \underline{y}_{k,n+1} + \frac{h}{12}[5\underline{f}(t_{k,n-1}, y_{k,n-1}(\alpha)) - 16\overline{f}(t_{k,n}, y_{k,n}(\alpha)) + 23\underline{f}(t_{k,n+1}, y_{k,n+1}(\alpha))],$$

$$\overline{y}_{k,n+2}(\alpha) = \overline{y}_{k,n+1} + \frac{h}{12}[5\overline{f}(t_{k,n-1}, y_{k,n-1}(\alpha)) - 16\underline{f}(t_{k,n}, y_{k,n}(\alpha)) + 23\overline{f}(t_{k,n+1}, y_{k,n+1}(\alpha))]$$

∘Adams-Moulton two-step method:

$$\underline{y}_{k,n+1}(\alpha) = \underline{y}_{k,n}(\alpha) - (\frac{h}{12})\overline{f}(t_{k,n-1}, y_{k,n-1}(\alpha)) + (\frac{2h}{3})\underline{f}(t_{k,n}, y_{k,n}(\alpha)) + (\frac{5h}{12})\underline{f}(t_{k,n+1}, y_{k,n+1}(\alpha)),$$

$$\overline{y}_{k,n+1}(\alpha) = \overline{y}_{k,n}(\alpha) - (\frac{h}{12})\underline{f}(t_{k,n-1}, y_{k,n-1}(\alpha)) + (\frac{2h}{3})\overline{f}(t_{k,n}, y_{k,n}(\alpha)) + (\frac{5h}{12})\overline{f}(t_{k,n+1}, y_{k,n+1}(\alpha))$$

.

4.3 Predictor-Corrector method

The following algorithm is based on the Adams-Bashforth three-step method as the Predictor and the Adams-Moulton two-step method as the Corrector.

ALGORITHM (Predictor-Corrector three-step method)

To approximate the solution of the following fuzzy initial value problem

$$\begin{cases} x'(t) = f(t, x(t), \lambda_k(x_k)), & t \in I_k = [t_k, t_{k+1}] \\ \underline{x}(t_{k,0}) = \underline{\alpha}_{k,0}, \underline{x}(t_{k,1}) = \underline{\alpha}_{k,1}, \underline{x}(t_{k,2}) = \underline{\alpha}_{k,2}, \\ \overline{x}(t_{k,0}) = \overline{\alpha}_{k,0}, \overline{x}(t_{k,1}) = \overline{\alpha}_{k,1}, \overline{x}(t_{k,2}) = \overline{\alpha}_{k,2}. \end{cases}$$

The positive integer N is chosen.

step 1. Let $k = j$.

step 2.

$$\underline{w}_{j,0}(\alpha) = \underline{\alpha}_{j,0}, \underline{w}_{j,1}(\alpha) = \underline{\alpha}_{j,1}, \underline{w}_{j,2}(\alpha) = \underline{\alpha}_{j,2},$$

$$\overline{w}_{j,0}(\alpha) = \overline{\alpha}_{j,0}, \overline{w}_{j,1}(\alpha) = \overline{\alpha}_{j,1}, \overline{w}_{j,2}(\alpha) = \overline{\alpha}_{j,2}.$$

step 3. Let $h = \frac{t_{j+1}-t_j}{N}$,

step 4. Let $i = 1$.

step 5. Let

$$\begin{cases} \underline{w}^{(0)}_{j,i+2}(\alpha) = \underline{w}_{j,i+1}(\alpha) + \frac{h}{12}[5\underline{f}(t_{j,i-1}, w_{j,i-1}(\alpha)) - 16\overline{f}(t_{j,i}, w_{j,i}(\alpha)) + 23\underline{f}(t_{j,i+1}, w_{j,i+1}(\alpha))], \\ \overline{w}^{(0)}_{j,i+2}(\alpha) = \overline{w}_{j,i+1}(\alpha) + \frac{h}{12}[5\overline{f}(t_{j,i-1}, w_{j,i-1}(\alpha)) - 16\underline{f}(t_{j,i}, w_{j,i}(\alpha)) + 23\overline{f}(t_{j,i+1}, w_{j,i+1}(\alpha))], \end{cases}$$

step 6. Let $t_{i+2} = t_0 + (i+2)h$,

step 7. Let

$$\begin{cases} \underline{w}_{j,i+2}(\alpha) = \underline{w}_{j,i+1} - (\frac{h}{12})\overline{f}(t_{j,i}, w_{j,i}(\alpha)) + (\frac{2h}{3})\underline{f}(t_{j,i+1}, w_{j,i+1}(\alpha)) + (\frac{5h}{12})\underline{f}(t_{j,i+2}, w^{(0)}_{j,i+2}(\alpha)), \\ \overline{w}_{j,i+2}(\alpha) = \overline{w}_{j,i+1} - (\frac{h}{12})\underline{f}(t_{j,i}, w_{j,i}(\alpha)) + (\frac{2h}{3})\overline{f}(t_{j,i+1}, w_{j,i+1}(\alpha)) + (\frac{5h}{12})\overline{f}(t_{j,i+2}, w^{(0)}_{j,i+2}(\alpha)), \end{cases}$$

step 8. $i = i+1$.

step 9. If $i \leq N-2$ go to step 5.

step 10. $j = j+1$.

step 11.

$$\underline{w}_{j+1,0}(\alpha) = \underline{w}_{j,N-2}(\alpha), \underline{w}_{j+1,1}(\alpha) = \underline{w}_{j,N-1}(\alpha), \underline{w}_{j+1,2}(\alpha) = \underline{w}_{j,N}(\alpha)$$

$$\overline{w}_{j+1,0}(\alpha) = \overline{w}_{j,N-2}(\alpha), \overline{w}_{j+1,1}(\alpha) = \overline{w}_{j,N-1}(\alpha), \overline{w}_{j+1,2}(\alpha) = \overline{w}_{j,N}(\alpha)$$

step 12. Go to step 3.

step 13. The algorithm will terminate and $(\underline{w}_{k,N}(\alpha), \overline{w}_{k,N}(\alpha))$ approximates the real value of $(\underline{x}(t_{k+1};\alpha), \overline{x}(t_{k+1};\alpha))$.

4.4 Convergence

To prove the of convergence of the approximations in (4.5), For a prefixed k and $\alpha \in [0,1]$, we need to prove:

$\lim_{h\to 0} \underline{y}_{k,N}(\alpha) = \underline{x}(t,\alpha) \qquad \lim_{h\to 0} \overline{y}_{k,N}(\alpha) = \overline{x}(t,\alpha)$

Lemma 4.4.1. *Let a sequence of numbers* $\{w_n\}_{n=0}^{N}$ *satisfy:*

$$|w_{n+1}| \leq A|w_n| + B|w_{n-1}|, \qquad 0 \leq n \leq N-1$$

for some given positive constants A and B. Then

$|w_n| \leq (A^{n-1} + \beta_1 A^{n-3}B + \beta_2 A^{n-5}B^2 + \ldots + \beta_s B^{[\frac{n}{2}]})|w_1| + (A^{n-2}B + \gamma_1 A^{n-4}B^2 + \ldots + \gamma_t AB^{[\frac{n}{2}]})|w_0| \qquad n \;\; odd$

and

$|w_n| \leq (A^{n-1} + \beta_1 A^{n-3}B + \beta_2 A^{n-5}B^2 + \ldots + \beta_s AB^{[\frac{n}{2}]-1})|w_1| + (A^{n-2}B + \gamma_1 A^{n-4}B^2 + \ldots + \gamma_t B^{[\frac{n}{2}]})|w_0| \qquad n \;\; even$

where β_s, γ_t are constants for all s and p.

The proof is straightforward, using mathematical induction is straightforward.

Theorem 4.4.2. *Consider system (4.4). For a fixed $K \in Z^+$ and $\alpha \in [0,1]$,*
$\lim_{h\to 0} \underline{y}_{k,N}(\alpha) = \underline{x}(t_{k+1}, \alpha)$,
$\lim_{h\to 0} \overline{y}_{k,N}(\alpha) = \overline{x}(t_{k+1}, \alpha)$.

Let $\{z_{i,n}(\alpha)\}_{n=0}^{N}$ be the Adams-Bashforth two-step approximation to the fuzzy IVP:

$$\begin{cases} x'(t) = f(t, x(t), \lambda_k(x_k)), & t \in I_k, \\ x(t_{k,1}) = \widetilde{\alpha}_{k,1}, \quad x(t_{k,2}) = \widetilde{\alpha}_{k,2} \end{cases} \tag{4.7}$$

If $\{y_{i,n}(\alpha)\}_{n=0}^{N}$ denotes the result of (4.5) for some $y_{i,0}(\alpha), y_{i,1}(\alpha)$ as the initial points, for each $l = 1, \cdots N$ we have

$$\underline{y}_{i,l}(\alpha) = \underline{y}_{i,l-1}(\alpha) - (\frac{h}{2})\overline{f}(t_{i,l-2}, y_{i,l-2}(\alpha)) + (\frac{3h}{2})\underline{f}(t_{i,l-1}, y_{i,l-1}(\alpha)),$$

$$\overline{y}_{i,l}(\alpha) = \overline{y}_{i,l-1}(\alpha) - (\frac{h}{2})\underline{f}(t_{i,l-2}, y_{i,l-2}(\alpha)) + (\frac{3h}{2})\overline{f}(t_{i,l-1}, y_{i,l-1}(\alpha)).$$

Also

$$\underline{z}_{i,l}(\alpha) = \underline{z}_{i,l-1}(\alpha) - (\frac{h}{2})\overline{f}(t_{i,l-2}, z_{i,l-2}(\alpha)) + (\frac{3h}{2})\underline{f}(t_{i,l-1}, z_{i,l-1}(\alpha)),$$

$$\overline{z}_{i,l}(\alpha) = \overline{z}_{i,l-1}(\alpha) - (\frac{h}{2})\underline{f}(t_{i,l-2}, z_{i,l-2}(\alpha)) + (\frac{3h}{2})\overline{f}(t_{i,l-1}, z_{i,l-1}(\alpha)).$$

By the Lipschitz condition we have

$$|\underline{z}_{i,l} - \underline{y}_{i,l}| \leq |\underline{z}_{i,l-1} - \underline{y}_{i,l-1}| + \frac{h}{2}L_1|\overline{z}_{i,l-2} - \overline{y}_{i,l-2}| + \frac{3h}{2}L_2|\underline{z}_{i,l-1} - \underline{y}_{i,l-1}|,$$

$$|\overline{z}_{i,l} - \overline{y}_{i,l}| \leq |\overline{z}_{i,l-1} - \overline{y}_{i,l-1}| + \frac{h}{2}L_3|\underline{z}_{i,l-2} - \underline{y}_{i,l-2}| + \frac{3h}{2}L_4|\overline{z}_{i,l-1} - \overline{y}_{i,l-1}|.$$

We put $L = \max\{L_1, L_2, L_3, L_4\}$, $s_{i,l} = \underline{z}_{i,l} - \underline{y}_{i,l}$ and $t_{i,l} = \overline{z}_{i,l} - \overline{y}_{i,l}$, then

$$|s_{i,l}| \leq (1 + \frac{3hL}{2})|s_{i,l-1}| + \frac{hL}{2}|t_{i,l-2}| \quad and \quad |t_{i,l}| \leq (1 + \frac{3hL}{2})|t_{i,l-1}| + \frac{hL}{2}|s_{i,l-2}|.$$

By adding the above equations and putting $|w_{i,l}| = |s_{i,l}| + |t_{i,l}|$ we obtain

$$|s_{i,l}|, |t_{i,l}| \leq |w_{i,l}| \leq (1 + \frac{3hL}{2})|w_{i,l-1}| + (\frac{hL}{2})|w_{i,l-2}|.$$

By using Lemma (4.4.1),

$$|w_{i,N}| \leq C|w_{i,1}| + D|w_{i,0}|, \tag{4.8}$$

where $C = ((1+\frac{3hL}{2})^{N-1} + \beta_1(1+\frac{3hL}{2})^{N-3}(\frac{hL}{2}) + \beta_2(1+\frac{3hL}{2})^{N-5}(\frac{hL}{2})^2 + \ldots + \beta_s(\frac{hL}{2})^{[\frac{N}{2}]})$ and $D = ((1+\frac{3hL}{2})^{N-2}(\frac{hL}{2}) + \gamma_1(1+\frac{3hL}{2})^{N-4}(\frac{hL}{2})^2 + \ldots + \gamma_t(1+\frac{3hL}{2})(\frac{hL}{2})^{[\frac{N}{2}]})$.

Also

$$|w_{i,N-1}| \leq E|w_{i,1}| + F|w_{i,0}|, \tag{4.9}$$

where $E = ((1+\frac{3hL}{2})^{N-2} + \beta_1(1+\frac{3hL}{2})^{N-4}(\frac{hL}{2}) + \beta_2(1+\frac{3hL}{2})^{N-6}(\frac{hL}{2})^2 + \ldots + \beta_s(\frac{hL}{2})^{[\frac{N-1}{2}]})$ and $F = ((1+\frac{3hL}{2})^{N-3}(\frac{hL}{2}) + \gamma_1(1+\frac{3hL}{2})^{N-5}(\frac{hL}{2})^2 + \ldots + \gamma_t(1+\frac{3hL}{2})(\frac{hL}{2})^{[\frac{N-1}{2}]})$

For $i = 0$, we have

$$\begin{cases} \underline{y}_{0,0}(\alpha) = \underline{z}_{0,0}(\alpha) = \underline{x}_{0,0}(\alpha), & \overline{y}_{0,0}(\alpha) = \overline{z}_{0,0}(\alpha) = \overline{x}_{0,0}(\alpha), \\ \underline{y}_{0,1}(\alpha) = \underline{z}_{0,1}(\alpha) = \underline{x}_{0,1}(\alpha), & \overline{y}_{0,1}(r) = \overline{z}_{0,1}(\alpha) = \overline{x}_{0,1}(\alpha). \end{cases} \tag{4.10}$$

By (4.8), (4.9), it is clear that

$$|w_{0,N-1}|, |w_{0,N}| = 0. \tag{4.11}$$

By using Theorem (2.5.2), and (4.11) we have

$$\begin{cases} \lim_{h\to 0} \underline{z}_{0,N-1} = \underline{x}_{1,0}, & \lim_{h\to 0} \overline{z}_{0,N-1} = \overline{x}_{1,0} \\ \lim_{h\to 0} \underline{z}_{0,N} = \underline{x}_{1,1}, & \lim_{h\to 0} \overline{z}_{0,N} = \overline{x}_{1,1} \end{cases} \tag{4.12}$$

then

$$\begin{cases} \lim_{h\to 0} \underline{y}_{0,N-1} = \underline{x}_{1,0}, & \lim_{h\to 0} \overline{y}_{0,N-1} = \overline{x}_{1,0} \\ \lim_{h\to 0} \underline{y}_{0,N} = \underline{x}_{1,1}, & \lim_{h\to 0} \overline{y}_{0,N} = \overline{x}_{1,1} \end{cases} \tag{4.13}$$

For $i = 1$, we have

$$\begin{cases} \underline{y}_{1,0}(\alpha) = \underline{y}_{0,N-1}(\alpha), & \overline{y}_{1,0}(\alpha) = \overline{y}_{0,N-1}(\alpha) \\ \underline{y}_{1,1}(\alpha) = \underline{y}_{0,N}(\alpha), & \overline{y}_{1,1}(\alpha) = \overline{y}_{0,N}(\alpha) \\ \underline{z}_{1,0} = \underline{x}_{1,0}, & \overline{z}_{1,0} = \overline{x}_{1,0} \\ \underline{z}_{1,1} = \underline{x}_{1,1}, & \overline{z}_{1,1} = \overline{x}_{1,1} \end{cases} \tag{4.14}$$

From (4.13), (4.14) we obtain

$$\begin{cases} \lim_{h\to 0} \underline{y}_{0,N-1} = \underline{x}_{1,0} = \underline{z}_{1,0}, & then \quad |\underline{y}_{1,0} - \underline{z}_{1,0}| \to 0, \\ \lim_{h\to 0} \overline{y}_{0,N-1} = \overline{x}_{1,0} = \overline{z}_{1,0}, & then \quad |\overline{y}_{1,0} - \overline{z}_{1,0}| \to 0, \\ \lim_{h\to 0} \underline{y}_{0,N} = \underline{x}_{1,1} = \underline{z}_{1,1}, & then \quad |\underline{y}_{1,1} - \underline{z}_{1,1}| \to 0, \\ \lim_{h\to 0} \overline{y}_{0,N} = \overline{x}_{1,1} = \overline{z}_{1,1}, & then \quad |\overline{y}_{1,1} - \overline{z}_{1,1}| \to 0, \end{cases} \tag{4.15}$$

then

$$|w_{1,N-1}|, |w_{1,N}| = 0. \tag{4.16}$$

In a similar way we obtain

$$|w_{1,N-1}|, |w_{1,N}| = 0. \tag{4.17}$$

Then by using Theorem (2.5.2), we obtain:

$$\begin{cases} \lim_{h\to 0} \underline{y}_{k,N}(\alpha) = \underline{x}(t_{k+1},\alpha), \\ \lim_{h\to 0} \overline{y}_{k,N}(\alpha) = \overline{x}(t_{k+1},\alpha). \end{cases} \tag{4.18}$$

4.5 Numerical example

Example 4.5.1. Consider the following hybrid fuzzy IVP,

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)), & t \in [t_k, t_{k+1}], t_k = k, k = 0, 1, 2, \cdots \\ x(0;\alpha) = [0.75 + 0.25r, 1.125 - 0.125r], & 0 \le \alpha \le 1, \end{cases} \tag{4.19}$$

where

$$m(t) = \begin{cases} 2(t(mod\ 1)), & if\ t(mod\ 1) \le 0.5 \\ 2(1 - t(mod\ 1)), & if\ t(mod\ 1) > 0.5 \end{cases}, \quad \lambda_k(\mu) = \begin{cases} \widehat{0} & if\ k = 0 \\ \mu, & if k \in \{1, 2, \cdots\} \end{cases}$$

The hybrid fuzzy IVP (4.20) is equivalent to the following system of fuzzy IVPs:

$$\begin{cases} x_0'(t) = x_0(t), & t \in [0, 1], \\ x_0(0;\alpha) = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha], & 0 \le \alpha \le 1, \\ x_i'(t) = x_i(t) + m(t)x_i(t_i), & t \in [t_i, t_{i+1}], \quad x_i(t_i) = x_{i-1}(t_i), i = 1, 2, \cdots . \end{cases} \tag{4.20}$$

In (4.20), $x(t)+m(t)\lambda_k(x(t_k))$ is continuous function of t, x, and $\lambda_k(x(t_k))$. Therefore by [20], for each $k = 0, 1, 2, \cdots$, the fuzzy IVP

$$\begin{cases} x'(t) = x(t) + m(t)\lambda_k(x(t_k)) & t \in [t_k, t_{k+1}], t_k = k, \\ x(t_k) = x_{t_k}, \end{cases} \tag{4.21}$$

has a unique solution on $[t_k, t_{k+1}]$. To numerically solve the hybrid fuzzy IVP (4.20)by using the Euler method for hybrid fuzzy differential equations with $N = 10$ to obtain

$y_1(2, \alpha)$. we have

$$y_1(1, \alpha) = [(0.75 + 0.25\alpha)(1 + \frac{1}{10})^{10}, (1.125 - 0.125\alpha)(1 + \frac{1}{10})^{10}], \quad 0 \leq \alpha \leq 1.$$

Then

$$\underline{y}_1(1.1, \alpha) = \underline{y}_1(1, \alpha) + \frac{1}{10}(\underline{y}_1(1, \alpha) + m(1)\underline{y}_1(1, \alpha)),$$

$$\overline{y}_1(1.1, \alpha) = \overline{y}_1(1, \alpha) + \frac{1}{10}(\overline{y}_1(1, \alpha) + m(1)\overline{y}_1(1, \alpha)).$$

$y_1(1+\frac{i}{10}, \alpha) = [(1+\frac{1}{10})\underline{y}_1(1+\frac{i-1}{10}, \alpha)+\frac{i-1}{50}\underline{y}_1(1, \alpha), (1+\frac{1}{10})\overline{y}_1(1+\frac{i-1}{10}, \alpha)+\frac{i-1}{50}\overline{y}_1(1, \alpha)],$

and for $i = 7, 8, 9, 10,$

$y_1(1+\frac{i}{10}, \alpha) = [(1+\frac{1}{10})\underline{y}_1(1+\frac{i-1}{10}, \alpha)+\frac{11-i}{50}\underline{y}_1(1, \alpha), (1+\frac{1}{10})\overline{y}_1(1+\frac{i-1}{10}, \alpha)+\frac{11-i}{50}\overline{y}_1(1, \alpha)],$

Let

$$c_2 = (1 + \frac{1}{10})^{10} + \sum_{k=1}^{5} \frac{1}{10}(0.2k)(1 + \frac{1}{10})^{9-k} + \sum_{k=1}^{4} \frac{1}{10}(0.2k)(1 + \frac{1}{10})^{k-1}$$

Then

$y_1(2, \alpha) = c_2 y_1(1, \alpha)$

$\qquad = [c_2(0.75 + 0.25\alpha)(1 + \frac{1}{10})^{10}, c_2(1.125 + 0.125\alpha)(1 + \frac{1}{10})^{10}],\ 0 \leq \alpha \leq 1.$

t	Euler,$N = 10$	RK,$N = 2$	Predictor-Corrector,$N = 10$	Actual
1.0	2.59	2.717	2.717	2.718
1.5	4.75	5.286	5.28	5.29
2.0	8.66	9.67	9.67	9.68

Conclusions and Future works

Taking into account that the convergence order of the Euler method is $O(h)$ (as given in [21]), a higher order of convergence is obtainable by using the methods proposed and laid out in this thesis, namely, that a predictor-corrector method of convergence order $O(h^m)$ be used where the Adams-Bashforth m-step method and the Adams-Moulton (m-1)-step method are considered as predictor and corrector, respectively. By following the ideas of [32], the proposed methods can solve the stiff problems.

Also in this work, three numerical methods based on fuzzy spline interpolation, i.e. Explicit three-step method, Implicit two-step method, and IPC three-step method, were discussed. We showed that our proposed IPC three-step method has more accuracy and gives a better approximation than the Adams-Bashforth, the Adams-Moulton and the predictor-corrector methods. It is worth mentioning that the IPC three-step method can be generalized to IPC m-step methods.

In addition, a numerical method is provided for solving hybrid fuzzy differential equations, and the convergence of the method is proved.

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